Carnegie Mellon University Instructor: Venkatesan Guruswami
15-252: More GITCS Term: Spring 2020

Problem Set 7
Due date: Friday, March 6

Instructions

- You are allowed to collaborate with one other student taking the class, or do it solo.
- Collaboration is defined as discussion of the lecture material and solution approaches to the problems. Please note that *you are not allowed to share any written material and you must write up solutions on your own.* You must clearly acknowledge your collaborator in the write-up of your solutions.
- Solutions must be typeset in LaTeX and emailed to the TA.
- You should not search for solutions on the web. More generally, you should try and solve the problems without consulting any reference material other than what we cover in class and any provided notes.
- Please start working on the problem set early. Though it is short, the problem(s) might take some time to solve.

1. Let $G = (\{1, 2, \ldots, n\}, E)$ be an undirected graph, not necessarily bipartite. Associate an $n \times n$ matrix $M = \{M_{ij}\}$ to the graph as follows:

$$M_{ij} = \begin{cases} x_{ij} & \text{if } \{i,j\} \in E, i < j \\ -x_{ji} & \text{if } \{i,j\} \in E, i > j \\ 0 & \text{otherwise.} \end{cases}$$

where $x_{ij}, i < j$, are indeterminates. Prove that the determinant of M is not identically zero (when expanded as a polynomial in the x_{ij}'s) if and only if G has a perfect matching.

Hint: Replace each edge by two directed edges. What terms in the determinant expansion are non-zero, and which ones cancel out?

2. Suppose that $G = (A, B, E)$ is a bipartite graph, and a subset $R \subseteq E$ of the edges are marked red. Let $|A| = |B| = n$, and $k \leq n$ be a positive integer. We are interested in whether there is a perfect matching between A and B using *exactly k red edges.*

Towards this, we first randomly assign weights $w : E \to \{0, 1, 2, \ldots, T-1\}$ to all the edges. Let $\mathcal{M}$ denote the set of perfect matchings in G with exactly k red edges. Assume that the weight function w is such that there is a unique perfect matching $M \in \mathcal{M}$ that has the least weight among perfect matchings in $\mathcal{M}$. This is achieved with high probability by choosing T large enough, courtesy the isolation lemma.

Give an algorithm to compute the weight of this unique perfect matching M (that has the least weight among all the perfect matchings with exactly k red edges).

In fact, we can use this subroutine to find if there is a perfect matching in G that uses exactly k red edges. Do attempt this, but it is not part of the submission.

Hint: Compute the determinant of the adjacency matrix after assigning 2^w weights to the edges, with a slight indeterminate addition for the edges that are red.