Carnegie Mellon University Instructor: Venkatesan Guruswami
15-252: More GITCS Term: Spring 2020

Problem Set 6

Due date: Friday, February 28

Instructions

- You are allowed to collaborate with one other student taking the class, or do it solo.
- Collaboration is defined as discussion of the lecture material and solution approaches to the problems. Please note that *you are not allowed to share any written material and you must write up solutions on your own*. You must clearly acknowledge your collaborator in the write-up of your solutions.
- Solutions must be typeset in LaTeX and emailed to the TA.
- You should not search for solutions on the web. More generally, you should try and solve the problems without consulting any reference material other than what we cover in class and any provided notes.
- Please start working on the problem set early. Though it is short, the problem(s) might take some time to solve.

1. An undirected graph $G = (V, E)$ is said to be bipartite if the vertex set V can be partitioned into sets L and R such that every edge of the graph has one end point in L and the other end point in R. Prove that a connected d-regular graph G is bipartite if and only if $\lambda_1(A_G) = -d$.

   Hint: Use the Rayleigh coefficient formulation of λ_1.

2. Fix a connected d-regular non-bipartite graph G. Let $d = \lambda_n > \lambda_{n-1} \geq \ldots \geq \lambda_1 > -d$ be the eigenvalues of A_G. We analyze a random walk on the graph. Start with a vertex $v_0 = i$ of the graph G, and v_1 is picked as a uniformly random vertex among the neighbors of i, and v_2 is then picked as a uniformly random neighbor of v_1 and so on.

   For example, if the underlying graph is the three vertex graph $G = (V, E)$ such that $V = \{1, 2, 3\}$ and $E = \{(1, 2), (2, 3), (3, 1)\}$. Starting with $v_0 = 1$, the vertex v_1 is uniformly distributed among $\{2, 3\}$, and the first three vertices of the random walk are uniformly random among the set $\{(1, 2, 1), (1, 2, 3), (1, 3, 1), (1, 3, 2)\}$.

   Let $p_j^{(t)}$ is the probability that the vertex v_t is equal to j. In the above example, $p^{(1)} = (0, 0.5, 0.5)$ and $p^{(2)} = (0.5, 0.25, 0.25)$.

   (a) Show that $p^{(t)} = (\frac{1}{d}A)^t v^{(i)}$, where A is the adjacency matrix of G, and $v^{(i)}$ is defined as follows:

   $$v^{(i)}(j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

   (b) If the graph has good conductance, the random walk converges to the uniform distribution very fast. To be precise, prove that

   $$\max_{j \in \{1,2,\ldots,n\}} |p_j^{(t)} - \frac{1}{n}| \leq \epsilon \text{ if } t \geq \frac{\log \frac{2}{\epsilon}}{\log \frac{d}{\lambda}}$$

   where $\lambda = \max\{|\lambda_1|, |\lambda_{n-1}|\}$.

Hint: Write $v^{(i)}$ as $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + u$

You might want to prove the following first - $\|A_G u\| \leq \lambda \|u\|$, where $\|u\| = \sqrt{u^T u}$. Then, use Rayleigh coefficient formulation of the second largest eigenvalue of A applied to the vector u.