Carnegie Mellon University Instructor: Venkatesan Guruswami
15-252: More GITCS Term: Spring 2020

Problem Set 3

Due date: Friday, February 7

Instructions

- You are allowed to collaborate with one other student taking the class, or do it solo.
- Collaboration is defined as discussion of the lecture material and solution approaches to the problems. Please note that *you are not allowed to share any written material and you must write up solutions on your own.* You must clearly acknowledge your collaborator in the write-up of your solutions.
- Solutions must be typeset in LaTeX and emailed to the TA.
- You should not search for solutions on the web. More generally, you should try and solve the problems without consulting any reference material other than what we cover in class and any provided notes.
- Please start working on the problem set early. Though it is short, the problem(s) might take some time to solve.

1. Suppose we have a lambda expression T with the following property: for any given lambda expression M, $(T\ M)$ evaluates to $\mathsf{TRUE} := \lambda xy.x$ if M β-reduces to a normal form in finitely many steps under normal order evaluation, and $(T\ M)$ evaluates to $\mathsf{FALSE} := \lambda xy.y$ otherwise.

   Consider the lambda expression
   $$\lambda f.((T\ (f\ f))\ ((\lambda x.(x\ x))\ (\lambda x.(x\ x)))\ (\lambda x.x)).$$
   By deriving a contradiction based on the above expression, conclude that such a purported lambda expression T cannot exist.

2. A common way to execute functional programs like the lambda calculus is to use *combinators*. A combinator is simply a closed lambda expression, i.e., a lambda experssion with no free variables. A combinator basis B is a set of combinators which can express *any* closed lambda expression solely by composition of the combinators in the basis B.

   (a) Show that
   $$I := \lambda x.x$$
   $$T := \lambda xy.x$$
   $$S := \lambda xyz.x\ z\ (y\ z)$$
   form a combinator basis. <u>Hint</u>: Show how a closed lambda calculus expression can be expressed by converting the innermost lambda subexpressions to combinators and working outwards.

   (b) Express the church numeral $1 = \lambda fx.(f\ x)$ in the above basis.

   In fact, we can prove that $\{S, T\}$ itself is a combinator basis. This follows by proving that we can express I via some composition of just S and T. (Do try this as an exercise, but you don't need to submit it.)