1. For every language $L$ defined on a finite alphabet, there is a unique minimal DFA that accepts $L$. That is, if there are two DFAs of the same minimum size that accept $L$, they are isomorphic in that there is a re-naming of the states of one DFA that results in the other DFA. However, it turns out that such a property is not true for NFAs. Construct an example of a language which has two minimal NFAs that are not isomorphic to each other. (Again, an NFA $N_1$ is said to be isomorphic to another NFA $N_2$ of the same size if there is a re-naming of the states of $N_1$ that results in $N_2$.)

Hint: Consider the following language $L \subseteq \{a, b, c\}^*$:

$$L = \{ab, ac, ba, bc, ca, cb\}.$$

2. Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA accepting the language $L$. Let $n = |Q|$. For every $a \in \Sigma$, we define a matrix $M_a \in \{0, 1\}^{n \times n}$ associated with the automata $N$ as follows:

$$(M_a)_{i,j} = \begin{cases} 1, & \text{if } j \in \delta(i, a). \\ 0, & \text{otherwise}. \end{cases}$$

We also define a Boolean matrix multiplication as follows: We say $C = A \cdot B$, where $A, B, C \in \{0, 1\}^{n \times n}$ if and only if for every $1 \leq i, j \leq n$, $C_{i,j} = \bigvee_{k \in 1, 2, \ldots, n} (A_{i,k} \land B_{k,j})$.

(a) For a word $w = a_1a_2\ldots a_t \in \Sigma^t$, let $M_w = M_{a_1} \cdot M_{a_2} \cdots M_{a_t}$. Show that $N$ accepts $w$ if and only if $u_1M_wu_F = 1$, where $u_1$ is a row vector of size $n$ which is equal to $(1, 0, 0, \ldots, 0)$ (here, we assume that state 1 is the initial state $q_0$ in $N$) and $u_F$ is a column vector of size $n$ where $(u_F)_i = 1$ if $i$ is a final state in $N$, and 0 otherwise.
(b) Consider the following construction of a new DFA $N' = (Q', \Sigma', \delta', q'_0, F')$ defined as follows:

i. The statespace $Q' = \{0, 1\}^{n \times n}$ is the set of all the Boolean matrices of dimension $n \times n$.

ii. The initial state $q'_0$ corresponds to the identity matrix $I_{n \times n}$.

iii. For every $a \in \Sigma$, $M_2 \in \delta'(M_1, a)$ if and only if $M_2 = M_1 \cdot M_a$ where $M_a$ is the matrix defined earlier.

Define the set of final states in $N'$ appropriately such that $N'$ accepts the following language:

$$\sqrt{L} = \{x \in \Sigma^* : xx \in L\}$$

where $xx$ is the string obtained by concatenating the string $x$ with itself. As a corollary, prove that $\sqrt{L}$ is regular.