Carnegie Mellon University Instructor: Venkatesan Guruswami
15-252: More GITCS Term: Spring 2020

Problem Set 2

Due date: Friday, January 31

Instructions

- You are allowed to collaborate with one other student taking the class, or do it solo.
- Collaboration is defined as discussion of the lecture material and solution approaches to the problems. Please note that *you are not allowed to share any written material and you must write up solutions on your own.* You must clearly acknowledge your collaborator in the write-up of your solutions.
- Solutions must be typeset in LaTeX and emailed to the TA.
- You should not search for solutions on the web. More generally, you should try and solve the problems without consulting any reference material other than what we cover in class and any provided notes.
- Please start working on the problem set early. Though it is short, the problem(s) might take some time to solve.

1. For every language L defined on a finite alphabet, there is a unique minimal DFA that accepts L. That is, if there are two DFAs of the same minimum size that accept L, they are isomorphic in that there is a re-naming of the states of one DFA that results in the other DFA. However, it turns out that such a property is not true for NFAs. Construct an example of a language which has two minimal NFAs that are not isomorphic to each other. (Again, an NFA N_1 is said to be isomorphic to another NFA N_2 of the same size if there is a re-naming of the states of N_1 that results in N_2.)

2. Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA accepting the language L. Let $n = |Q|$. For every $a \in \Sigma$, we define a matrix $M_a \in \{0,1\}^{n\times n}$ associated with the automata N as follows:

$$(M_a)_{i,j} = \begin{cases} 1, & \text{if } j \in \delta(i,a). \\ 0, & \text{otherwise.} \end{cases}$$

We also define a Boolean matrix multiplication as follows: We say $C = A \cdot B$, where $A, B, C \in \{0,1\}^{n\times n}$ if and only if for every $1 \le i, j \le n$, $C_{i,j} = \bigvee_{k\in 1,2,\ldots,n} (A_{i,k} \wedge B_{k,j})$.

(a) For a word $w = a_1 a_2 \ldots a_t \in \Sigma^t$, let $M_w = M_{a_1} \cdot M_{a_2} \cdots M_{a_t}$. Show that N accepts w if and only if $u_1 M_w u_F = 1$, where u_1 is a row vector of size n which is equal to $(1, 0, 0, \ldots, 0)$ (here, we assume that state 1 is the initial state q_0 in N.) and u_F is a column vector of size n where $(u_F)_i = 1$ if i is a final state in N, and 0 otherwise.

(b) Consider the following construction of a new DFA $N' = (Q', \Sigma, \delta', q_0', F')$ defined as follows:

i. The statespace $Q' = \{0,1\}^{n\times n}$ is the set of all the Boolean matrices of dimension $n \times n$.

ii. The initial state q_0' corresponds to the identity matrix $I_{n \times n}$.

iii. For every $a \in \Sigma$, $M_2 \in \delta'(M_1, a)$ if and only if $M_2 = M_1 \cdot M_a$ where M_a is the matrix defined earlier.

Define the set of final states in N' appropriately such that N' accepts the following language:

$$\sqrt{L} = \{x \in \Sigma^* : xx \in L\}$$

where xx is the string obtained by concatenating the string x with itself. As a corollary, prove that $\sqrt{L}$ is regular.