Carnegie Mellon University Instructor: Venkatesan Guruswami
15-252: More GITCS Term: Spring 2020

Problem Set 14
No need to turn in!
All the best on your final courseworks, and have a great, safe summer!

We will close out our problem sets for the semester with the following question on graph 3-coloring (a perennial favorite of your instructor).

Recall that a graph $G = (V, E)$ is said to be 3-colorable if there is a coloring of its vertices $\chi : V \to \{1, 2, 3\}$ such that the endpoints of every edge receive distinct colors, i.e., $\forall (u, v) \in E$, $\chi(u) \neq \chi(v)$, and such a coloring χ is called a valid 3-coloring.

The input to the 3-Coloring problem is a (simple, undirected) graph $G = (V, E)$ on n vertices, and the goal is to find a valid 3-coloring of its vertices if G is 3-colorable, or assert that no valid 3-coloring exists when G is not 3-colorable.

Below the notation $O^*(f(n))$ stands for $O(f(n)n^c)$ for some constant c. When dealing with exponential runtimes $f(n)$, extra polynomial factors are like noise, so we suppress them with the $O^*(\cdot)$ notation for clarity.

1. *(Warm-up)* The most naive brute-force algorithm for 3-Coloring will take $O^*(3^n)$ time. It is relatively easy to improve this to get an $O^*(2^n)$ time algorithm. Do you see how? (You may assume, wlog, that the input graph is connected.)

2. *(3 is only one more than 2)* Give an algorithm for 3-Coloring with running time $O^*\left(\binom{n}{\lfloor n/3 \rfloor}\right)$.

   (One can show that $\binom{n}{\lfloor n/3 \rfloor} < (1.9)^n$ for large n, so this is an improvement over the first part.)

3. *(3 is only one more than 2, and randomness is your friend)* Give a randomized algorithm for 3-Coloring with runtime $O^*((1.5)^n)$ which finds a valid 3-coloring with probability at least $1 - 2^{-n}$ if the input graph G is 3-colorable (and always outputs "no valid 3-coloring exists" when G is not 3-colorable).

   Hint: For an (unknown) valid 3-coloring χ, suppose you knew for each $v \in V$ a value $\chi(v)$ *doesn't* take... And when you don't know something, you could take a guess...?