Markov Decision Processes and Reinforcement Learning

Readings:

- Mitchell, chapter 13
- for much more, see *Reinforcement Learning, an Introduction* by Sutton and Barto on our class website.

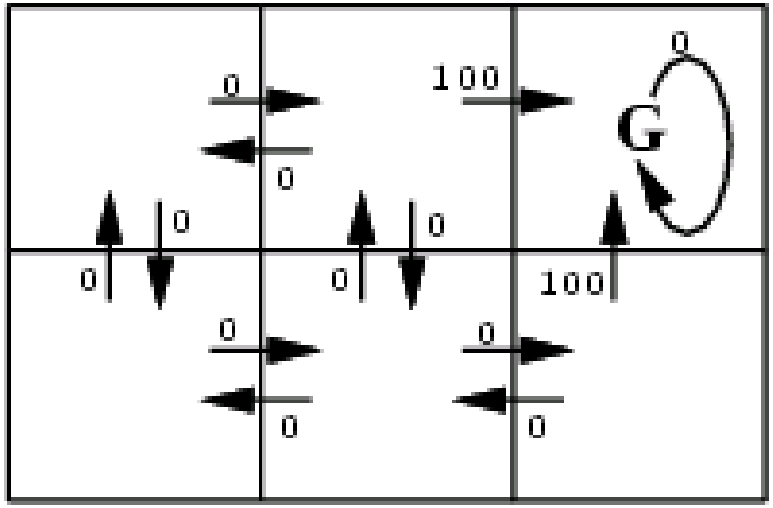
Machine Learning 10-601

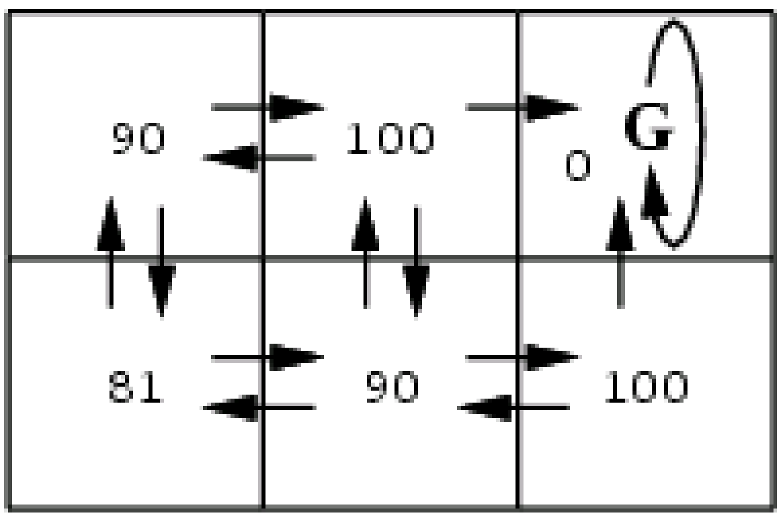
April 21, 2008

Tom M. Mitchell
Machine Learning Department
Carnegie Mellon University

Reinforcement Learning

[Sutton and Barto 1981; Samuel 1957; ...]



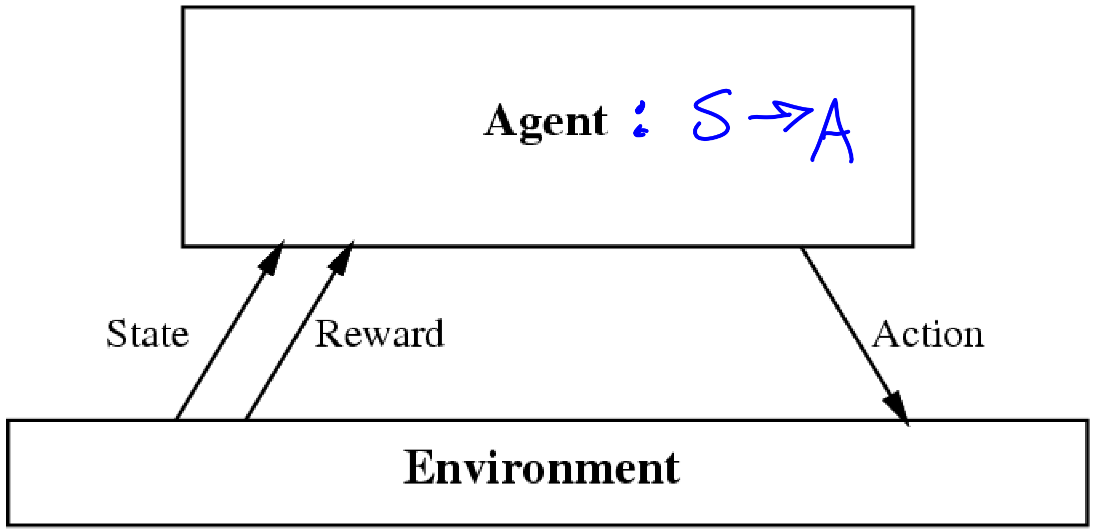


$$V^*(s) = \boxed{E[r_t + \gamma\, r_{t+1} + \gamma^2 r_{t+2} + \ldots]}$$

Outline

- Learning control strategies
  - Credit assignment and delayed reward
  - Discounted rewards
- Markov Decision Processes
  - Solving a known MDP
- Online learning of control strategies
  - When next-state function is known: value function $V^*(s)$
  - When next-state function unknown: learning $Q^*(s,a)$
- Role in modeling reward learning in animals

Reinforcement Learning Problem



$$s_0 \xrightarrow[r_0]{a_0} s_1 \xrightarrow[r_1]{a_1} s_2 \xrightarrow[r_2]{a_2} \dots$$

Goal: Learn to choose actions that maximize

$$r_0 + \gamma r_1 + \gamma^2 r_2 + \dots \ , \text{ where } 0 \leq \gamma < 1$$

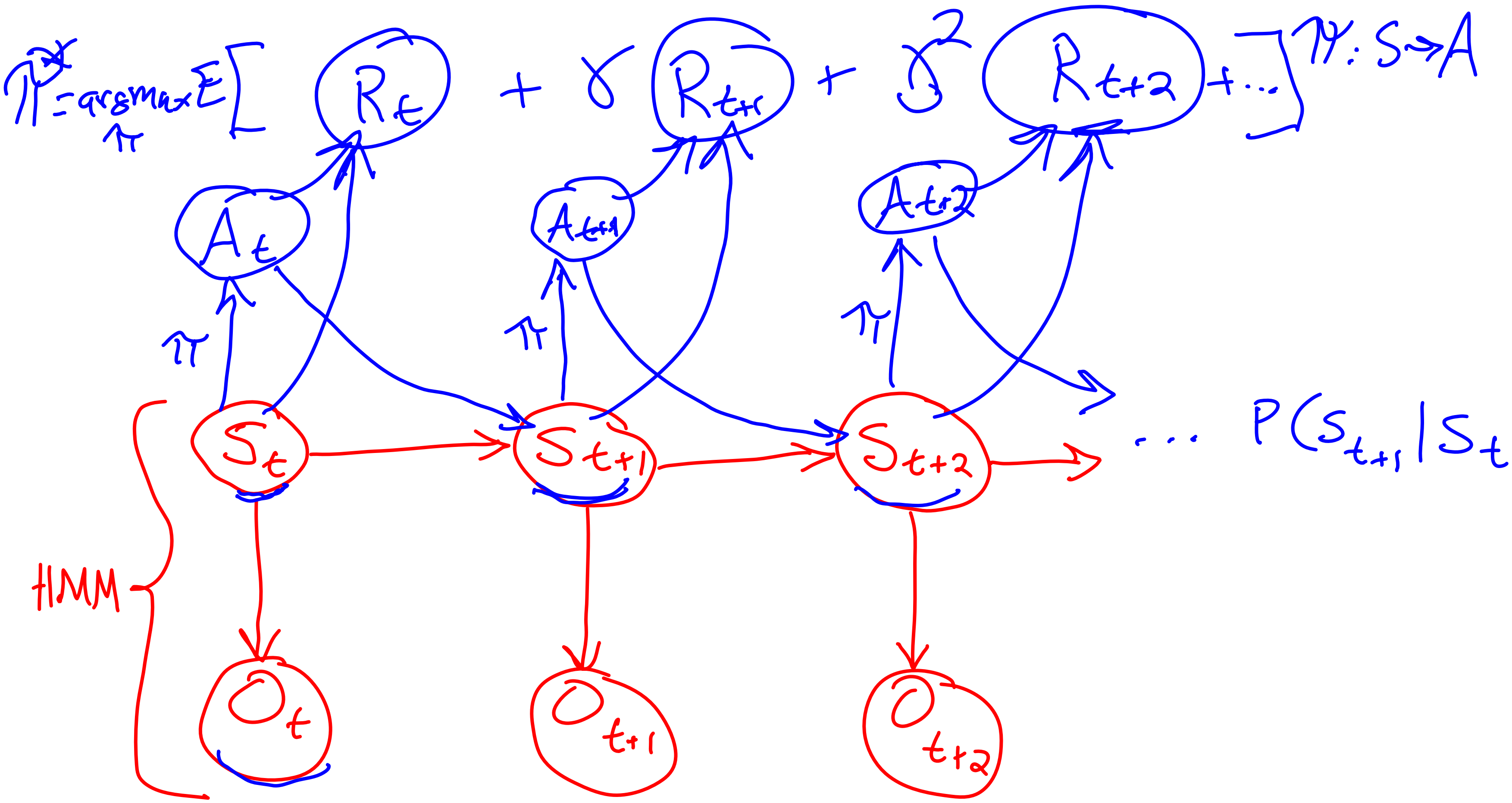
Markov Decision Process

- Set of states S
- Set of actions A
- At each time, agent observes state $s_t \in$ S, then chooses action $a_t \in$ A
- Then receives reward r_t , and state changes to s_{t+1}
- Markov assumption: $P(s_{t+1} \mid s_t, a_t, s_{t-1}, a_{t-1}, ...) = P(s_{t+1} \mid s_t, a_t)$
- Also assume $P(r_t \mid s_t, a_t, s_{t-1}, a_{t-1},...) = P(r_t \mid s_t, a_t)$

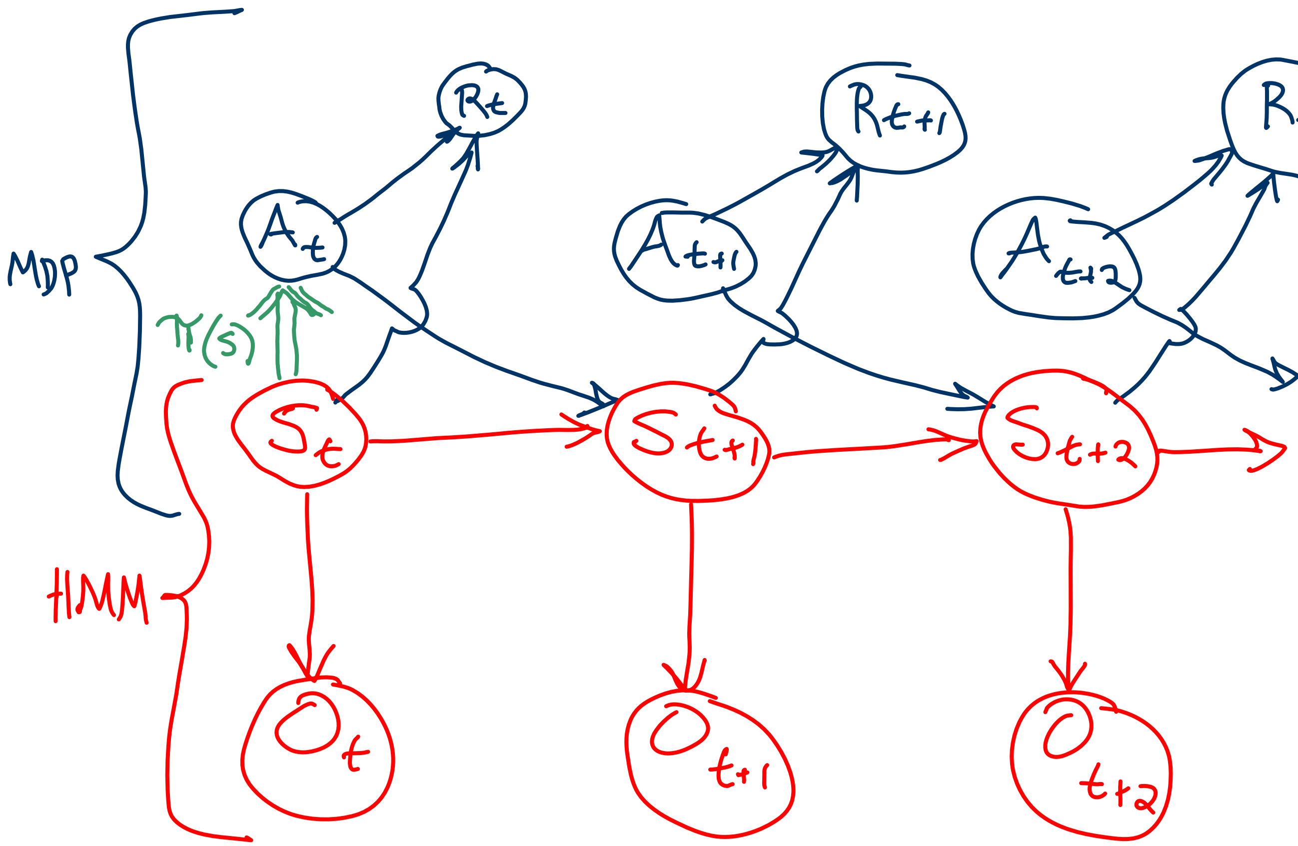
- The task: learn a policy π: S → A for choosing actions that maximizes $E[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + ...]$

$$0 < \gamma \leq 1$$

HMM, Markov Process, Markov Decision Process



HMM, Markov Process, Markov Decision Process



Reinforcement Learning Task for Autonomous Agent

Execute actions in environment, observe results, and

- Learn control policy π: S→A that maximizes $\sum_{t=0}^{\infty} \gamma^t E[r_t]$ from every state s $\in$ S
- Where $0 < \lambda \cdot 1$ is the discount factor for future rewards

Note:

- Function to be learned is π: S→A
- But training examples of the form < <s,a>, r >
- → available training experience is not input-output pairs of the function to be learned !

Value Function for each Policy

- Given a policy $\pi : S \rightarrow A$, define

$$V^\pi(s) = E[\sum_{t=0}^{\infty} \gamma^t r_t]$$

assuming actions are chosen according to π

- Then we want the policy π^* where

$$\pi^* = \arg\max_\pi V^\pi(s), \quad (\forall s)$$

- For any MDP, such a policy exists
- We'll write $V^*(s) = V^{\pi^*}(s)$
- Note if we have $V^*(s)$ and $P(s_{t+1}|s_t,a)$, we can compute $\pi^*(s)$

MACHINE LEARNING DEPARTMENT

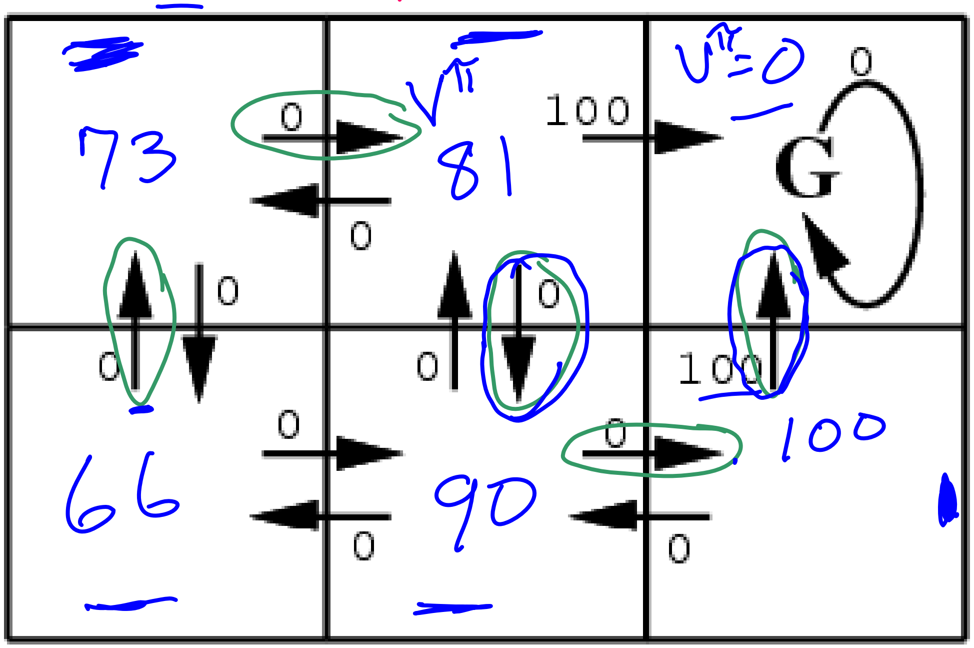
Value Function – what are the $V^{\pi}(s)$ values?

$$V^{\pi}(s) = E[\sum_{t=0}^{\infty} \gamma^t r_t]$$

$$\boxed{V^{\pi}(\underbrace{s}_{t}) = r(\underbrace{s}_{t}, \underbrace{\pi(s)}_{a_t}) + \gamma V(\underbrace{\pi(s)}_{a_t})}$$

Suppose π is shown by circled action from each state

Suppose $\gamma = 0.9$



$0 + \gamma 0 + \gamma^2 100$

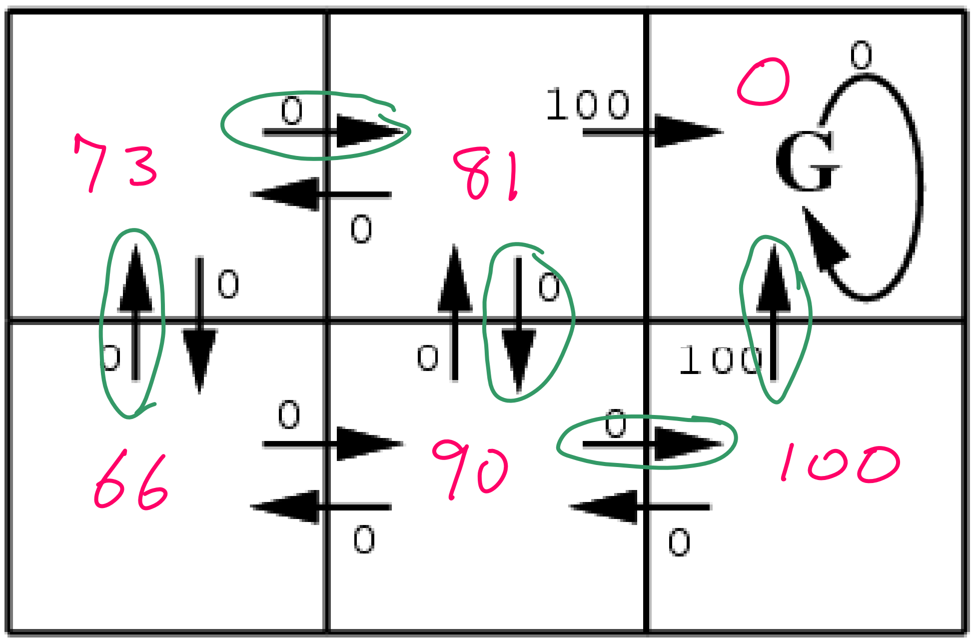
$r(s, a)$ (immediate reward)

Value Function – what are the $V^\pi(s)$ values?

$$V^\pi(s) = E[\sum_{t=0}^{\infty} \gamma^t r_t]$$

Suppose π is shown by circled action from each state

Suppose $\gamma = 0.9$



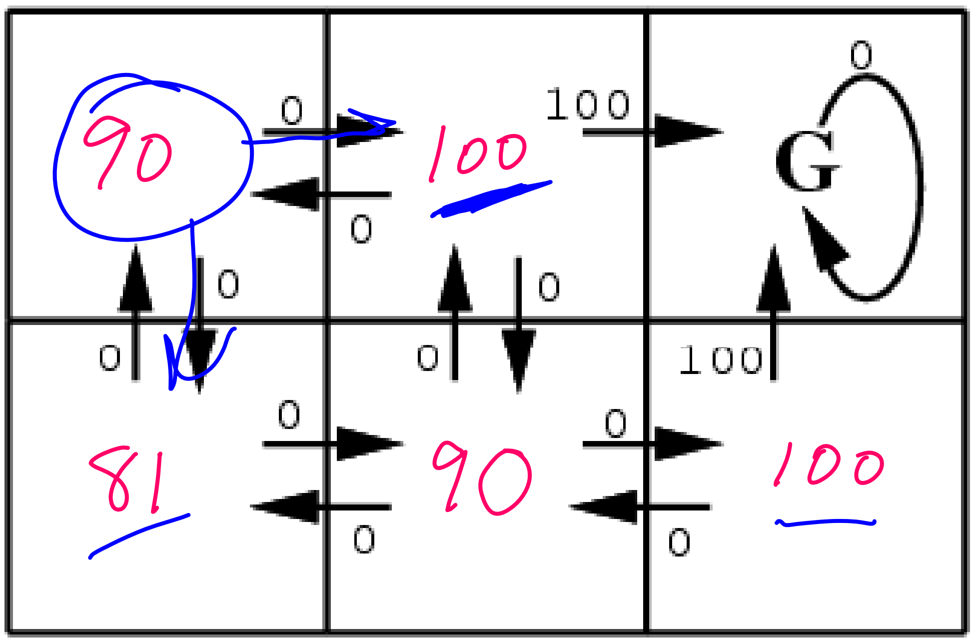
$r(s, a)$ (immediate reward)

Value Function – what are the V*(s) values?

$$V^{\pi}(s) = E[\sum_{t=0}^{\infty} \gamma^t r_t]$$

$V^{\pi^*} \equiv V^*$

optimal policy

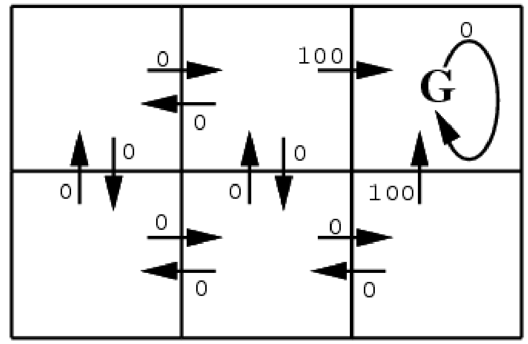


$r(s,a)$ (immediate reward)

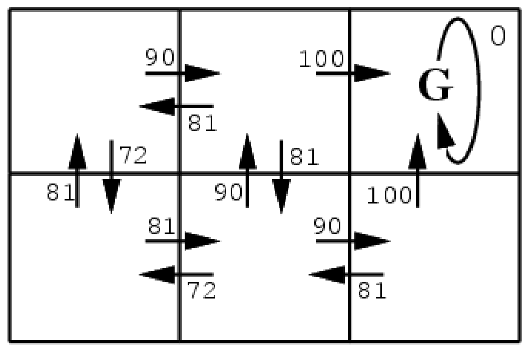
Immediate rewards r(s,a)

State values V*(s)

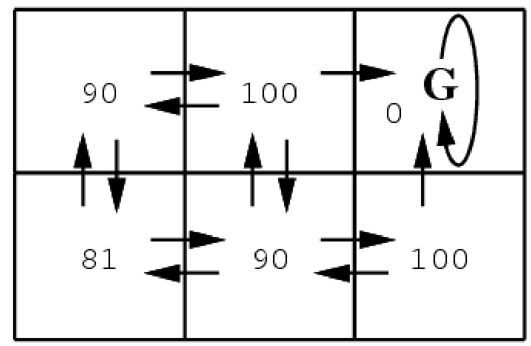
State-action values Q*(s,a)



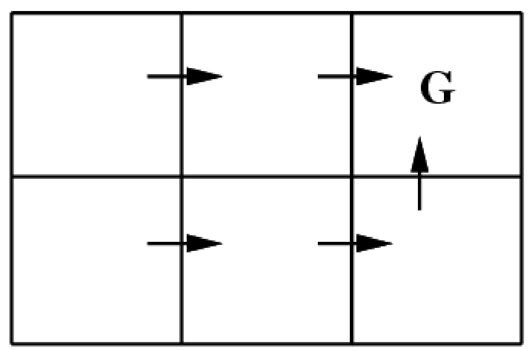
$r(s, a)$ (immediate reward) values



$Q(s, a)$ values



$V^*(s)$ values



One optimal policy

Recursive definition for V*(S)

$$V^*(s) = E[\sum_{t=0}^{\infty} \gamma^t r_t]$$

assuming actions are chosen according to the optimal policy, π^*

$$V^*(s_1) = E[r(s_1, a_1)] + E[\gamma r(s_2, a_2)] + E[\gamma^2 r(s_3, a_3)] + \ldots]$$

$$V^*(s_1) = E[r(s_1, a_1)] + \gamma E_{s_2|s_1,a_1}[V^*(s_2)]$$

a_t

s_{t+1} s_t

$$V^*(s) = E[r(s, \pi^*(s))] + \gamma E_{s'|s,\pi^*(s)}[V^*(s')]$$

P

Value Iteration for learning V* : assumes $P(S_{t+1}|S_t, A)$ known

to zeros

Initialize V(s) arbitrarily

Loop until policy good enough

Loop for s in S

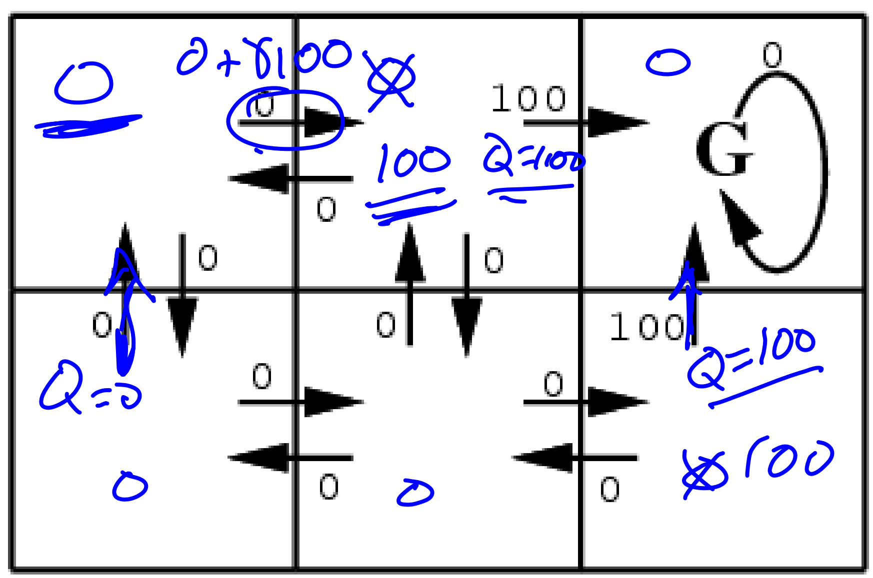
Loop for a in A

- $$Q(s,a) \leftarrow r(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V(s')$$

$$V(s) \leftarrow \max_a Q(s,a)$$

End loop

End loop



V(s) converges to V*(s).

Same alg works if we randomly traverse the environment, as long as visit every transition repeatedly

Value Iteration

Interestingly, value iteration works even if we randomly traverse the environment instead of looping through each state and action methodically

- but we must still visit each state infinitely often on an infinite run
- For details: [Bertsekas 1989]
- Implications: online learning as agent randomly roams

If max (over states) difference between two successive value function estimates is less that ε, then the value of the greedy policy differs from the optimal policy by no more than $2\epsilon\gamma/(1-\gamma)$

So far: learning optimal policy when we know $P(s_t \mid s_{t-1}, a_{t-1})$

What if we don't?

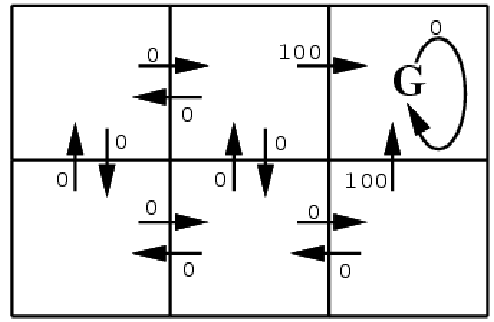
Immediate rewards r(s,a)

State values V*(s)

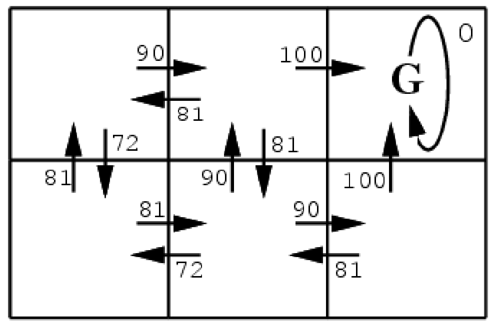
State-action values Q*(s,a)

$$V^*(s) = E[r(s,\pi^*(s))] + \gamma E_{s'|s,\pi^*(s)}[V^*(s')]$$

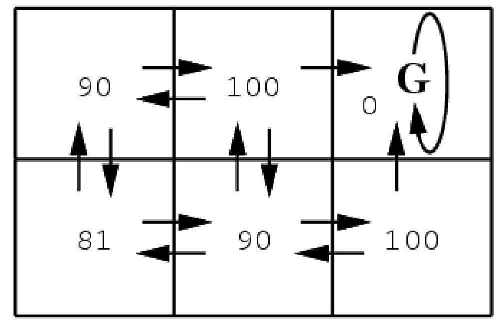
Bellman equation.



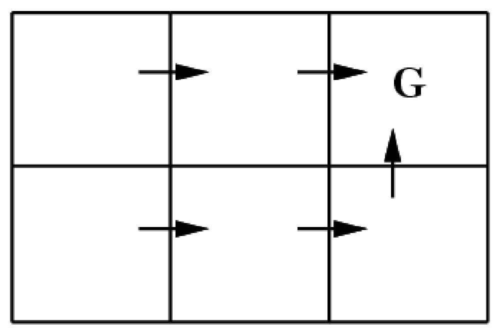
$r(s,a)$ (immediate reward) values



$Q(s,a)$ values



$V^*(s)$ values



One optimal policy

Q learning

Define new function, closely related to V*

$$V^*(s) = E[r(s,\pi^*(s))] + \ \gamma E_{s'|s,\pi^*(s)}[V^*(s')]$$

$$Q(s,a) = E[r(s,a)] + \ \gamma E_{s'|s,a}[V^*(s')]$$

If agent knows Q(s,a), it can choose optimal action without knowing $P(s_{t+1}|s_t,a)$!

$$\pi^*(s) = \arg\max_a Q(s,a) \qquad V^*(s) = \max_a Q(s,a)$$

And, it can <u>learn</u> Q without knowing $P(s_{t+1}|s_t,a)$

Q Function

Consider first the deterministic case. P(s'| s,a) deterministic, denoted δ(s,a)

Define new function very similar to V^*

$$Q(s,a) \equiv r(s,a) + \gamma V^*(\delta(s,a))$$

If agent learns Q, it can choose optimal action even without knowing δ!

$$\pi^*(s) = \underset{a}{\operatorname{argmax}}[r(s,a) + \gamma V^*(\delta(s,a))]$$

$$\pi^*(s) = \underset{a}{\operatorname{argmax}}\, Q(s,a)$$

Q is the evaluation function the agent will learn

Training Rule to Learn Q

Note Q and V^* closely related:

$$V^*(s) = \max_{a'} Q(s, a')$$

Which allows us to write Q recursively as

$$\begin{aligned} Q(s_t, a_t) &= r(s_t, a_t) + \gamma V^*(\delta(s_t, a_t))) \\ &= r(s_t, a_t) + \gamma \max_{a'} Q(s_{t+1}, a') \end{aligned}$$

Nice! Let $\hat{Q}$ denote learner's current approximation to Q. Consider training rule

$$\hat{Q}(s, a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s', a')$$

where s' is the state resulting from applying action a in state s

Q Learning for Deterministic Worlds

For each s, a initialize table entry $\hat{Q}(s, a) \leftarrow 0$

Observe current state s

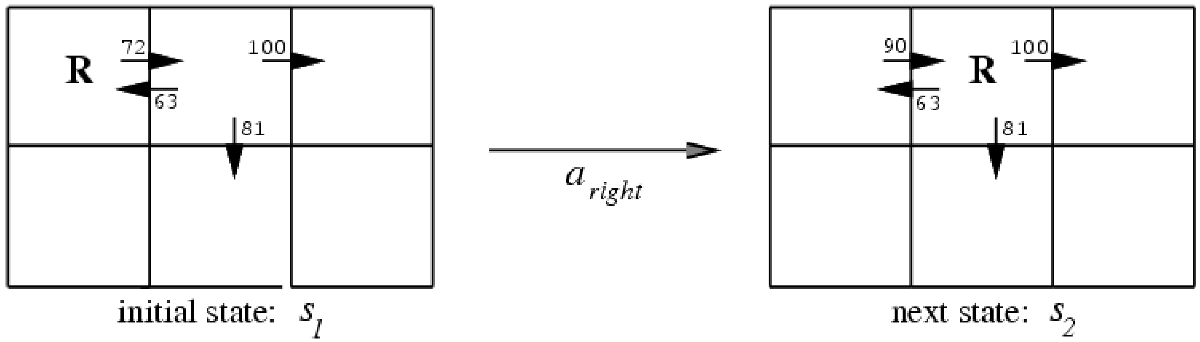
Do forever:

- Select an action a and execute it
- Receive immediate reward r
- Observe the new state s'
- Update the table entry for $\hat{Q}(s, a)$ as follows:

$$\hat{Q}(s, a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s', a')$$

- $s \leftarrow s'$

Updating $\hat{Q}$



$$\begin{aligned}\hat{Q}(s_1, a_{right}) &\leftarrow r + \gamma \max_{a'} \hat{Q}(s_2, a') \\ &\leftarrow 0 + 0.9\ \max\{63, 81, 100\} \\ &\leftarrow 90\end{aligned}$$

notice if rewards non-negative, then

$$(\forall s, a, n)\ \ \hat{Q}_{n+1}(s, a) \geq \hat{Q}_n(s, a)$$

and

$$(\forall s, a, n)\ \ 0 \leq \hat{Q}_n(s, a) \leq Q(s, a)$$

$\hat{Q}$ converges to Q. Consider case of deterministic world where see each $\langle s, a\rangle$ visited infinitely often.

Proof: Define a full interval to be an interval during which each $\langle s, a\rangle$ is visited. During each full interval the largest error in $\hat{Q}$ table is reduced by factor of γ

Let $\hat{Q}_n$ be table after n updates, and Δ_n be the maximum error in $\hat{Q}_n$; that is

$$\Delta_n = \max_{s,a} |\hat{Q}_n(s,a) - Q(s,a)|$$

For any table entry $\hat{Q}_n(s,a)$ updated on iteration $n+1$, the error in the revised estimate $\hat{Q}_{n+1}(s,a)$ is

$$\begin{aligned}
|\hat{Q}_{n+1}(s,a) - Q(s,a)| &= |(r + \gamma \max_{a'} \hat{Q}_n(s',a')) \\
&\quad -(r + \gamma \max_{a'} Q(s',a'))| \\
&= \gamma |\max_{a'} \hat{Q}_n(s',a') - \max_{a'} Q(s',a')| \\
&\leq \gamma \max_{a'} |\hat{Q}_n(s',a') - Q(s',a')| \\
&\leq \gamma \max_{s'',a'} |\hat{Q}_n(s'',a') - Q(s'',a')| \\
|\hat{Q}_{n+1}(s,a) - Q(s,a)| &\leq \gamma \Delta_n
\end{aligned}$$

Use general fact:

$$|\max_a f_1(a) - \max_a f_2(a)| \leq \max_a |f_1(a) - f_2(a)|$$

Nondeterministic Case

Q learning generalizes to nondeterministic worlds

Alter training rule to

$$\hat{Q}_n(s,a) \leftarrow (1-\alpha_n)\hat{Q}_{n-1}(s,a) + \alpha_n[r + \max_{a'} \hat{Q}_{n-1}(s',a')]$$

where

$$\alpha_n = \frac{1}{1 + visits_n(s,a)}$$

Can still prove convergence of $\hat{Q}$ to Q [Watkins and Dayan, 1992]

Temporal Difference Learning

Q learning: reduce discrepancy between successive Q estimates

One step time difference:

$$Q^{(1)}(s_t, a_t) \equiv r_t + \gamma \max_a \hat{Q}(s_{t+1}, a)$$

Why not two steps?

$$Q^{(2)}(s_t, a_t) \equiv r_t + \gamma r_{t+1} + \gamma^2 \max_a \hat{Q}(s_{t+2}, a)$$

Or n?

$$Q^{(n)}(s_t, a_t) \equiv r_t + \gamma r_{t+1} + \cdots + \gamma^{(n-1)} r_{t+n-1} + \gamma^n \max_a \hat{Q}(s_{t+n}, a)$$

Blend all of these:

$$Q^{\lambda}(s_t, a_t) \equiv (1-\lambda) \left[Q^{(1)}(s_t, a_t) + \lambda Q^{(2)}(s_t, a_t) + \lambda^2 Q^{(3)}(s_t, a_t) \right.$$

Temporal Difference Learning

$$Q^{\lambda}(s_t, a_t) \equiv (1-\lambda)\left[Q^{(1)}(s_t, a_t) + \lambda Q^{(2)}(s_t, a_t) + \lambda^2 Q^{(3)}(s_t, a_t)\right.$$

Equivalent expression:

$$\begin{aligned} Q^{\lambda}(s_t, a_t) &= r_t + \gamma[\ (1-\lambda)\max_a \hat{Q}(s_t, a_t) \\ &\qquad +\lambda\ Q^{\lambda}(s_{t+1}, a_{t+1})] \end{aligned}$$

TD(λ) algorithm uses above training rule

- Sometimes converges faster than Q learning
- converges for learning V^* for any $0 \leq \lambda \leq 1$ (Dayan, 1992)
- Tesauro's TD-Gammon uses this algorithm

MACHINE LEARNING DEPARTMENT

Subtleties and Ongoing Research

- Replace $\hat{Q}$ table with neural net or other generalizer
- Handle case where state only partially observable
- Design optimal exploration strategies
- Extend to continuous action, state
- Learn and use $\hat{\delta} : S \times A \rightarrow S$
- Relationship to dynamic programming