10-601
Machine Learning

Bayesian networks: Inference

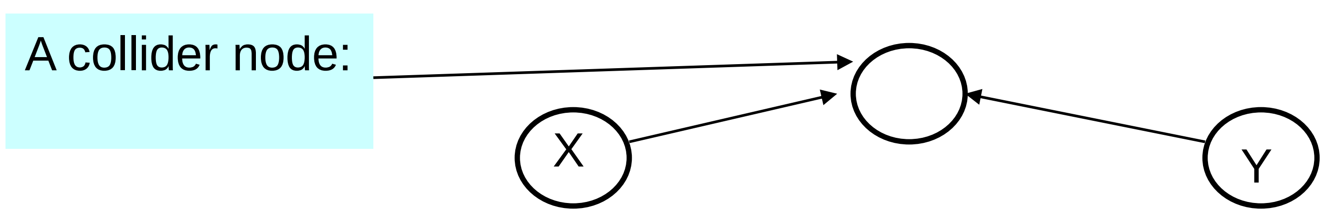
Reading: Bishop 8.1 and 8.2.2

d-separation

- We will give rules to identify d-connected variables. Variables that are not d-connected are d-separated.
- The following three rules can be used to determine if x and y are d-connected given Z:

1. If Z is empty then x and y are d-connected if there exists a path between them does not contain a collider.
2. x and y are d-connected given Z if there exists a path between them that does not contain a collider and does not contain any member of Z
3. If Z contains a collider or one of its descendents then if a path between x and y contains this node they are d-connected

3\. (revised) If all colliders on an undirected path between x and y are in Z or have a descendent in Z, then they are d-connected



Variables

- An alarm system

  B – Did a burglary occur?

  E – Did an earthquake occur?

  A – Did the alarm sound off?

  M – Mary calls

  J – John calls

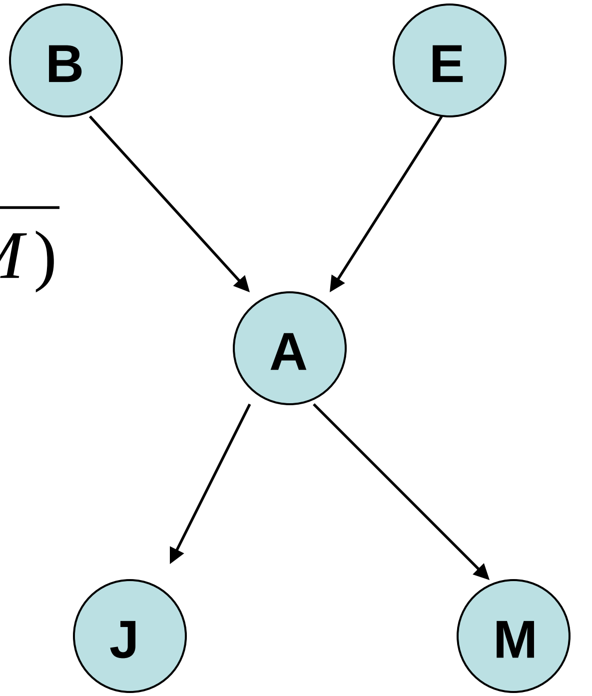
- Lets use our knowledge of the domain!

Inference

- We are interested in queries of the form: $P(B \mid J, \neg M)$
- This can also be written as a joint:

$$P(B \mid J, \neg M) = \frac{P(B, J, \neg M)}{P(B, J, \neg M) + P(\neg B, J, \neg M)}$$

- How do we compute the new joint?



Inference in Bayesian networks

- We will discuss three methods:

1. Enumeration
2. Variable elimination
3. Stochastic inference

Computing: $P(B,J,\neg M)$

$P(B,J,\neg M) =$

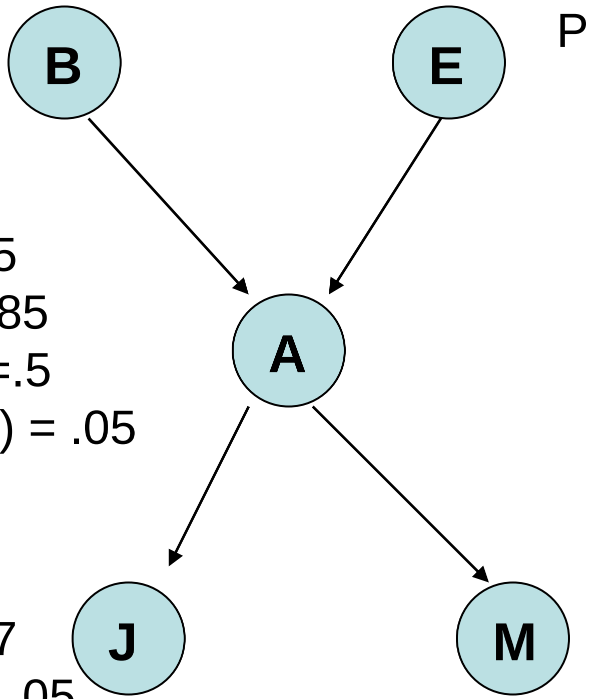
$P(B,J,\neg M,A,E)+$

$P(B,J,\neg M,\neg A,E) + P(B,J,\neg M,A,\neg E) + P(B,J,\neg M,\neg A,\neg E) =$

$0.0007+0.00001+0.005+0.0003 = 0.00601$

$P(B)=.05$

$P(E)=.1$



$P(A|B,E)=.95$
$P(A|B,\neg E) = .85$
$P(A|\neg B,E)=.5$
$P(A|\neg B,\neg E) = .05$

$P(J|A)=.7$
$P(J|\neg A) = .05$

$P(M|A)=.8$
$P(M|\neg A) = .15$

Computing partial joints

$$P(B \mid J, \neg M) = \frac{P(B, J, \neg M)}{P(B, J, \neg M) + P(\neg B, J, \neg M)}$$

Sum all instances with these settings (the sum is over the possible assignments to the other two variables, E and A)

- This method can be improved by re-using calculations (similar to dynamic programming)
- Still, the number of possible assignments is exponential in the unobserved variables?
- That is, unfortunately, the best we can do. General querying of Bayesian networks is NP-complete

Inference in Bayesian networks if NP complete (sketch)

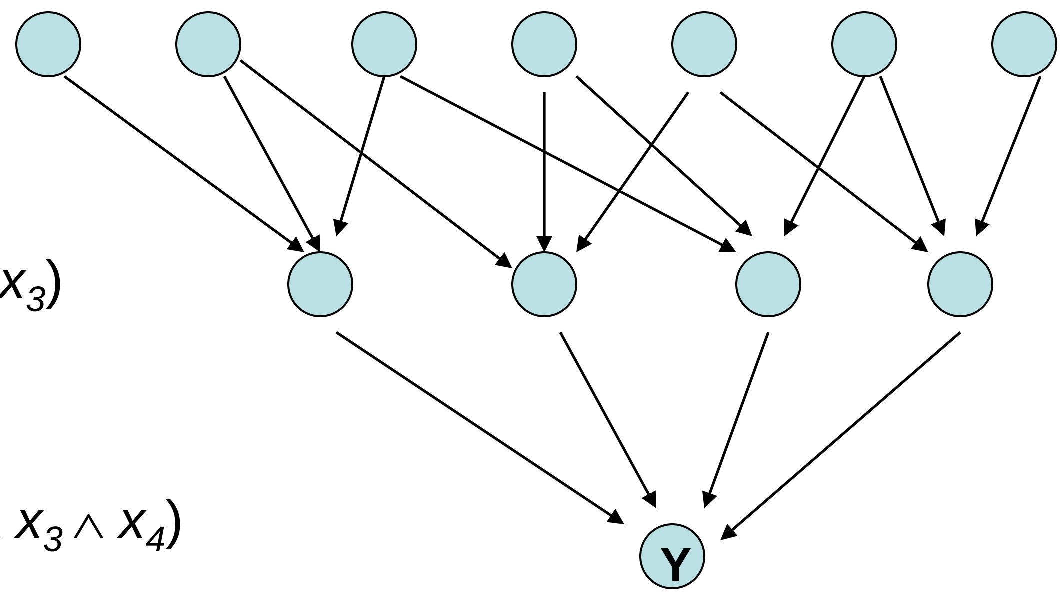
- Reduction from 3SAT
- Recall: 3SAT, find satisfying assignments to the following problem: $(a \vee b \vee c) \wedge (d \vee \neg b \vee \neg c) \ldots$

What is P(Y=1)?

$P(x_i=1) = 0.5$

$P(x_i=1) = (x_1 \vee x_2 \vee x_3)$

$P(Y=1) = (x_1 \wedge x_2 \wedge x_3 \wedge x_4)$



Inference in Bayesian networks

- We will discuss three methods:

1. Enumeration
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Variable elimination

P(B,J, ¬M) =

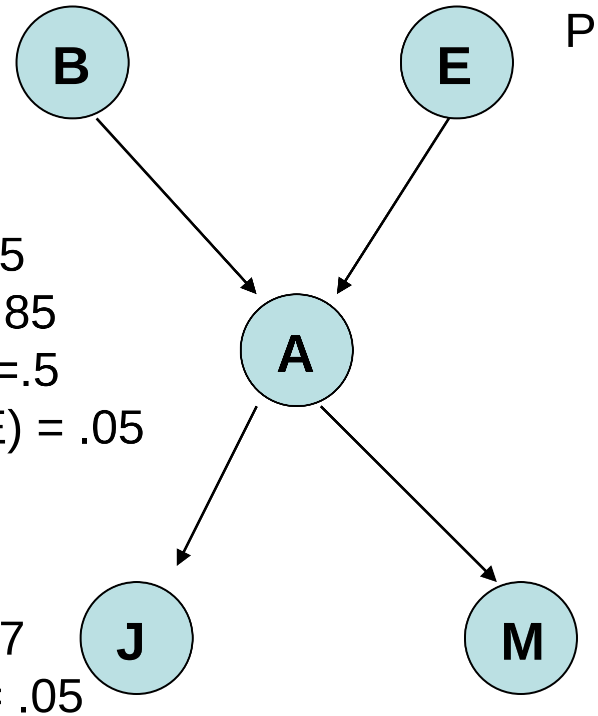
P(B,J, ¬M,A,E)+

P(B,J, ¬M, ¬ A,E) +
P(B,J,¬M,A, ¬ E) + P(B,J, ¬M,
¬ A, ¬ E) =

0.0007+0.00001+0.005+0.0003
= 0.00601

P(B)=.05

P(E)=.1



P(A|B,E))=.95
P(A|B,¬E) = .85
P(A| ¬ B,E))=.5
P(A| ¬ B, ¬ E) = .05

P(J|A))=.7
P(J|¬A) = .05

P(M|A))=.8
P(M|¬A) = .15

Reuse computations rather than recompute probabilities

Computing: P(B,J, ¬M)

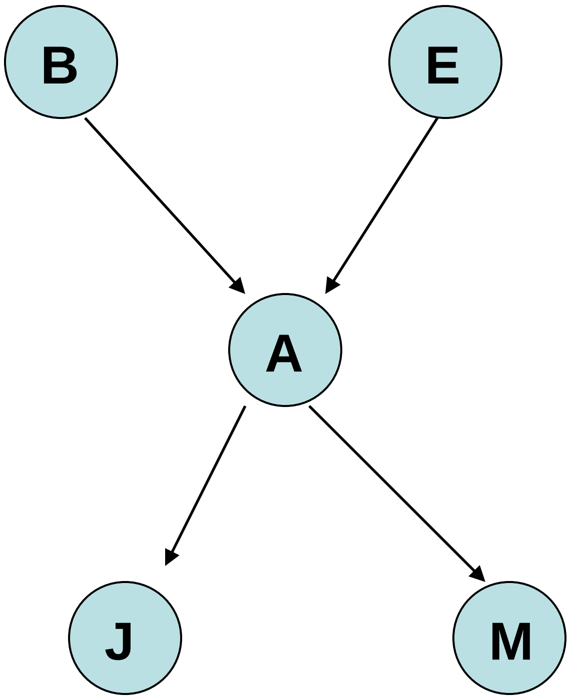
P(B,J, ¬M) =

P(B,J, ¬M,A,E)+

P(B,J, ¬M, ¬ A,E) + P(B,J, ¬M,A, ¬ E) + P(B,J, ¬M, ¬ A, ¬ E) =

$$\sum_{a}\sum_{e} P(B)P(e)P(a \mid B,e)P(M \mid a)P(J \mid a)$$

Store as a function of a and use whenever necessary (no need to recompute each time)



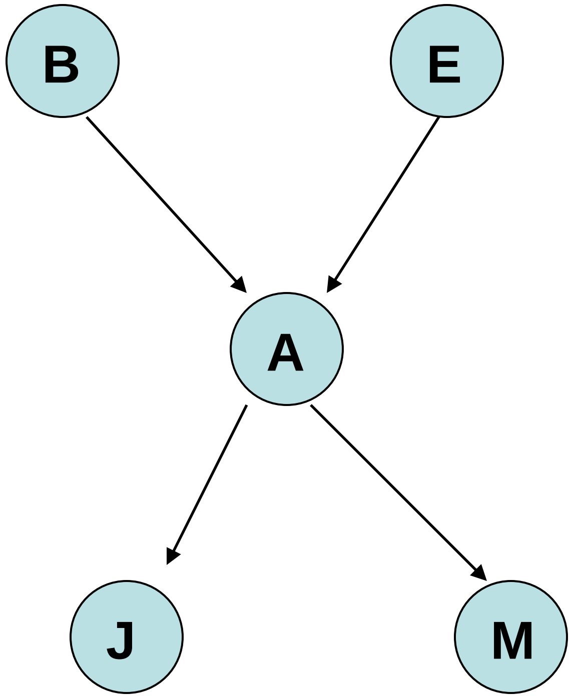
Variable elimination

$$P(B,J,M) = \sum_a \sum_e P(B)P(e)P(a \mid B,e)P(M \mid a)P(J \mid a)$$

$$= P(B)\sum_e P(e)\sum_a P(a \mid B,e)P(M \mid a)P(J \mid a)$$

Set: $f_M(A) = \begin{pmatrix} P(M \mid A) \\ P(M \mid \neg A \end{pmatrix}$

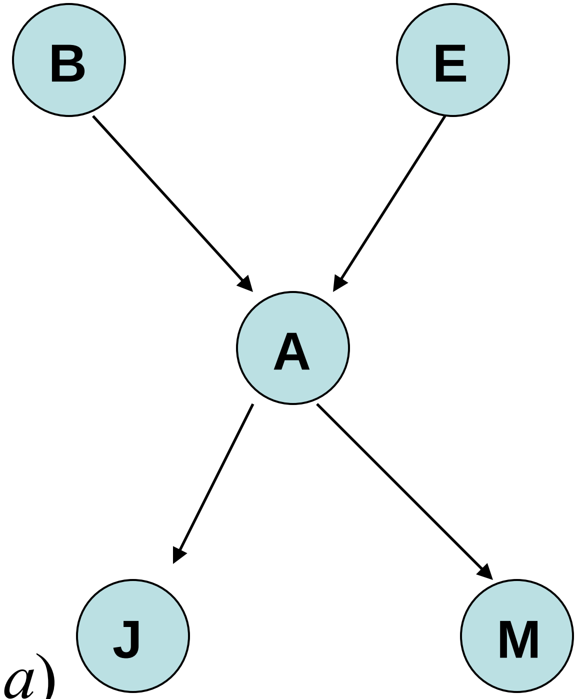
$$f_J(A) = \begin{pmatrix} P(J \mid A) \\ P(J \mid \neg A \end{pmatrix}$$



Variable elimination

$$P(B,J,M) = \sum_a \sum_e P(B)P(e)P(a \mid B,e)P(M \mid a)P(J \mid a)$$

$$= P(B)\sum_e P(e)\sum_a P(a \mid B,e)P(M \mid a)P(J \mid a)$$



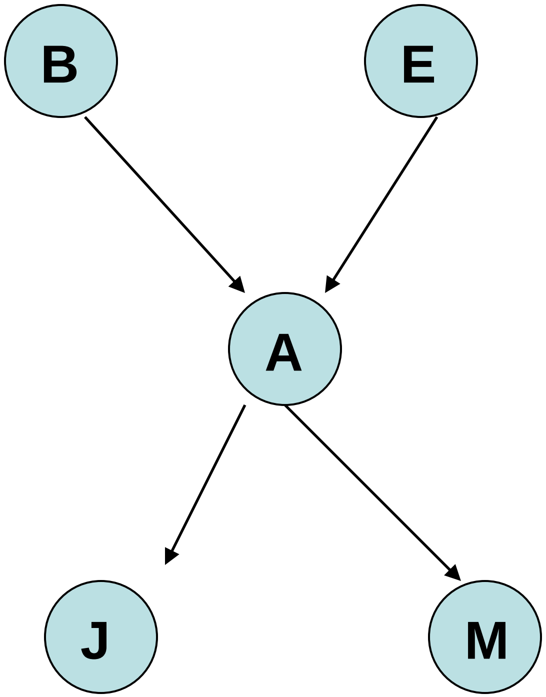
Set: $f_M(A) = \begin{pmatrix} P(M \mid A) \\ P(M \mid \neg A) \end{pmatrix}$

$$f_J(A) = \begin{pmatrix} P(J \mid A) \\ P(J \mid \neg A) \end{pmatrix}$$

$$P(B,J,M) = P(B)\sum_e P(e)\sum_a P(a \mid B,e) f_M(a) f_J(a)$$

Variable elimination

$$= P(B)\sum_{e} P(e)\sum_{a} P(a \mid B,e) f_M(a) f_J(a)$$



Lets continue with these functions:

$$f_A(a,B,e) = P(a \mid B,e)$$

We can now define the following function:

$$f_{A,J,M}(B,e) = \sum_{a} f_A(a,B,e) f_J(a) f_M(a)$$

And so we can write:

$$P(B,J,M) = P(B)\sum_{e} P(e) f_{A,J,M}(B,e)$$

Variable elimination

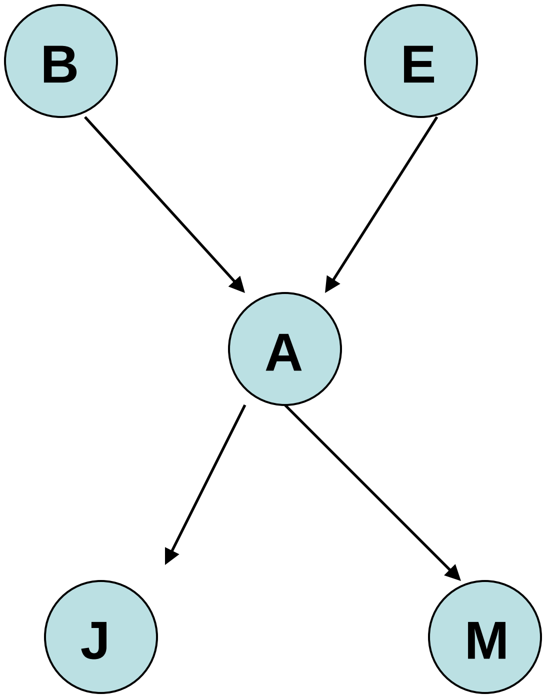
$$P(B,J,M) = P(B)\sum_{e} P(e) f_{A,J,M}(B,e)$$

Lets continue with another function:

$$f_{E,A,J,M}(B) = \sum_{e} P(e) f_{A,J,M}(B,e)$$

And finally we can write:

$$P(B,J,M) = P(B) f_{E,A,J,M}(B)$$



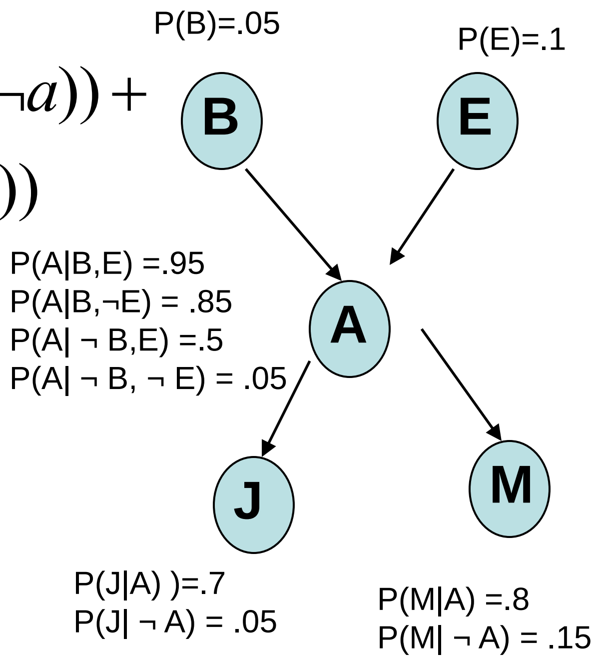
Example

$$P(B,J,M) = P(B) f_{E,A,J,M}(B)$$

$$= 0.05 \sum_e P(e) f_{A,J,M}(B,e) = 0.05(0.1 f_{A,J,M}(B,e) + 0.9 f_{A,J,M}(B,\neg e))$$

$$0.05(0.1(0.95 f_J(a) f_M(a) + 0.05 f_J(\neg a) f_M(\neg a)) +$$

$$0.9(.85 f_J(a) f_M(a) + .15 f_J(\neg a) f_M(\neg a)))$$



Calling the same function multiple times

Final computation (normalization)

$$P(B \mid J, \neg M) = \frac{P(B, J, \neg M)}{P(B, J, \neg M) + P(\neg B, J, \neg M)}$$

Algorithm

- *e* - evidence (the variables that are known)
- *vars* - the conditional probabilities derived from the network in reverse order (bottom up)
- For each *var* in *vars*

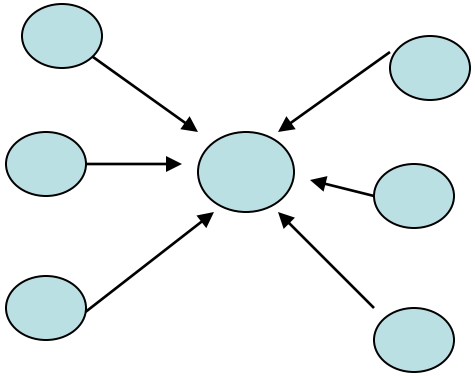
  \- *factors* <- make_factor (*var,e*)

  \- if *var* is a hidden variable then create a new factor by summing out *var*
- Compute the product of all factors
- Normalize

Computational complexity

- We are reusing computations so we are reducing the running time.
- However, there are still cases in which this algorithm we lead to exponential running time.
- Consider the case of $f_x(y_1 \ldots y_n)$. When factoring x out we would need to account for all possible values of the y's.

Variable elimination can lead to significant costs saving but its efficiency depends on the network structure



Inference in Bayesian networks

- We will discuss three methods:

1. Enumeration
2. Variable elimination
3. Stochastic inference

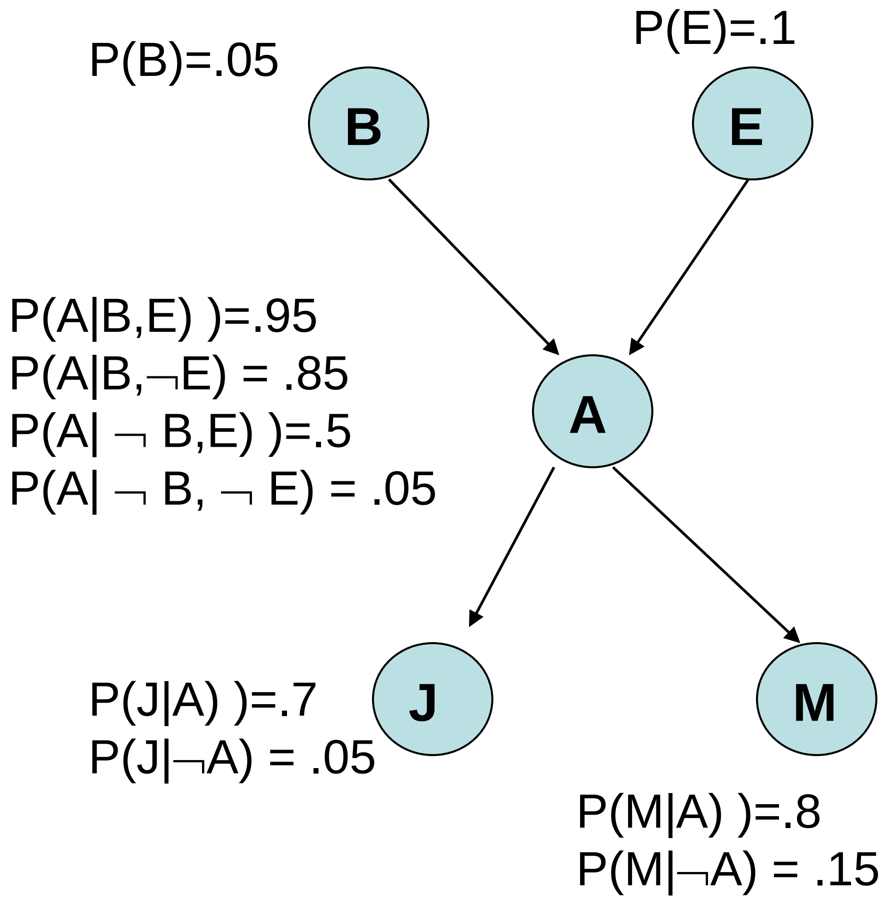
Stochastic inference

- We can easily sample the joint distribution to obtain possible instances

1. Sample the free variable
2. For every other variable:

   - If all parents have been sampled, sample based on conditional distribution

We end up with a new set of assignments for B,E,A,J and M which are a random sample from the joint

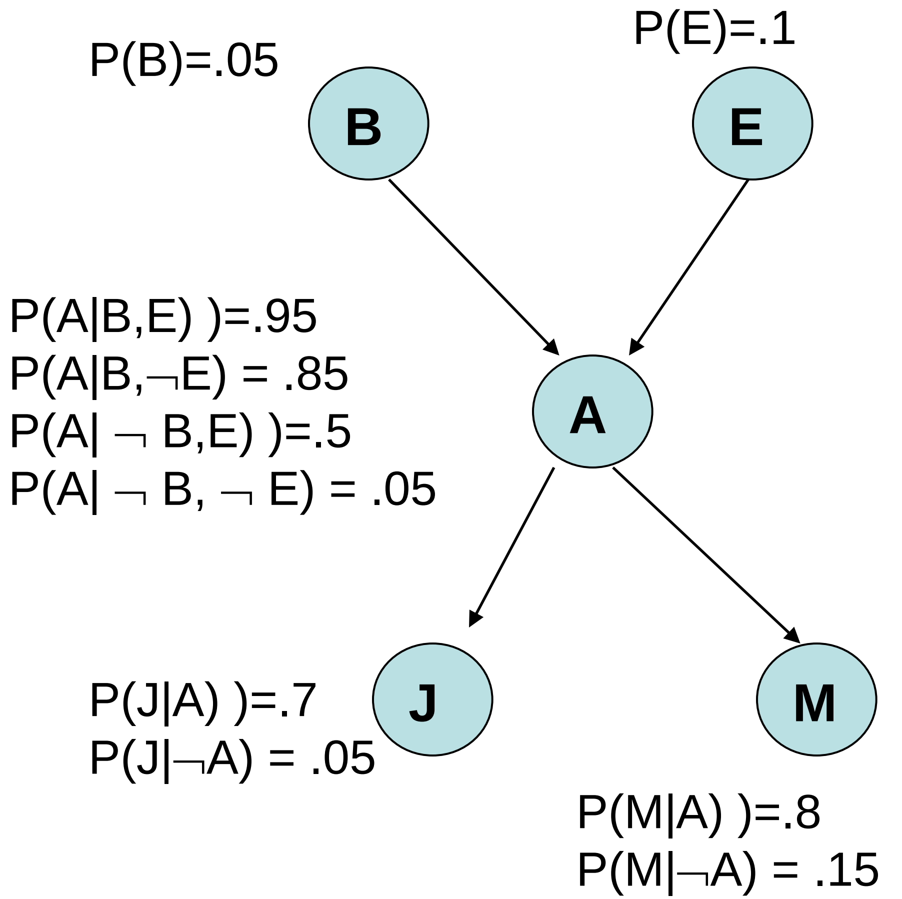


Stochastic inference

- We can easily sample the joint distribution to obtain possible instances

1. Sample the free variable
2. For every other variable:
   - If all parents have been sampled, sample based on conditional distribution

Its always possible to carry out this sampling procedure, why?



Using sampling for inference

- Lets revisit our problem: Compute P(B | J,¬M)
- Looking at the samples we can count:

  - N: total number of samples

  - N_c : total number of samples in which the condition holds (J,¬M)

  - N_B: total number of samples where the joint is true (B,J,¬M)
- For a large enough N
  - N_c / N ≈ P(J,¬M)
  - N_B / N ≈ P(B,J,¬M)
- And so, we can set

$$P(B \mid J,\neg M) = P(B,J,\neg M) / P(J,\neg M) \approx N_B / N_c$$

Using sampling for inference

- Lets revisit our problem: Compute $P(B \mid J, \neg M)$
- Looking at the samples we can cound:

  - N: total number o

  - N_c : total number

  - N_B: total number

Problem: What if the condition rarely happens?

We would need lots and lots of samples, and most would be wasted

- For a large enoug

  - $N_c / N \approx P(J, \neg M)$

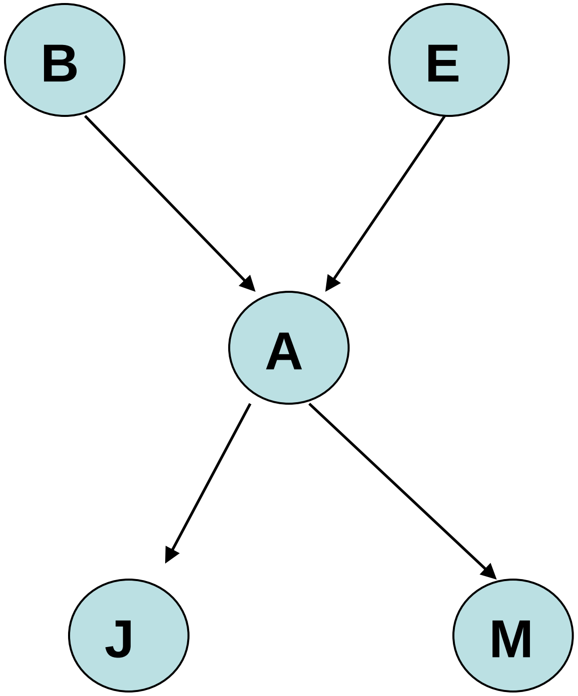
  - $N_B / N \approx P(B, J, \neg M)$

- And so, we can set

$$P(B \mid J, \neg M) = P(B, J, \neg M) / P(J, \neg M) \approx N_B / N_c$$

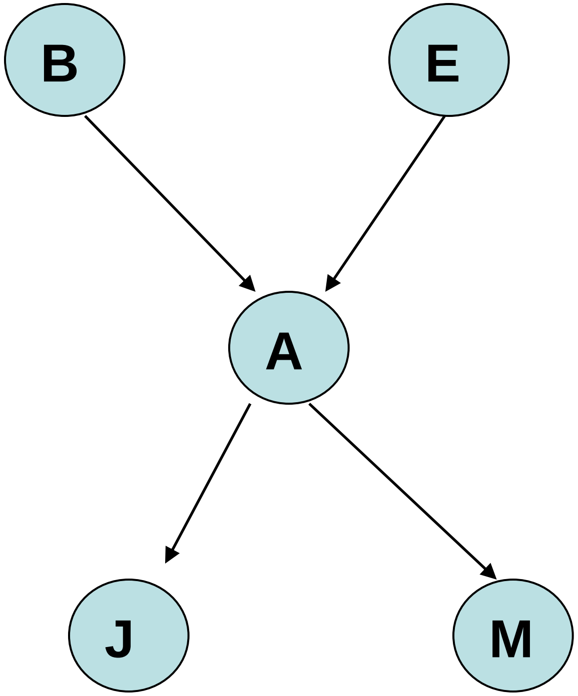
Weighted sampling

- Compute $P(B \mid J, \neg M)$
- We can manually set the value of J to 1 and M to 0
- This way, all samples will contain the correct values for the conditional variables
- Problems?



Weighted sampling

- Compute P(B | J,¬M)
- Given an assignment to parents, we assign a value of 1 to J and 0 to M.
- We record the *probability* of this assignment ($w = p_1 * p_2$) and we weight the new joint sample by *w*



Weighted sampling algorithm for computing P(B | J,¬M)

- Set $N_B, N_c = 0$
- Sample the joint setting the values for *J* and *M*, compute the weight, *w,* of this sample
- $N_c = N_c + w$
- If $B = 1$, $N_B = N_B + w$

- After many iterations, set

$$P(B \mid J, \neg M) = N_B / N_c$$

Important points

- Bayes rule
- Joint distribution, independence, conditional independence
- Attributes of Bayesian networks
- Constructing a Bayesian network
- Inference in Bayesian networks

Other inference methods

- Convert network to a polytree

  - In a polytree no two nodes have more than one path between them

  - We can convert arbitrary networks to a polytree by clustering (grouping) nodes. For such a graph there is a algorithm which is linear in the number of nodes

  - However, converting into a polytree can result in an exponential increase in the size of the CPTs