The "Dutch Book" argument, tracing back to independent work by F.Ramsey (1926) and B.deFinetti (1937), offers prudential grounds for action in conformity with personal probability.

Under several structural assumptions about combinations of stakes

– that is, assumptions about the combination of wagers –

your betting policy is *undominated* in payoffs (***coherent***) if and only if your *fair-odds* are probabilities.

Let's review the elementary *Dutch Book* argument.

A bet on/against event E, at odds of r:(1-r) with total stake S > 0 (say, bets are in $ units), is specified by its payoffs, as follows.

	E	E^c
bet on E	win (1-r)S	lose rS
bet against E	lose (1-r)S	win rS
abstain from betting	status quo	status quo

Alternatively, by permitting S < 0 we can reverse *betting on* and *betting against*.

We assume that the *status quo* (the consequence of abstaining) represents <u>no</u> net change in wealth. It is depicted by a 0 payoff in the units of the stake.

The *structural* assumptions about bets require conditions (*a*) — (*c*), below:

(a) Given an event *E*, a betting rate r:(1-r) and stake S, your preferences satisfy exactly one of three profiles.

Here $<$ designates strict preference and $\approx$ designates indifference.

	betting on $<$ abstaining $<$ betting against *E*,
or	betting on $\approx$ abstaining $\approx$ betting against *E*,
or	betting against *E* $<$ abstaining $<$ betting on *E*.

(b) The (finite) conjunction of *favorable* / *fair* / or *unfavorable* bets is again *favorable* / *fair* / or *unfavorable*.

A conjunction of bets is ***favorable*** if it is preferred to (>) abstaining, ***unfavorable*** if dispreferred to (<) abstaining, and ***fair*** if indifferent to (≈) abstaining.

(*c*) Your preference for outcomes is continuous in rates, in particular, each event *E* carries a unique *fair-odds* r_E for betting on *E*.

Note: It follows from these assumptions that your attitude towards a simple bet is independent of the size of the stake.

Dutch Book **Theorem**

Subject to these assumptions on betting —

- **Your fair betting odds are probabilities**

  that is, they satisfy the three axioms of probability

  (1) $0 \leq P(A) \leq 1$.

  (2) $P(X) = 1$, *where X is the "sure event"*

  (3) *If* $A \cap B = \phi$, then $P(A) + P(B) = P(A \cup B)$.

if and only if

- **Your preferences are *coherent*.**

If your fair odds do not satisfy the three axioms of probability, then there is some combination of *fair* bets which is **dominated** by *abstaining* in a partition of payoff states.

In other words, then there is some (finite) combination of bets each of which you judge *fair*, where you lose for sure in each state of a finite partition.

The Dutch Book result can be extended to include "conditional probability," $P(A \mid B)$, by using "called-off" bets.

The *called-off* bet on A (given B), is where *status quo* results in case B fails to occur.

A *called-off bet* on/against event A, given B, at odds of r:(1-r) with total stake S (> 0) is specified by its payoffs, as follows.

	AB	$\mathbf{A^cB}$	$\mathbf{B^c}$
on A	**(1-r)S**	**-rS**	**0**
against A	**-(1-r)S**	**rS**	**0**

Then coherent betting, including "called-off" bets, entails

Axiom 4: $$P(A|B) \times P(B) = P(A \cap B).$$

We can strengthen the coherence requirement to require a respect for *strict dominance* (admissibility).

- ***Strict Coherence*: Avoid betting so as to allow no chance for winning yet some chance for losing.**

This corresponds to *admissibility* with respect to abstaining from betting, using the partition of state-payoffs.

***Theorem*: Your fair odds (and called-off odds) are strictly coherent *if and only if* they are probabilities (and conditional probabilities) for a probability that is positive on each possible event.**

Robust Bayesian analysis begins with a relaxation of the simple-minded betting model – structural assumption (***a***) is weakened as follows:

A decision maker may have one-sided betting odds – that is, there may be distinct odds for *betting on* versus *betting against* an event.

This relates to having a **different** price for buying an option than for selling it, without there being a single *fair* price at which you will both buy and sell.

The generalized Dutch Book theorem that results, says:

- **A set of one-sided bettings odds is *coherent***

  (no *Dutch Book is possible*)   if and only if

  these one-sided odds are represented by a (convex) set *P* of

  probability distributions, as follows:

  - The lower probability (w.r.t *P*) $P_*(E)$ gives the odds for *betting on E*.

  - The upper probability (w.r.t *P*) $P^*(E)$ gives the odds for *betting against E*.

Thus, as **buying** a *bet on E* is the same as **selling** a *bet against E*,

- for a coherent agent, $P_*(E) = 1 - P^*(E^c)$.

When we have:

a statistical model, parameterized by θ,

a statistical decision problem characterized by a loss function ***L***,

and action space ***A***,

what is the relation between classical admissibility and Bayes-decisions?

Definition: A class ***C*** of decision rules is *complete* if each for each decision rule not in ***C***, it is inadmissible against some decision rule in ***C***.

C is *minimally complete* if no proper subset is complete.

Theorem**: Suppose that Θ has only finitely many states, that the loss function *L* is bounded (below), and that the *risk set* (i.e., the risks functions associated with action space *A*) is closed from below.**

Then:

- **(i) the class of Bayes decisions is complete;**
- **(ii) the *admissible* Bayes decisions is a minimal complete class.**
- **(iii) each Minimax solution is Bayes for some (*worst case*) prior.**

Note 1: If the prior $\pi(\theta)$ over Θ assigns each parameter state positive probability, then each Bayes decision with respect to π is admissible.

Note 2: When the parameter space is infinite, these results extend by allowing for limits of Bayes' solutions, e.g., *improper* priors. Often then, the *worst case* (minimax) prior is *improper*.