CS329 Probabilistic Robotics
Today’s Topic: 3D Mapping with EM

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The Problem
- Forward-pointed laser (2D information)
- Upward-pointed laser (3D information)
- Panoramic camera

The (Almost) Raw Range Data

3D Mapping
- Observation: World mostly composed of planar surfaces
- Problem: How to generate multi-plane model?

The 3D Multi-Plane Mapping Problem
- Entails a range of problems
  - Parameter estimation: Each surface presently possesses 6 parameters (x, y, z, θ, φ, ψ)
  - Outlier identification: Shall we explain every single measurement by a planar surface?
  - Correspondence: What measurement corresponds to what surface?
  - Model selection: How many surfaces should there be?
- Robot localization: Noise in robot motion has a systematic effect on multiple measurements (this spoils any independence assumption)

Concurrent Mapping and Localization: To be Discussed in Later Classes
The 3D Multi-Plane Mapping Problem

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3D Structure Mapping (Raw Data)

3D Texture Mapping (Raw Data)

Fine-Grained Structure

- Can we do better?

Fine-Grained Structure (Wall data): Scatter plot of surface normals

By W. Burgard and D. Hähnel

The Remaining Problem

Data

Mostly Multi-Planar model
Clustering Data with K-Means

EM: Generalized K-Means

EM for Fitting Multi-Planar Models

Probabilistic Model

Measurement Model

- Case 1: Measurement $z_i$ caused by plane $\theta$
  $$p(z_i | \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_z - \beta)^2}{2\sigma^2}}$$

- Case 2: $z_i$ caused by non-planar object
  $$p(z_i | \theta) = \frac{1}{z_{max}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_z - \beta)^2}{2\sigma^2}}$$

Measurement Model

$$p(z_i | \theta, c_1, ..., c_j, c_{n}) = \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{(y_z - \beta)^2}{2\sigma^2}} \prod_{j=1}^{n} c_{j}$$

correspondence variables $C$:
$$c_j, c_j \in [0,1]$$
$$c_j + \sum_{j=1}^{n} c_j = 1$$
$$\Rightarrow p(Z|\theta, C) = \prod_{j=1}^{n} c_{j}$$
Expected Log-Likelihood Function

\[ p(Z|\theta, C) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2} \sum_{j} (z_{ij} - \mu)^2} \]

...after some simple math

\[ E[\ln p(Z,C|\theta)] = \sum_{i} \left( \ln \frac{1}{(J+1)\sqrt{2\pi \sigma^2}} - \frac{1}{2\sigma^2} E[c_{ij}|Z,\theta] \ln \frac{2\pi \sigma^2}{J} \right) \]

The EM Algorithm

- Define:
  \[ Q(\theta|\vartheta) = E[\ln p(Z,C|\theta|Z,\theta)] = \sum_{i} \left( \ln \frac{1}{(J+1)\sqrt{2\pi \sigma^2}} - \frac{1}{2\sigma^2} E[c_{ij}|Z,\theta] \ln \frac{2\pi \sigma^2}{J} \right) \]

- Initialization: Pick random \( \theta \)

- E-Step: Calculate expectations in \( Q(\theta^{(n+1)}|\vartheta^{(n)}) \)

- M-Step: Generate New Map \( \theta^{(n+1)} = \arg\max_{\theta} Q(\theta|\vartheta^{(n)}) \)

The E (Expectation) Step

E-Step

- Calculate expectations with fixed model \( \theta \):

  \[ E[c_{ij}|Z,\theta^{(n)}] = e^{-\frac{1}{\sigma^2} (z_{ij} - \beta_{j})^2} / \sum_{j} e^{-\frac{1}{\sigma^2} (z_{ij} - \beta_{j})^2} \]

  \[ E[c_{ij}|Z,\theta^{(n)}] = e^{-\frac{1}{\sigma^2} (z_{ij} - \beta_{j})^2} \cdot \left( \frac{1}{J} \right) \]

  (normalize so that \( \sum_{i} E[c_{ij}|Z,\theta^{(n)}] = 1 \))

The M (Maximization) Step

M-Step

- Maximize

  \[ Q(\theta|\vartheta) = E[\ln p(Z,C|\theta|Z,\theta)] = \sum_{i} \left( \ln \frac{1}{(J+1)\sqrt{2\pi \sigma^2}} - \frac{1}{2\sigma^2} E[c_{ij}|Z,\theta] \ln \frac{2\pi \sigma^2}{J} \right) \]

  subject to \( \alpha^{i} \cdot \alpha_{j} = 1 \)

- Is equivalent to minimizing

  \[ \sum_{i} \sum_{j} E[c_{ij}|Z,\theta^{(n)}] (\alpha^{i} \cdot \beta_{j}) \cdot \left( \frac{1}{J} \right) \]

  subject to \( \alpha^{i} \cdot \alpha_{j} = 1 \)
M-Step: Solve via Lagrange Multipliers

Notice: the notation has been simplified

\[
\sum_{j} \sum_{i} E[c_{ij}] (\alpha_i, z - \beta_j)^2 \rightarrow \min \quad \forall j : \alpha_j = 1
\]

Define

\[
L = \sum_{j} \sum_{i} E[c_{ij}] (\alpha_i, z - \beta_j)^2 + \sum_{j} \lambda \alpha_j \alpha_j
\]

And observe that

\[
\frac{\partial L}{\partial \alpha_i} = E[c_{ij}] (\alpha_i, z - \beta_j) z_i - \lambda \alpha_i = 0
\]

\[
\frac{\partial L}{\partial \beta_j} = E[c_{ij}] (\alpha_i, z - \beta_j) = 0
\]

\[
\alpha_j \alpha_j = 1
\]

The 3D Multi-Plane Mapping Problem

- Entails a range of problems
  - Parameter estimation: Each surface presently possesses parameters \((x, y, z, \theta, \phi, \psi)\)
  - Outlier detection: How do we explain every single measurement to a surface?
  - Correspondence: What measurement corresponds to what surface?
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Choosing the “Right” Number of Planes: AIC

\[
\log p(J | d) = \text{const} + \log p(d | J) + \log p(J)
\]

Model Selection

Approximately every 20 iterations of EM:

- Start new surfaces
  - Near any set of collinear measurements
- Terminate unsupported surfaces
  - If not supported by enough measurements
  - If density of measurements too low
  - If two planes are too close to each other

Quality of 3D model

Without EM

With EM
Quality of 3D model

![Without EM](image1) ![With EM](image2)

Numerical Results

- \( J \) (number of surfaces in \( \theta \))
- Fraction measurements explained by \( \theta \)

Advantages and Limitations

- Increased compactness, accuracy
- Not real-time
- Only flat surfaces

Towards Online EM

- Each Measurement is only considered in \( \leq N \) E-steps
- When doing so, each measurement only compared to \( \leq K \) surfaces
- Each surface calculated from \( \leq M \) measurements
- New surfaces only started from recent measurements, discarded if not growing fast
- …with small \( N, K, \) and \( M \)

Online EM and Model Selection

- raw data
- mostly planar map with Online EM

Relation to Bayes Filters??

- Maximizes posterior likelihood
  \[
  \arg\max_{\theta} E_{z} \{ \log p(z_{1:T}, \theta) | \theta \} + \log p(\theta) \\
  = \arg\max_{\theta} p(\theta | z_{1:T}) 
  \]
  instead of computing posterior incrementally
  \[
  p(\theta | z_{1:T}) = f(\theta | z_{T}), \bar{z}
  \]
- What does this imply?