The Limits of Computation

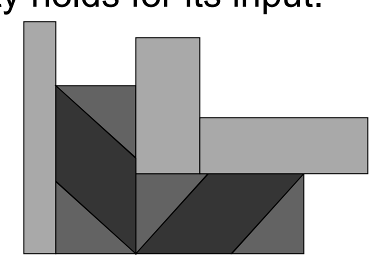
7A

Intractability

Decision Problems

- A specific set of computations are classified as decision problems.
- An algorithm describes a **decision problem** if its output is simply YES or NO, depending on whether a certain property holds for its input.
- Example:

  Given a set of N shapes, can these shapes be arranged into a rectangle?

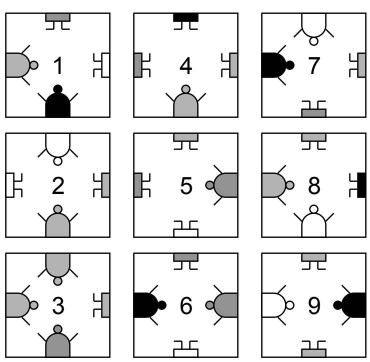


Monkey Puzzle Problem

- Given:
  - A set of N square cards whose sides are imprinted with the upper and lower halves of colored monkeys.
  - N is a square number, such that $N = M^2$.
  - Cards cannot be rotated.
- Problem:
  - Determine if an arrangement of the N cards in an M X M grid exists such that each adjacent pair of cards display the upper and lower half of a monkey of the same color.

Source: www.dwheeler.com (2002)

Example



Images from: **Simonas Šaltenis,** Aalborg University, simas@cs.auc.dk

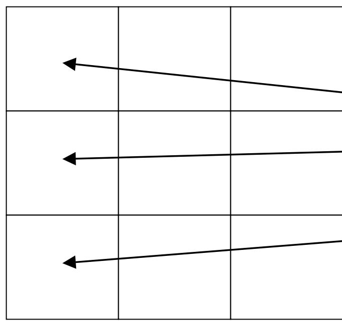
Solution

Analysis

Simple algorithm:

- Pick one card for each cell of M X M grid.
- Verify if each pair of touching edges make a full monkey of the same color.
- If not, try another arrangement until a solution is found or all possible arrangements are checked.
- Answer "YES" if a solution is found. Otherwise, answer "NO" if all arrangements are analyzed and no solution is found.

Analysis



If there are N = 9 cards (M = 3):

To fill the first cell, we have 9 card choices.

To fill the second cell, we have 8 card choices left.

To fill the third cell, we have 7 card choices remaining.

etc.

The total number of unique arrangements for N = 9 cards is:

9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1 = 362,880

Analysis

For N cards, the number of arrangements to examine is N! (N factorial)

If we can analyze one arrangement in a microsecond:

N	Time to analyze all arrangements
9	362,880 μs = 0.36288 s
16	20,922,789,888,000 μs ≈ 242 days
25	15,511,210,043,330,985,984,000,000 μs ≈ 491,520,585,955 years

Map Coloring

- Given a map of N territories, can the map be colored using K colors such that no two adjacent territories are colored with the same color?
- K=4: Answer is always yes. (See Chap 5)
- K=2: Only if the map contains no point that is the junction of an odd number of territories.

Map Coloring

Map Coloring

- Given a map of 48 territories, can the map be colored using **3** colors such that no two adjacent territories are colored with the same color?
  - Pick a color for California (3 choices)
  - Pick a color for Nevada (3 choices)
  - ...
- There are 3^{48} = 79,766,443,076,872,509,863,361 possible colorings.
- No one has come up with a better algorithmic solution that works in general for any map, so far.

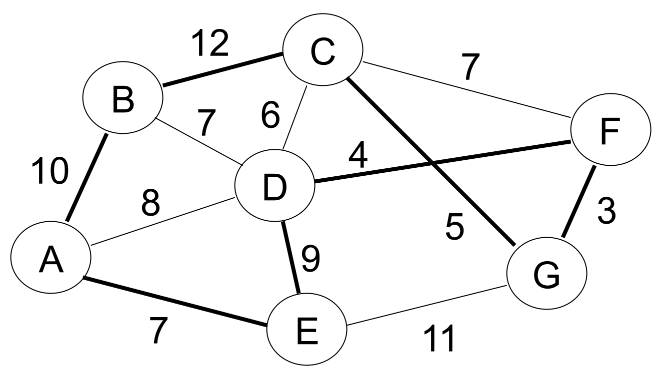
Classifications

- Algorithms that are $O(N^k)$ for some fixed k are **polynomial-time** algorithms.
  - $O(1)$, $O(\log N)$, $O(N)$, $O(N \log N)$, $O(N^2)$
  - reasonable, **tractable**
- All other algorithms are **super-polynomial-time** algorithms.
  - $O(2^N)$, $O(N^N)$, $O(N!)$
  - unreasonable, **intractable**

Traveling Salesperson

- Given: a weighted graph of nodes representing cities and edges representing flight paths (weights represent cost)
- Is there a route that takes the salesperson through every city and back to the starting city with cost no more than K?
  - The salesperson can visit a city only once (except for the start and end of the trip).

Traveling Salesperson



Is there a route with cost at most 52? YES (Route above costs 50.)
Is there a route with cost at most 48? YES? NO?

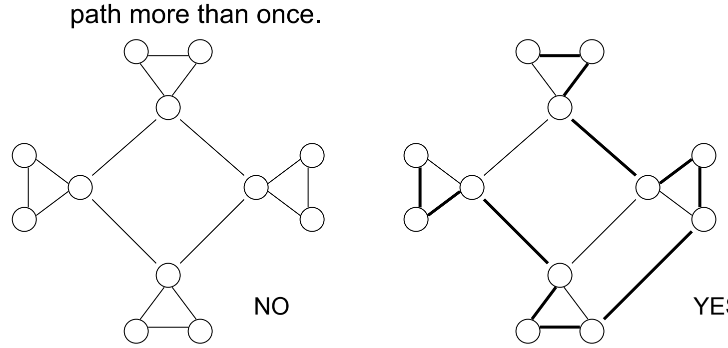
Traveling Salesperson

- If there are N cities, what is the maximum number of routes that we might need to compute?
- Worst-case: There is a flight available between every pair of cities.
- Compute cost of every possible route.
  - Pick a starting city
  - Pick the next city (N-1 choices remaining)
  - Pick the next city (N-2 choices remaining)
  - ...

  how to build a route
- Maximum number of routes: ___________

Hamiltonian Paths

- Given an undirected graph of N nodes, is there a path that passes through all nodes exactly once?
  - A path cannot have a cycle since a node would be on the path more than once.



Hamiltonian Path

- If there are N nodes, what is the maximum number of paths that we might need to examine?
  - Pick a starting node (N choices)
  - Pick the next node (N-1 choices remaining)
  - Pick the next node (N-2 choices remaining)
  - ...
- Maximum number of paths: _________

Satisfiability

- Propositional calculus
- Operations on boolean (logical) values:

P	*Q*	*P & Q*	*P v Q*	*P → Q*
F	F	F	F	T
F	T	F	T	T
T	F	F	T	F
T	T	T	T	T

P	*~P*
F	T
T	F

T = True
F = False

Satisfiability

- Given a sentence in the propositional calculus using the operators &, v, →, ~ :
  - Is there an assignment of boolean values for the symbols so that the sentence reduces down to T (true)? (Is the sentence satisfiable?)
- Example: (A & B) v (~C → A)
  - Truth assignment: A = True, B = False, C = False.
- How many assignments do we need to check for N symbols?
  - Each symbol has 2 possibilities .... ___ assignments

P and NP

- The class **P** consists of all those decision problems that can be solved on a deterministic sequential machine in an amount of time that is polynomial in the size of the input
- The class **NP** consists of all those decision problems whose positive solutions can be verified in polynomial time given the right information, or equivalently, whose solution can be found in polynomial time on a non-deterministic machine.

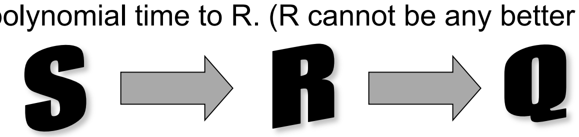
  from Wikipedia

NP Complete

- The class **NPC** consists of all those problems in NP that are least likely to be in P.
  - Each of these problems is called **NP Complete**.
  - Monkey puzzle, Traveling salesperson, Hamiltonian path, map coloring, satisfiability are all in NPC.
- Every problem in NPC can be transformed to another problem in NPC.
  - If there were some way to solve one of these problems in polynomial time, we should be able to solve all of these problems in polynomial time.

Reductions

- To show a new problem R is NP-Complete, we must show:
  - R can be reduced in polynomial time to another NP-Complete problem Q. (R cannot be any worse than Q.)
  - An NP-Complete problem S can be reduced in polynomial time to R. (R cannot be any better than S.)

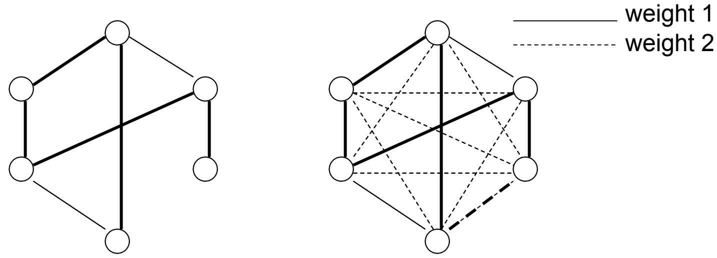


But since S and Q are NP-Complete, R must be also.

  - First NP Complete problem: Satisfiability problem (1971, Cook's Theorem)

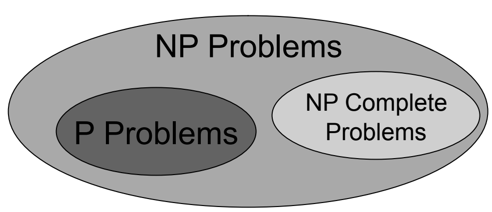
Reduction Example

- Reduce the Hamiltonian path problem to the traveling salesperson problem:



A graph G with N nodes has a Hamiltonian path if and only if the corresponding traveling salesperson graph has a route with a total cost of at most N+1.

Complexity Classes



If P ≠ NP, then all decision problems can be broken down into this classification scheme.
If P = NP, then all three classes are one and the same.

The Clay Mathematics Institute is offering a $1M prize for the first person to prove P = NP or P ≠ NP.
(http://www.claymath.org/millennium/P_vs_NP/)