Advanced Optimization

(10-801: CMU)

Lecture 22

Fixed-point theory; nonlinear conic optimization

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Suvrit Sra

Fixed-point theory

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Three types of results

1. **Geometric:** Banach contraction and relatives
2. **Order-theoretic:** Knaster-Tarski
3. **Topological:** Brouwer, Schauder-Leray, etc.

Fixed-point theory – main concerns

- existence of a solution
- uniqueness of a solution
- stability under small perturbations of parameters
- structure of solution set (failing uniqueness)
- algorithms / approximation methods to obtain solutions
- rate of convergence analyses

Fixed-point theory – Banach contraction

Some conditions under which the nonlinear equation

$$x = Tx, \qquad x \in M \subset X,$$

can be solved by iterating

$$x_{k+1} = Tx_k, \qquad x_0 \in M, \quad k = 0, 1, \ldots.$$

Fixed-point theory – Banach contraction

Theorem (Banach 1922.) Suppose (i) $T : M \subseteq X \to M$; (ii) M is closed, nonempty set in a complete metric space (X, d); (iii) T is q-contractive, i.e.,

$$d(Tx, Ty) \leq qd(x, y), \qquad \forall x, y \in M, \text{ constant } 0 \leq q < 1.$$

Then, we have the following:

(i) $Tx = x$ has exactly one solution (T has a unique FP in M)

(ii) The sequence $\{x_k\}$ converges to the solution for any $x_0 \in M$

(iii) A priori error estimate

$$d(x_k, x^*) \leq q^k (1 - q)^{-1} d(x_0, x_1)$$

(iv) A posterior error estimate

$$d(x_{k+1}, x^*) \leq q(1 - q)^{-1} d(x_k, x_{k+1})$$

(v) (Global) linear rate of convergence: $d(x_{k+1}, x^*) \leq qd(x_k, x^*)$

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- Several other variations of maps have been studied for Banach spaces (see e.g., book by Bauschke, Combettes (2012))

Banach contraction – proof

Blackboard

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Summary:

- d must be positive-definite, i.e, $d(x, y) = 0$ iff $x = y$
- (X, d) must be complete (contain all its Cauchy sequences)
- $T : M \to M$, M must be closed
- But M need not be compact!
- Contraction is often a rare luxury; nonexpansive maps are more common (we've already seen several)

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- Any algorithm for computing a Brouwer FP based on function evaluations only must in the worst case perform a number of function evaluations exponential in both the number of digits of accuracy and the dimension.
- Contrast with $n = 1$, where bisection yields $|f(\hat{x}) - f^*| \leq 2^{-\delta}$ in $O(\delta)$

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Set-valued map

$$F : M \to 2^M, \qquad x \in M \mapsto F(x) \in 2^M, \text{ i.e. } F(x) \subseteq M.$$

Closed-graph

$$\{(x, y) \mid y \in F(x)\} \text{ is a closed subset of} X \times X$$

(i.e., $x_k \to x$, $y_k \to y$ and $y_k \in F(x_k) \implies y \in F(x)$)

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Theorem (S. Kakutani 1941.) Let $M \subset \mathbb{R}^n$ be nonempty, convex, compact. Let $F : M \to 2^M$ be a set-valued map with a **closed graph**; also for all $x \in M$, let $F(x)$ be non-empty and convex. Then, F has a fixed point.

Application: See proof of Nash equilibrium on Wikipedia

Brouwer FP – example

- Consider a Markov transition matrix $A \in \mathbb{R}_+^{n \times n}$
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How to compute such an x?

Conic optimization

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Max-min gauges

$$\begin{aligned} M_K(x/y) &:= \inf \{\beta \in \mathbb{R} \mid x \leq \beta y\} \\ m_K(x/y) &:= \sup \{\alpha \in \mathbb{R} \mid \alpha y \leq x\}. \end{aligned}$$

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▶ **Parts:** We have an equivalence relation $x \sim_K y$ on K if x dominates y and vice versa. The equivalence classes are called parts of the cone.

Hilbert projective metric

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Proposition. *Let K be a cone in V; (K, d_H) satisfies:*

- $d_H(x, y) \geq 0$, *and* $d_H(x, y) = d_H(y, x)$ *for all* $x, y \in K$
- $d_H(x, z) \leq d_H(x, y) + d_H(y, z)$ *for all* $x \sim_K y \sim_K z$, *and*
- $d_H(\alpha x, \beta y) = d_H(x, y)$ *for all* $\alpha, \beta > 0$ *and* $x, y \in K$.

If K is closed, then $d_H(x, y) = 0$ iff $x = \lambda y$ for some $\lambda > 0$. In this case, if $X \subset K$ satisfies that for each $x \in K \setminus \{0\}$ there is a unique $\lambda > 0$ such that $\lambda x \in X$ and P is a part of K, then $(P \cap X, d_H)$ is a genuine metric space.

Proof: on blackboard

Nonexpansive maps with d_H

Def. (OPSH maps.) Let $K \subseteq V$ and $K' \subseteq V'$ be closed cones. The $f : K \to K'$ is called order preserving if for $x \leq_K y$, $f(x) \leq_{K'} f(y)$. It is homogeneous of degree r if $f(\lambda x) = \lambda^r f(x)$ for all $x \in K$ and $\lambda > 0$. It is subhomogeneous if $\lambda f(x) \leq f(\lambda x)$ for all $x \in K$ and $0 < \lambda < 1$.

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Exercise: Prove that if $f : K \to K'$ is OPH of degree $r > 0$ then

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- In particular, if $r = 1$, then f is nonexpansive (in d_H)

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Theorem (Birkhoff.) Let $\Delta(L) := \sup \{d_H(Lx, Ly) \mid Lx \sim_K Ly\}$ be the projective diameter of L. Then

$$\kappa(L) = \tanh(\tfrac{1}{4}\Delta(L))$$

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- If $\Delta(L) < \infty$, then we have a strict contraction!

Application to Pagerank eigenvector

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- Column stochastic: $a_{ij} \geq 0$ and $\sum_i a_{ij} = 1$ for $1 \leq j \leq n$

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- ▶ Suppose $\Delta(A) < \infty$ – (next slide)

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- ▶ Invoke Banach contraction theorem.

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- Linear rate of convergence

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Lemma *If $A \in \mathbb{R}^{m \times n}_+$. If there exists $J \subset [n]$ s.t. $Ae_i \sim_{K'} Ae_j$ for all $i, j \in J$, and $Ae_i = 0$ for all $i \notin J$ then the projective diameter*

$$\Delta(A) = \max_{i,j \in J} d_H(Ae_i, Ae_j) < \infty.$$

More applications

- Geometric optimization on the psd cone

  Sra, Hosseini (2013). *"Conic geometric optimisation on the manifold of positive definite matrices."* arXiv:1312.1039.

- MDPs, Stochastic games, Nonlinear eigenvalue problems, etc.

References

♠ *Nonlinear functional analysis–Vol.1 (Fixed-point theorems). E. Zeidler.*

♠ *Nonlinear Perron-Frobenius theory. Lemmens, Nussbaum (2013).*