The Lattice of β Recursively Enumerable Sets

Klaus Sutner

Abstract

In this paper we study RE_β, the lattice of the β-r.e. sets under inclusion, where β is an arbitrary limit ordinal. The ℓ-finite sets in RE_β are a generalization of finite sets in classical recursion theory: a β-r.e. set is ℓ-finite iff all its β-r.e. subsets are β-recursive. A β-r.e. set is thin iff its Σ_1 cardinality is less than $\beta^\star$, the Σ_1 projectum of β. We will show that every thin set is β-recursive. Hence thin sets and ℓ-finite sets coincide. Using this result we will also show that every β-r.e. set that is not ℓ-finite has a β-recursive subset that is not ℓ-finite. Furthermore, we establish Friedberg Splitting for all limit ordinals. Lastly, we obtain various non-existence results for maximal sets in RE_β.

Contents

1 Introduction **2**

2 Reflection and Σ_1 Normal Forms **3**

2.1 Projecta, Cofinalities and β-Cardinality 3

2.2 Reflection for L_β 5

2.3 Normal form Σ_1 definitions 9

3 Thin Sets are β-Recursive **12**

3.1 Characterization of ℓ-finite sets 12

3.2 Proof of the main lemma 15

4 Applications: Recursive Subsets, Friedberg Splitting, and Maximal Sets **21**

4.1 Recursive Subsets 21

4.2 Friedberg Splitting 23

4.3 Maximal Sets 25

1 Introduction

In this paper we study some basic properties of the collection of β-r.e. sets construed as a distributive lattice under set theoretic inclusion. We denote this lattice by RE_β. In classical recursion theory the collection of finite sets forms a definable ideal: a recursively enumerable set $X \subseteq \omega$ is finite iff the principal ideal generated by X consists entirely of recursive sets (and thus forms a boolean lattice). It was suggested by Maass to adopt this as a definition of generalized finite sets in β-recursion theory, see [7]. Thus a β-r.e. set X is called *ℓ-finite* iff its principal ideal consists entirely of β-recursive sets. It is easy to see that the ℓ-finite sets form an ideal in RE_β. We write $\Im_\beta$ for this ideal.

For admissible ordinals α Lerman has given ample evidence that $\Im_\alpha$ is the appropriate generalization of the notion of "finite" from RE_ω to RE_α, see [Le1,Le2]. In particular, Lerman has shown that for admissible ordinals α the ℓ-finite sets are exactly the α^*-finite sets, i.e., those elements of L_α that have α-cardinality less than α^*. He also proves that the elementary theory of the quotient lattice $\mathsf{RE}^*_\alpha := \mathsf{RE}_\alpha/\Im_\alpha$ is equidecidable with the elementary theory of RE_α. It is not hard to see that Lerman's proof also covers the inadmissible case and shows the following:

- $\Im_\beta$ is the only ideal of RE_β that is definable over RE_β, consists only of β-recursive sets and contains all the really finite sets of RE_β.
- The elementary theories of RE_β and $\mathsf{RE}^\star_\beta$ are equidecidable.

The ideal $\Im_\beta$ is therefore a natural starting point for the investigation of the lattice RE_β. In the weakly admissible case the ℓ-finite sets are exactly those β-finite sets that have Σ_1-cardinality less than $\beta^\star$. So Lerman's characterization carries over to these ordinals. However, for strongly inadmissible ordinals there always are ℓ-finite sets that fail to be β-finite; thus β-cardinality is no longer sufficient to distinguish the ℓ-finite sets. We therefore use the Σ_1-cardinality of a β-r.e. set to measure its size, in particular we call a β-r.e. set *thin* if it has Σ_1-cardinality less than $\beta^\star$. It is easy to see that a β-r.e. set is ℓ-finite iff it is thin and β-recursive. But for weakly admissible β every thin set is actually β-finite and therefore trivially β-recursive. Thus, for weakly admissible ordinals, the ℓ-finite and the thin sets coincide. This is not at all obvious in the strongly inadmissible situation, in particular if $\beta^\star$ has no regularity properties. So the question arises whether there exists any thin β-r.e. set that fails to be β-recursive. In our main lemma we will answer this question in the negative. Thus the thin sets and the ℓ-finite sets coincide for arbitrary limit ordinals.

Thin sets play a natural role as nullsets of RE_β. In addition they also crop up in recursion theoretic arguments. Consider for example the Friedberg Splitting theorem. The standard construction can easily be lifted to weakly admissible ordinals. For strongly inadmissible ordinals, however, a new difficulty occurs: the injury sets in the construction are no longer β-finite. They are, however, easily seen to be thin. By our main lemma, the injury sets are ℓ-finite and one can show that the construction succeeds even in the strongly inadmissible case. As another application we will show that every β-r.e. set that fails to be ℓ-finite has a β-recursive subset that also fails to be ℓ-finite (this is non-trivial for strongly inadmissible β). Thin sets can also be used to derive various non-existence results for maximal sets in RE_β.

This paper is organized as follows. In section 2 we review the basic definitions and a couple of useful facts about the constructible hierarchy. We will use the classical Gödel hierarchy rather than

the Jensen hierarchy since we feel it to be more natural. Our results can easily be transferred to the J-hierarchy. In particular we will prove a reflection principle for L_β and list several properties of β-pseudo stable ordinals. Section 3 contains characterizations of ℓ-finite and thin sets and the somewhat lengthy proof of our main lemma: every thin β-r.e. set is β-recursive. Section 4 is devoted to applications of this result to the lattice of β-r.e. sets. In particular we will show that Friedberg splitting holds for all limit ordinals and derive various non-existence results for maximal sets.

2 Reflection and Σ_1 Normal Forms

In this section we establish two technical results that will be helpful later. First we show that the following reflection principle holds for arbitrary limit ordinals β.

Reflection principle for L_β
Let $\delta < \eta < \beta$, η a β-cardinal, and $p \subseteq \delta$. If $p \in L_\beta$ then $p \in L_\eta$.

Furthermore we introduce a special normal for Σ_1 definitions in L_β. This normal form will be crucial in showing that thin sets are β-recursive. We begin with a brief review of terminology and list a few standard results about the constructible hierarchy that will be used in the sequel without reference.

To keep notation manageable we assume $V = L$ throughout this paper. Background material on the constructible hierarchy and the Levy hierarchy can be found for example in [1, 2]. Occasionally we will slightly abuse notation and not differentiate between various language levels. As a typical example, suppose X is a Σ_1 substructure of L_β (in symbols $X \prec_1 L_\beta$), $\gamma \in X \cap \beta$, $\phi(u, v)$ is a parameterless Δ_0 formula and $p \in L_\gamma \cap X = (L_\gamma)^X$ a parameter in X. To express the fact that X "believes" that L_γ is a model of $\exists x \phi(x, p)$ we will simply write $X \models (L_\gamma \models \exists x \phi(x, p))$. Now consider the Mostowski collapse $\pi : X \to L_\delta$. Set $\bar{\gamma} := \pi(\gamma)$ and $\bar{p} := \pi(p)$. Then $L_\delta \models (L_\gamma \models \exists x \phi(x, p))$. To see this note that the satisfaction relation $\models$ is primitive recursive and π is compatible with Σ_1 substructures: $X \prec_1 L_\beta$ implies that X is closed under the primitive recursive definition of $\models$. Hence we may infer that $\phi(x_0, \bar{p})$ holds for some $x_0 \in L_\gamma$ since L_γ is a transitive substructure of L_β and thus absolute for Δ_0 formulae. Arguments of this kind will be used frequently in what follows.

(x, y) will denote set-theoretical pairing, for emphasis we will sometimes write $(x, y)_s$. $Pa(X)$ will be the closure of X under pairs. For a function f the notation $f : X \leftrightarrow Y$ indicates that f is a bijection with domain X and range Y. The symbol $\Box$ denotes the end of a proof.

2.1 Projecta, Cofinalities and β-Cardinality

Let $\gamma \leq \beta$ and let $n > 0$ be a natural number. The Σ_n-L_β projectum respectively the Σ_n-L_β cofinality of γ are defined as follows:

$$\begin{aligned} \sigma n\mathsf{prj}^\beta(\gamma) &:= \min(\delta \mid \exists f\, \Sigma_n\text{-}L_\beta(f : \gamma \to \delta \wedge f \text{ injective}) \\ \sigma n\mathsf{cof}^\beta(\gamma) &:= \min(\delta \mid \exists f\, \Sigma_n\text{-}L_\beta(f : \delta \to \gamma \wedge \mathsf{rg}(f) \subseteq \gamma \text{ is cofinal }). \end{aligned}$$

As usual we write $\beta^\star$ for $\sigma 1 p^\beta(\beta)$ and κ for $\sigma 1 \mathsf{cof}^\beta(\beta)$. Let further $\widehat{\beta} = \max(\beta^\star, \kappa)$. β is admissible iff $\beta = \kappa$, weakly admissible iff $\kappa \geq \beta^\star$, and strongly inadmissible otherwise. Note that β is admissible iff $\beta = \widehat{\beta}$, weakly admissible iff $\widehat{\beta} = \kappa$, and strongly inadmissible iff $\widehat{\beta} = \beta^\star > \kappa$. Also let $\mathsf{cof}^\beta(\gamma) := \min(\delta \mid \exists f \in L_\beta(f : \delta \to \gamma \wedge rg(f) \subseteq \gamma \text{ is cofinal })$ and set $\sigma 0\, \mathsf{cof}^\beta(\gamma) := \mathsf{cof}^\beta(\gamma)$. γ is called Σ_n-L_β regular, $n \geq 0$, if $\sigma n \mathsf{cof}^\beta(\gamma) = \gamma$.

The Σ_n-L_β cardinality $|X|^{\beta,n}$ of a Σ_n-L_β set X is defined by

$$|X|^{\beta,n} := \min\big(\, \delta \leq \beta^\star \mid \exists f \Sigma_n\text{-}L_\beta\ (f : \delta \leftrightarrow \gamma)\,\big)$$

We will be mostly interested in the case $n = 0$ and $n = 1$. Define the β-cardinality of $X \in L_\beta$ by $|X|^\beta := |X|^{\beta,0}$. It will be shown shortly that this yields the same result as the somewhat more traditional definition $|X|^\beta = \min\big(\, \delta \mid \exists f \in L\beta(f : \delta \leftrightarrow \gamma)\,\big)$. An ordinal $\eta < \beta$ is a β-cardinal if $|\eta|^\beta = \eta$. $gc(\beta)$ will denote the largest β-cardinal if it exists and β otherwise. Limit, successor and regular β-cardinal have their obvious meaning. Clearly for $n > 0$ and $\gamma \leq \beta$ we have: $\sigma n \mathsf{prj}^\beta(\gamma)$ is a β-cardinal (or β) and $\sigma n \mathsf{cof}^\beta(\gamma)$ is a regular β-cardinal (or β). Lastly, $X \in L_\beta$ is called η-finite where η is a β-cardinal if $|X|^\beta < \eta$.

As a general notational convention we will usually omit the superscript β in a context like $\sigma n \mathsf{prj}^\beta(\gamma)$ if no confusion is possible.

The following definitions are due to Friedman and Sacks and are rather straightforward generalizations from α-recursion theory. A subset $X \subseteq L_\beta$ is β-recursively enumerable (β-r.e.) iff X is $\Sigma_1 - L_\beta$. X is β-recursive (β-rec) iff X is $\Delta_1 - L_\beta$. Most importantly X is β-finite iff $X \in L_\beta$. A function f such that both its domain and range are subsets of L_β is partial β-recursive iff its graph is β-r.e. The function f is β-recursive iff in addition its domain is β-recursive.

Suppose X is β-r.e. ; we will sometimes abuse X to denote a corresponding Σ_1 formula that defines X over L_β. Thus we may write $L_\beta \models a \in X$.

The following facts are well known.

- Σ_n Separation

  According to Jensen the Σ_n projecta have the following alternative characterizations:

  $$\begin{aligned} \sigma n \mathsf{prj}(\beta) &= \min(\delta \mid \exists f\, \Sigma_n\text{-}L_\beta(\mathsf{dom}(f) \subseteq \delta \wedge \mathsf{rg}(f) = L_\beta) \\ &= \min(\delta \mid \exists X\, \Sigma_n\text{-}L_\beta(X \subseteq \delta \wedge X \notin L_\beta). \end{aligned}$$

  Therefore Σ_1 separation is available below $\beta^\star$.

- Δ_1 Separation

  Δ_1 separation holds below $\widehat{\beta}$: $\widehat{\beta} = \min(\delta \mid \exists X\, \beta\text{-rec}(X \subseteq \delta \wedge X \notin L_\beta)) = |L_\beta|^{\beta,1}$.

- Σ Recursion

  β-recursive functions are closed under recursion up to κ. To be more explicit, let $G : \kappa \times L_\beta \to L_\beta$ be β-recursive and define by recursion a new function $f : \kappa \to L_\beta$ where $f(\nu) := G(\nu, f \restriction \nu)$. Then f is also β-recursive and $\mathsf{dom}(f) = \kappa$. As an application of this principle one can show that there is a strictly increasing, continuous β-recursive function $q : \kappa \to \beta$ that has range cofinal in β. We shall reserve q as a name for such a function.

- Enumerations

  Maass has shown that $X \subseteq L_\beta$ is β-r.e. iff there is a β-recursive function f such that $f : \delta \leftrightarrow X$ where $\delta \leq \beta$; see [8] for a proof. It follows that for any β-r.e. set X we have $|X|^{\beta,1} < \beta$ and, as a matter of fact, $|X|^{\beta,1} \leq \widehat{\beta}$. f is frequently called an enumeration of X; however, we will sometimes use the term enumeration with a slightly different meaning. Let $\Phi(v)$ be a Σ_1 formula with parameters from L_β that defines X over L_β, so that $x \in X$ iff $L_\beta \models \Phi(x)$ for any $x \in L_\beta$. Let q be the special cofinality function. For $\sigma < \kappa$ set $X^\sigma := \{\, x \in L_\beta \mid L_{q(\sigma)} \models \Phi(x) \,\}$. Clearly X^σ is β-finite and the map $\sigma \mapsto X^\sigma$ is β-recursive. The sequence $(X^\sigma \mid \sigma < \kappa)$ will be referred to as an enumeration of X.

- Universal Predicates

  There is a universal Σ_n–L_β predicate $U_n^\beta(u, v)$ uniformly for all β. In particular for $n = 1$ there is a simultaneous enumeration of all β-r.e. sets using only indices less than $\beta^\star$. To see this let $P : L_\beta \to \beta^\star$ be a β-recursive projection (i.e. an injective function). For $e < \beta^\star$ set $W_e := \{\, x \in L_\beta \mid \exists \bar{e}(P(\bar{e}) \simeq e \wedge U_1^\beta(\bar{e}, x)) \,\}$. Note, however, that there is a hidden parameter: P is in general not Σ_1^β–$\emptyset$.

- Σ_n Uniformization

  This is one of the celebrated results of fine structure theory due to Jensen. Let $R \subseteq L_\beta \times L_\beta$ be a Σ_n-L_β relation. Then there is a Σ_n-L_β function r that uniformizes R: $\mathsf{dom}(r) = \mathsf{dom}(R)$ and $\forall x \in \mathsf{dom}(R)\big(R(x, r(x))\big)$. In the special case $n = 1$ one can easily see that L_β is actually uniformly Σ_1 uniformizable for all β.

- Σ_n Definitions

  Suppose $a \in L_\beta$, $X \subseteq L_\beta$. a is called Σ_n definable over L_β with parameters from X, $n \geq 1$, iff there is parameterless Σ_n formula $\Phi(u, v_1, \ldots, v_n)$ and $x_1, \ldots, x_n \in X$ such that $L_\beta \models \exists! u \Phi(u, x_1, \ldots, x_n) \wedge \Phi(a, x_1, \ldots, x_n)$. There is a parameter $p \in L_\beta$ such that every $a \in L_\beta$ is Σ_1 definable over L_β with parameters from $\beta^\star \cup \{p\}$. Let $p(\beta)$ be the $<_\beta$-minimal such p.

- Σ_1 Cardinality

  Let $a \in L_\beta$ and set $\eta := |a|^\beta$ and $\bar{\eta} := |a|^{\beta,1}$. Then $\eta \leq \widehat{\beta}$ or $\bar{\eta} < \widehat{\beta}$ implies that $\eta = \bar{\eta}$. Hence the β-cardinality and the Σ_1 cardinality of $a \in L_\beta$ almost always coincide. (Of curse, if η is a β-cardinal such that $\widehat{\beta} < \eta < \beta$ then $|\eta|^{\beta,1} < |\eta|^\beta$).

  Proof. Clearly $\eta \geq \bar{\eta}$, so assume for the sake of a contradiction that $\eta > \bar{\eta}$. Now $\bar{\eta} = |a|^{\beta,1} = |\eta|^{\beta,1}$, so there is a β-recursive bijection $g : \bar{\eta} \leftrightarrow \eta$. g cannot be β-finite as $\bar{\eta} < \eta$ and both $\bar{\eta}$ and η are β-cardinals. Hence $\bar{\eta} \geq \kappa$ for otherwise one could use Σ collection to show that g is β-finite. Together with our hypothesis $\bar{\eta} \leq \widehat{\beta}$ this implies $\kappa \leq \bar{\eta} < \widehat{\beta}$. Hence $\widehat{\beta} = \beta^\star$ and it follows from the definition of $\beta^\star$ that $\eta = \mathsf{rg}(g) < \beta^\star$. But then $g \subseteq \bar{\eta} \times \eta \subseteq L_\eta \subseteq L_\beta^\star$ is β-finite by Σ separation and we have the desired contradiction.

2.2 Reflection for L_β

We will now show that reflection holds for arbitrary limit ordinals. So suppose $p \subseteq \delta$ is a β-finite set where $\delta < \eta < \beta$ and η a β-cardinal. We have to show that $p \in L_\eta$. To this end we introduce a

special type of Skolem function for successor levels of the constructible hierarchy to deal with the pathological case $\beta = \lambda + \omega$ and $p \notin L_\lambda$

First a few facts concerning Skolem functions at limit levels. A more thorough treatment including proofs can again be found in [1]. Let f be a partial Σ_n-L_β function with $\mathsf{dom}(f) \subseteq \omega \times L_\beta$ and $\mathsf{rg}(f) \subseteq L_\beta$. f is called a Σ_n Skolem function for L_β iff for all $x \in L_\beta$ and every $\Sigma_n^\beta - \{x\}$ relation R we have: $R \neq \emptyset$ implies $\exists i < \omega R(f(i,x))$. The existence of Σ_n Skolem functions follows from the Σ_n uniformization theorem of Jensen, the proof being elementary in the case $n = 1$. In particular there exists a Σ_1 Skolem function h^β for L_β that is uniformly definable for all $\beta > \omega$ by a parameterless Σ_1 formula. We will refer to h^β as the standard Skolem function and frequently omit the superscript. Also let $h_a(i,x) := h(i,(a,x)_s)$ for a $a \in L_\beta$. We use h_a to build Σ_1 Skolem hulls of subsets of L_β:

$$SH_1(X,a;L_\beta) := h_a^\beta[\omega \times Pa(X)].$$

Proposition 2.1 *Let $X \subseteq L_\beta$ be closed under pairs, a $\in L_\beta$. Then $SH_1(X,a;L_\beta)$ is a Σ_1 substructure of L_β.*

The crucial property of Σ_1 substructures is for our purposes is condensation: Let $\pi : X \leftrightarrow Y$ be the Mostowski collapse of $X \prec_1 L_\beta$. Then there exists $\gamma \leq \beta$ such that $Y = L_\gamma$.

An Example

As a typical application of Skolem hulls and condensation we will show how to construct a weakly admissible ordinal β whose Σ_1 cofinality lies strictly between $\beta^\star$ and β. A moments reflection shows that this is a rather rare situation. So define $X := SH_1(L_\omega, \aleph_1; L_{\aleph_2})$ and let $\pi : X \leftrightarrow L_\beta$ be the Mostowski collapse. Note that there are exactly two β-cardinals, namely $\omega = \pi(\omega)$ and $\eta := \pi(\aleph_1)$. We claim that $\beta^\star = \omega < \rho = \kappa$.

To see this let $f := h^{\aleph_2} \restriction \omega \times L_\omega$ and $\bar{f} := \pi[f]$. X is a Σ_1 substructure of L_β, so f is Σ_1-X and $\bar{f}$ is Σ_1-L_β. But $\mathsf{dom}(\bar{f}) \subseteq \omega \times L_\omega$ and $\mathsf{rg}(\bar{f}) = L_\beta$, so $\bar{f}$ really is a projection and we are through. It follows that $\beta > \eta = gc(\beta) > \beta^\star$, whence β is inadmissible. Now κ is a β-cardinal, therefore $\kappa \in \{\omega, \eta\}$ and we only have to show that $\kappa \neq \omega$. So assume for the sake of a contradiction that $\kappa = \omega$. Let $\bar{q} : \omega \to \beta$ be a corresponding Σ_1-L_β cofinality function and set $q := \pi^{-1}[\bar{q}] : \omega \to X \cap \aleph_2$. $\aleph_2$ is regular, so $L_{\aleph_2} \models \exists \lambda (L_\lambda \models \forall x < \omega \exists y (q(x) \simeq y))$. But X is a Σ_1 substructure of $L_{\aleph_2}$, so for some $\lambda \in X \cap \aleph_2$ we have $X \models (L_\lambda \models \forall x < \omega \exists y (q(x) \simeq y))$. Letting $\bar{\lambda} := \pi(\lambda)$ we get $L_\beta \models (L_{\bar{\lambda}} \models \forall x < \omega \exists y (\bar{q}(x) \simeq y))$. But then $\mathsf{rg}(\bar{q}) \subseteq \bar{\lambda} < \beta$, a contradiction. $\square$

Let us now return to our proof of the reflection principle for L_β. Since η is a β-cardinal we may assume without loss of generality that $\delta < \eta$ is a limit ordinal. As long as β is the limit of limit ordinals we can adopt an argument similar to Gödel's proof of the generalized continuum hypothesis in L: let $\lambda < \beta$ be a limit ordinal such that $p \in L_\lambda$. Set $X := SH_1(L_\delta, p; L_\lambda)$ and let $\pi : X \leftrightarrow L_\gamma$ be the Mostowski collapse where $\gamma \leq \lambda$. L_δ is transitive, so $p = \pi(p) \in L_\gamma$. As in the preceding example it follows that $\sigma 1 p^\gamma(\gamma) \leq \delta$. But $\gamma \leq \lambda < \beta$, so $|\gamma|^\beta \leq \delta < \eta$ and thus $p \in L_\gamma \subseteq L_\eta$.

If β fails to be the limit of limit ordinals a more subtle approach is needed. We define Skolem functions for successor levels $L_{\lambda+n}$, $0 \leq n < \omega$. To this end we use of a "flat" pairing function rather than the usual Kuratowski pair. For $1 \leq n < \omega$ let

$$[a_1, \ldots, a_n] := \{(a_1, 1)_s, \ldots, (a_n, n)_s\}.$$

Note that for $a_i \in L_\gamma$, $\omega \leq \gamma$, $[a_1, \ldots, a_n] \in L_{\gamma+3}$. Further let

$$Seq(a) := \{\, [a_1, \ldots, a_n] \mid a_i \in a, i \leq n < \omega \,\}.$$

For the decoding function we use the convention

$$[b]_i := \begin{cases} b_i & \text{if } b = [b_1, \ldots, b_n], \\ \emptyset & \text{otherwise.} \end{cases}$$

Lastly, let $\circ$ denote concatenation of these sequences. So $[a_1, \ldots, a_n] \circ [b_1, \ldots, b_m] = [a_1, \ldots, a_n, b_1, \ldots, b_m]$.

Now fix a recursive listing $(\Psi_i \mid i < \omega)$ of the Δ_0-ZF formulae using some standard arithmetization of the language of set theory. For an arbitrary limit ordinal ν and a natural number n define a partial function $H^{\nu+n}$ from $\omega \times L_{\nu+n}$ into $L_{\nu+n}$ by

> $H^{\nu+n}(i, x) \simeq y$ iff
> there is a $w \in Seq(L_{\nu+n})$ such that $w = [w_1, \ldots, w_n]$ is
> $<_L$-minimal with $L_{\nu+n} \models \Psi_i(x, w_1, \ldots, w_n)$ and $y = w_1$.

We will refer to w as a witness for y. To keep notation reasonable, we will write $\Psi_i(x, w)$ from now on. Given an element a of $L_{\nu+n}$ one can correspondingly define $H_a^{\nu+n}$ like $H^{\nu+n}$ but with a as additional parameter. The definition of $H^{\nu+n}$ just given could clearly be written uniformly as a $\Sigma_1 - \emptyset$ formula $\Phi(i, x, y)$. Similarly $H_a^{\nu+n}$ can be defined by a $\Sigma_1 - \{a\}$ formula $\Phi_a(i, x, y)$. The relations $<_L$ and $\models$ used in Φ (or rather their defining formulae) are absolute for transitive initial segments of the constructible hierarchy, therefore Φ is upward persistent: $L_{\nu+n} \models \Phi(i, x, y)$ implies $L_{\nu+m} \models \Phi(i, x, y)$ for all $m \geq n$.

Note that for $n = 0$ there is a recursive function r such that $h^\nu(i, x) \simeq H^\nu(r(i), x)$. By way of contrast, $H^{\nu+n}$ is not even first order definable over $L_{\nu+n}$ for $n > 0$. However, the syntactical machinery necessary to define $H^{\nu+n}$ will crop up in the constructible hierarchy after finitely many steps: for some m, $n < m < \omega$, Φ defines $H^{\nu+n}$ over $L_{\nu+m}$.

We can now prove the following generalization of proposition 2.1 to successor levels.

Lemma 2.1 *Let $X \subseteq L_\lambda$ be closed under pairs, $a \in L_{\lambda+n}$ where $0 \leq n < \omega$. Set $Y := H_a^{\lambda+n}[\omega \times X]$ and $Y_0 := Y \cap L_\lambda$. Then Y_0 is a Σ_1 substructure of L_λ.*

Proof. Using the Tarski criterion for Σ_1 substructures it suffices to show the following: Let $\boldsymbol{b} = b_1, \ldots, b_k$ be an k-tuple of elements of Y_0 and Ψ a $\Delta_0 - \emptyset$ formula such that $L_\beta \models \exists z \Psi(\boldsymbol{b}, z)$. Then there exists $z_0 \in Y_0$ such that $L_\beta \models \Psi(\boldsymbol{b}, z_0)$. For the sake of simplicity we assume that $k = 2$, so $L_\beta \models \exists z \Psi(b_1, b_2, z)$. X is closed under pairs, so for some $p \in X$ and some natural numbers i_1, i_2 we have $b_\mu = H_a^{\lambda+n}(i_\mu, p)$ for $\mu = 1, 2$.

Let w^1, w^2 be corresponding witnesses, say $L_\beta \models \Psi_{i_\nu}(p, a, w^\mu)$. Define a $\Delta_0 - \{a\}$ formula Φ by

$$\Phi(x, a, z, \boldsymbol{u}, \boldsymbol{v}) := \Psi_{i_0}(x, a, \boldsymbol{u}) \wedge \Psi_{i_1}(x, a, \boldsymbol{v}) \wedge \Psi(u_1, v_0, z).$$

Let j be the Gödel number of Φ and let $c \in L_\lambda$ be $<_\beta$-minimal such that $L_\beta \models \Psi(b_1, b_2, c)$. Lastly define $w := [c] \circ w^1 \circ w^2$.

Then w is a witness for c: $H_a^{\lambda+n}(j, p) \simeq c$ since $L_{\lambda+n} \models \Phi(p, a, w)$ and $[w]_1 = c$. Thus $c \in Y_0$ and we are done. $\square$

Lemma 2.2 (The Reflection Principle) *Reflection holds for arbitrary limit ordinals β: $\delta < \eta < \beta$, η a β-cardinal, $p \subseteq \delta$ and $p \in L_\beta$ implies $p \in L_\eta$.*

Proof. We may assume without loss of generality that we are in the difficult case $\beta = \lambda + \omega$, $p \in L_{\lambda+n} - L_\lambda$ for some natural number n. Let Φ_p and $m > n$ be as in the remark preceding the last lemma. Define a partial function f from $\omega \times L_\delta$ to $L_{\lambda+n}$ by $f(i,x) \simeq y$ iff $L_{\lambda+m} \models \Phi_p(i,x,y)$ where $i < \omega$, $x \in L_\delta$ and $y \in L_{\lambda+n}$. Then f is Σ_1 definable over $L_{\lambda+m}$ with parameters p and δ; indeed $f = H_p^{\lambda+n} \restriction \omega \times L_\delta$. Now set $Y := \mathsf{rg}(f)$, $Y_0 := Y \cap L_\lambda$. Y is extensional, so we may use the Mostowski collapse $\pi : Y \leftrightarrow Z$ where Z is transitive. Then $P[Y_0] = L_\gamma$ for some $\gamma \leq \lambda$. Hence $Z \subseteq L_{\lambda+n}$ and it suffices to show that $\gamma < \eta$. Note that $p = \pi[p] \in L_{\lambda+n}$. Now define $\bar{f}$ over $L_{\lambda+m}$ in the same fashion as f over $L_{\lambda+n}$. Z is transitive and therefore a Δ_0 substructure of $L_{\lambda+m}$. Hence $L_\gamma \subseteq \mathsf{rg}(\bar{f})$. But clearly $\bar{f} \in L_{\lambda+m+1} \subseteq L_\beta$, so $|\gamma|^\beta < \eta$ and we are through. □

Our first application of the reflection principle is to show that the β-cardinality of a β-finite set is always less than β. First a technical lemma.

Lemma 2.3 *Let λ and β be limit ordinals, $\lambda < \beta$, and n a nonnegative integer. Then there is β-finite injective function from $L_{\lambda+n}$ into λ.*

Proof. We proceed by induction on n.

The case $n = 0$ follows from a result in [2]: there is an onto λ-recursive function $\bar{f} : \lambda \to L_\lambda$. Let f be the Σ_1 uniformization of $\bar{f}^{-1}$. Then f is injective by definition and β-finite because of $\lambda < \beta$.

For $n = m + 1$ let $g : L_{\lambda+m} \to \lambda$ be injective β-finite according to the induction hypothesis. As before, let $(\Psi_i \mid i < \omega)$ be a recursive listing of all ZF formulae. Define functions

$$
\begin{aligned}
&N : L_{\lambda+n} \to \omega \\
&N(x) := \min\big(i < \omega \mid \exists p \in L_{\lambda+m}(\Psi_i(v,p) \text{ defines } x \subseteq L_{\lambda+m} \text{ over } L_{\lambda+m}) \big) \\
&P : L_{\lambda+n} \to L_{\lambda+m} \\
&P(x) := \min\big(p \in L_{\lambda+m} \mid \Psi_{N(x)}(v,p) \text{ defines } x \subseteq L_{\lambda+m} \text{ over } L_{\lambda+m} \big).
\end{aligned}
$$

Then we have for $x \in L_{\lambda+n}$: $x = \{ z \in L_{\lambda+m} \mid L_{\lambda+m} \models \Psi_{N(x)}(v, P(x)) \}$. But N and P are clearly β-finite. Therefore $f : L_{\lambda+n} \to L_\lambda$, $f(x) := (N(x), P(x))_s$, is also β-finite. But f is an injective map from $L_{\lambda+n}$ into L_λ and we are done. □

Corollary 2.1 *For all limit ordinals β every β-finite set has β-cardinality less than β.*

Proof. For admissible β there is nothing to show, so assume β is inadmissible. Thus $gc(\beta) < \beta$. By lemma 2.3 it is safe to assume that $a \subseteq gc(\beta)$. Let $f : \mathsf{otp}(a) \leftrightarrow a$ be the usual order isomorphism. One can show by induction on $\nu < \mathsf{otp}(a)$ that $f \restriction \nu \in L_{gc(\beta)}$ using reflection at limit stages. Now pick $\gamma < \beta$ such that a $\in L_\gamma$ and $gc(\beta) < \beta$. Then f is $\Sigma_1 - L_\gamma$ and therefore β-finite. Hence $|a|^\beta \leq \mathsf{otp}(a) \leq \gamma < \beta$. □

A similar argument can be used to show that every β-cardinal η is β-stable, i.e., that L_η is a Σ_1 substructure of L_β. One should note that the problems in the proof of lemma 2.2 cannot be avoided

by using the Jensen hierarchy instead of the Gödel hierarchy: here the difficulties arise at levels $J_{\nu+1}$.

2.3 Normal form Σ_1 definitions

We will now introduce a convenient normal for Σ_1 definitions of elements in L_β. Every set in L_β has a Σ_1 definition over L_β involving only parameters from $\beta^\star \cup \{p(\beta)\}$, thus we may restrict our attention to ordinals less than $\beta^\star$. The first step is to consider ordinals less than $\beta^\star$ that cannot be defined in terms of smaller ones. Friedman called these ordinals β-pseudo stable, see [4, 5]. To be more precise, let $p := p(\beta)$ be the standard parameter and $\gamma < \beta^\star$. Then γ is β-pseudo stable iff $\gamma \notin SH_1(\gamma, p; L_\beta)$. It is clear that any element in L_β has a Σ_1 definition that uses only p and a finite number of β-pseudo stable ordinals as parameters. This fact will be exploited below to define a natural tuple of parameters for every ordinal less than $\beta^\star$.

First we have to derive a few basic properties of β-pseudo stable ordinals. Let us say that a β-pseudo stable ordinal is limit β-pseudo stable iff it is the supremum of lesser β-pseudo stable ordinals and successor β-pseudo stable otherwise. Let $\mathcal{P}_\beta$ denote the set of all β-pseudo stable ordinals. Let $\gamma < \beta^\star$ be arbitrary and consider the Skolem hull $X := SH_1(\gamma, p; L_\beta)$ and its collapse $\pi : X \leftrightarrow L_\delta$, $\delta < \beta^\star$. Note that δ is uniquely determined by γ, so there is a collapse map $\mathcal{C}$ defined by

$$\begin{aligned} &\mathcal{C} : \beta^\star \to \beta^\star, \\ &\mathcal{C}(\gamma) := \delta \end{aligned}$$

The collapse $\mathcal{C}(\gamma)$ can be used to characterize β-pseudo stable ordinals.

Lemma 2.4 *Let $\gamma < \beta^\star$, γ not a β-cardinal. Then γ is β-pseudo stable iff $L_{\mathcal{C}(\gamma)} \models (\gamma$ is a cardinal$)$.*

Proof. First suppose γ is β-pseudo stable and consider $X := SH_1(\gamma, p; L_\beta)$, $\pi : X \leftrightarrow L_\delta$ the Mostowski collapse where $\delta = \mathcal{C}(\gamma)$. Let $\alpha := \min(\xi \in X \mid \xi > \gamma)$. As γ is β-pseudo stable we must have $\pi(\alpha) = \gamma$ and it suffices to show that α is a β-cardinal. So assume for a contradiction that $\eta := |\alpha|^\beta < \alpha$. X is Σ_1 substructure of L_β, so there is a bijection $f \in X$, $f : \eta \leftrightarrow \alpha$. But $\mathsf{dom}(f) \subseteq X$, hence $\mathsf{rg}(f) \subseteq X$ and $\gamma \in X$ contradicting the β-pseudo stability of γ.

For the opposite direction let X and π as above. By our assumption γ is a δ-cardinal but not a β-cardinal. So $\eta := \pi^{-1}(\gamma)$ must be a β-cardinal since $X \prec_1 L_\beta$ and the notion of being a cardinal is Π_1 and thus $\eta > \gamma$. But then γ cannot lie in X, for this would imply $\pi^{-1}(\gamma) = \gamma$. Hence γ is β-pseudo stable. □

A few comments are in order.

- For any β-pseudo stable ordinal γ and $X := SH_1(\gamma, p; L_\beta)$, $\pi : X \leftrightarrow L_\delta$, as above we have $\gamma = \pi(\rho)$ where $\rho := \min\big(\xi \in X \mid \zeta < \xi\big)$ is the least β-cardinal larger than γ in X.
- Using Skolem hull arguments as in the last proof it is possible to show that $\mathcal{P}_\beta$ is almost a closed unbounded subset of $\beta^\star$. To be more explicit, if $\beta^\star$ is a successor β-cardinal then $\mathcal{P}_\beta$ is indeed closed unbounded in $\beta^\star$. On the other hand, if $\beta^\star$ is a limit β-cardinal then $\mathcal{P}_\beta$ is still

unbounded in $\beta^\star$ but no longer closed: some of the β-cardinals less than $\beta^\star$ are not β-pseudo stable.

To see this let us first consider the case where $\beta^\star$ is a limit β-cardinal. Let $\gamma_0 < \beta^\star$ such that $\beta^\star$ is Σ_1 definable over L_β with parameters in $\gamma_0 \cup \{p(\beta)\}$. Pick two consecutive β-cardinals $\eta < \eta^+$ such that $\eta \leq \gamma_0 < \eta^+ \leq \beta^\star \in SH_1(\gamma_0, p; L_\beta)$. It clearly suffices to show that $\mathcal{P}_\beta \cap [\eta, \eta^+[$ is closed unbounded in η^+. So let γ be arbitrary in $[\gamma_0, \eta^+[$. Set $X := SH_1(\gamma + 1, p; L_\beta)$ and $\zeta := X \cap \eta^+$. Again let $\pi : X \leftrightarrow L_\delta$ be the Mostowski collapse where $\delta = \mathcal{C}(\gamma + 1)$.

Claim 1: ζ is an ordinal.

For let $\zeta_0 \in \zeta$. There is a map $f : \eta \leftrightarrow \zeta_0$ and f is Σ_1 definable over L_β, hence there is a $\Sigma_1 - X$ map $\bar{f} : \eta \cap X \leftrightarrow \zeta_0 \cap X$. But η lies in the transitive part of X. So $\zeta_0 = \mathsf{rg}(f) = \mathsf{rg}(\bar{f}) \subseteq X$ and $\zeta_0 \subseteq \zeta$.

Claim 2: $\zeta < \eta^+$.

Now $|X|^{\beta,1} = \eta$, so there is a $\Sigma_1 - L_\delta$ map f from a subset of $\omega \times \eta$ onto L_δ. As $\gamma + 1 < \eta^+ < \beta^\star$ we must have $\delta = \mathcal{C}(\gamma + 1) < \beta$. Hence f is β-finite and a witness to $|\delta|^\beta \leq \eta$; indeed $|\delta|^\beta = \eta$. So $\zeta < \delta < \eta^+$.

It follows that $X = SH_1(\zeta, p; L_\beta)$. Since $\zeta \notin X$ this shows that ζ is β-pseudo stable. Hence $\mathcal{P}_\beta \cap [\eta, \eta^+[$ is unbounded in η^+. To see that this set is also closed let γ, $\eta < \gamma < \eta^+$, be the limit of lesser β-pseudo stable ordinals. Clearly $SH_1(\gamma, p; L_\beta) = \bigcup \{ SH_1(\zeta, p; L_\beta) \mid \zeta \in \mathcal{P}_\beta \cap \gamma \}$ so it suffices to show that for all $\zeta \in \mathcal{P}_\beta \cap \gamma : \gamma \notin SH_1(\zeta, p; L_\beta)$. Assume otherwise for the sake of a contradiction, say $\gamma \in SH_1(\zeta, p; L_\beta) := X$. As in claim 1 one can show that actually $\gamma \subseteq X \cap \eta^+$, whence $\zeta \in \gamma \subseteq X$ contradicting the β-pseudo stability of ζ.

In the case where $\beta^\star = \eta^+$ is a successor β-cardinal the argument is entirely similar.

- It is obvious from the definitions that $\mathcal{P}_\beta$ is $\Pi_1 - L_\beta$. As one might expect, this set fails to be $\Sigma_1 - L_\beta$. For assume $\Phi(x; a, p)$ is a Σ_1 formula with only parameters a $< \beta^\star$ and $p = p(\beta)$ that defines $\mathcal{P}_\beta$ over L_β. Pick an ordinal γ and ζ such that $a < \gamma < \zeta < \beta^\star$ where γ fails to be β-pseudo stable and ζ is the least β-pseudo stable ordinal larger than γ. Set $X := SH_1(\zeta, p; L_\beta)$ and consider the β-cardinal $\rho := \min\big(\xi \in X \mid \zeta < \xi \big)$.

  We have just seen that $\mathcal{P}_\beta$ is unbounded in ρ, whence $L_\beta \models \exists x (\gamma < x < \rho \wedge \Phi(x; a, p))$. Since all parameters lie in $X \prec_1 L_\beta$ we have $X \models \exists x (\gamma < x < \xi \wedge \Phi(x; a, p))$. Therefore there is a β-pseudo stable ordinal ζ' between γ and ρ that lies in X. However, $X \cap [\zeta, \rho[= \emptyset$, so actually $\gamma < \zeta' < \zeta$. But there are no β-pseudo stable ordinals in the interval $[\gamma, \zeta[$ and we have the desired contradiction.

Lemma 2.4 motivates the following definition. For $\gamma < \beta^\star$ let

$$\begin{aligned} &\mathcal{C}' : \beta^\star \to \beta^\star + 1, \\ &\mathcal{C}'(\gamma) := \max\big(\xi \leq \beta^\star \mid L_\xi \models (\gamma \text{ is a cardinal}) \big) \end{aligned}$$

It follows from the lemma that $\mathcal{C}(\gamma) \leq \mathcal{C}'(\gamma)$ for all β-pseudo stable γ. Indeed, in most cases we have $\mathcal{C}(\gamma) = \mathcal{C}'(\gamma)$. Equality holds in particular for all successor β-pseudo stable ordinals γ.

Lemma 2.5 *Let $\xi < \beta^\star$ be β-pseudo stable and $\delta = \mathcal{C}(\xi)$. Then ξ is successor β-pseudo stable iff $\sigma 1 p^\delta(\xi) < \xi$.*

Proof. First assume ξ is successor β-pseudo stable. Then there is a $\gamma < \xi$ such that $SH_1(\gamma, p; L_\beta) = SH_1(\xi, p; L_\beta) =: X$. As usual let $\pi : X \leftrightarrow L_\delta$ be the Mostowski collapse. Set $f := \pi[h_p \restriction \omega \times L_\gamma]$ where h_p is the standard Skolem function with additional parameter $p = p(\beta)$. f shows that $\sigma 1 p^\delta(\xi) \leq \gamma < \xi$.

For the opposite direction it suffices to show that for every limit β-pseudo stable ξ we have $\sigma 1 p^\delta(\xi) = \xi$. Assume for the sake of a contradiction that there is a partial surjective δ-recursive function f with $\mathsf{dom}(f) \subseteq \xi_0$ and $\mathsf{rg}(f) = \xi$ where $\xi_0 < \xi$. Let X and π as above and set $\bar{f} := \pi^{-1}[f]$; so $\bar{f}$ is $\Sigma_1 - X$. Now pick $\bar{\xi}$ β-pseudo stable such that $\xi_0 < \bar{\xi} < \xi$ and $\bar{f}$ is $\Sigma_1 - SH_1(\bar{\xi}, p; L_\beta)$. But then $\mathsf{rg}(\bar{f}) \subseteq \bar{\xi}$ and therefore $\mathsf{rg}(f) \subseteq \pi(\bar{\xi}) = \bar{\xi} < \xi$, contradiction to f surjective. □

We are now ready to introduce a canonical normal form for Σ_1 definitions.

Definition 2.1 *Let $\xi < \beta^\star$ and $\xi \geq \xi_1 > \xi_2 \ldots > \xi_n$. Then $\xi_1, \ldots, \xi_n$ is a trace of ξ iff for some $i < \omega$ we have $h_p(i, \xi_1, \ldots, \xi_n) \simeq \xi$ and ξ_i is β-pseudo stable for all $i = 1, \ldots, n$.*

It is easy to show by induction over ξ that every $\xi < \beta^\star$ has a trace. For if ξ is β-pseudo stable it is its own trace. Otherwise there are $\xi_1, \ldots, \xi_k < \xi$ such that for some suitable Gödel number i we have $h_p(i, (\xi_1, \ldots, \xi_k)) \simeq \xi$. By our induction hypothesis each ξ_i has a trace and it is easy to piece these traces together to produce a trace of ξ. Traces are not uniquely determined. However, there is a natural way to select minimal trace for each ordinal $\xi < \beta^\star$. Think of all the set of all traces of ξ as an ordered tree, where the successors of a node are sorted from left to right in increasing order. All branches in the tree are traces and we may select the leftmost one as the *minimal trace* of ξ. To be more precise, define by induction on $n \geq 1$ a sequence of β-pseudo stable ordinals as follows:

$$\begin{aligned} \xi_n \quad := \quad & \min\big(\zeta \;\big|\; \exists m, \zeta_{n+1}, \ldots, \zeta_m \, (\xi_1, \ldots, \xi_{n-1}, \zeta, \zeta_{n+1}, \ldots, \zeta_m \\ & \qquad \text{is a trace of } \xi \text{ and } \xi_{n-1} > \zeta > \zeta_j \text{ for } n < j \leq m \big). \end{aligned}$$

Since the sequence of ordinals so defined is strictly descending we obtain a uniquely determined trace $\xi_1, \ldots, \xi_k$ of ξ. A Σ_1 substructure contains ξ if and only if it contains the minimal trace of ξ:

Lemma 2.6 *Let $\xi < \beta^\star$ and $\boldsymbol{\xi}$ its minimal trace. Let X be a Σ_1 substructure of L_β that contains $p(\beta)$. Then $\xi \in X$ iff $\boldsymbol{\xi} \in X$.*

Proof. Clearly $\boldsymbol{\xi} \in X$ implies $\xi \in X$. So assume $\xi \in X$; we have to show that the minimal trace $\boldsymbol{\xi}$ also lies in X. To keep notation manageable we will only consider the case $\boldsymbol{\xi} = \xi_1, \xi_2$ where $\xi > \xi_1 > \xi_2$; the argument in the general situation is quite similar. Let $i < \omega$ such that $h_p(i, (\xi_1, \xi_2)) \simeq \xi$ and set $\xi' := \min\big(\zeta \in X \mid \exists w < \zeta (h_p(i, (\zeta, w)) \simeq \xi) \big)$. ξ' exists as $\xi, p \in X \prec_1 L_\beta$.

We claim that $\xi' = \xi$. For $\xi' < \xi_0$ would immediately contradict the definition of a minimal trace. So assume $\xi' > \xi_1$. Then $X \models \exists \zeta < \xi_1, w < \zeta (h_p(i, (\zeta, w)) \simeq \xi)$ since L_β is a model of this formula, $X \prec_1 L_\beta$ and all the parameters lie in X. But this contradicts the definition of ξ'. A similar argument shows that ξ_2 lies in X and we are done. □

3 Thin Sets are β-Recursive

There are numerous ways to generalize the notion of finiteness from classical recursion theory to β recursion theory. If we consider finite sets as nullsets in RE_ω, then the ℓ-finite sets are the appropriate generalization to RE_β. A different aspect a recursively enumerable set being small is captured by the following definition.

Definition 3.1 *Let $X \subseteq L_\beta$ be a β-r.e. set. X is* thin *iff $|X|^{\beta,1} < \beta^\star$.*

In classical recursion theory the thin sets are of course exactly the finite sets or, equivalently, the ℓ-finite sets. Similarly in α recursion theory the thin sets are exactly the ℓ-finite sets. The crucial step in showing that this characterization carries over to arbitrary limit ordinals is our main lemma:

Lemma 3.1 (Main Lemma)
Every thin β-recursively enumerable set is β-recursive.

The proof is somewhat lengthy and takes up most of this section.

3.1 Characterization of ℓ-finite sets

Before turning to the proof of the main lemma we establish some basic properties of ℓ-finite and thin sets. To this end let $\mathsf{RE}_\beta(X) := \{\, Y \cap X \mid Y \in \mathsf{RE}_\beta \,\}$ where $X \subseteq L_\beta$ be the principal ideal generated by X. $\mathsf{RE}_\beta(X)$ is a sublattice of RE_β iff X is β-recursively enumerable. Observe that for any β-r.e. set X the lattices $\mathsf{RE}_\beta(X)$ and $\mathsf{RE}_\beta(|X|^{\beta,1})$ are isomorphic.

Lemma 3.2 *Let $X \subseteq L_\beta$ be β-recursively enumerable. The following are equivalent:*

1. *X is ℓ-finite,*
2. *X is β-recursive and $\mathsf{RE}_\beta(X)$ is a boolean algebra,*
3. *X is β-recursive and thin,*
4. *for some β-recursive permutation $P : L_\beta \leftrightarrow L_\beta$ and some $\delta < \beta^\star$: $X = P[\delta]$.*

Proof. $(1 \Rightarrow 2)$: This follows immediately from the fact that for X β-recursive and $Y \in \mathsf{RE}_\beta(X)$: Y has a complement in $\mathsf{RE}_\beta(X)$ iff Y has a complement in RE_β iff Y is β-recursive.

$(2 \Rightarrow 3)$: Assume for the sake of a contradiction that $\delta := |X|^{\beta,1} \geq \beta^\star$. Choose an enumeration $f : \delta \leftrightarrow X$ and let $C \subseteq \beta^\star$ be an arbitrary β-r.e. set that fails to be β-recursive. Set $X_0 := f[C] \subseteq X$. Since $\mathsf{RE}_\beta(X)$ is a boolean algebra, X_0 has a complement in $\mathsf{RE}_\beta(X)$. By the last remark X_0 is β-recursive. But then C is also β-recursive, contradiction.

$(3 \Rightarrow 4)$: Let $\delta := |X|^{\beta,1} < \beta^\star$ and fix an enumeration $f : \delta \leftrightarrow X$ of X.

Claim: $|L_\beta - \delta|^{\beta,1} = |L_\beta - X|^{\beta,1} = \widehat{\beta}$.

To see this let $A, B \subseteq L_\beta$ be two β-r.e. sets. Then $||A \cup B|^{\beta,1} \leq |A|^{\beta,1} + |B|^{\beta,1}$ whence $\widehat{\beta} = |L_\beta|^{\beta,1} \geq |L_\beta - \delta|^{\beta,1} + |\delta|^{\beta,1}$ and so indeed $\widehat{\beta} = |L_\beta - \delta|^{\beta,1}$ as $|X|^{\beta,1} = \delta < \beta^\star \leq \widehat{\beta}$. Analogously one shows $\widehat{\beta} = |L_\beta - X|^{\beta,1}$ and we are done. Hence there is a β-recursive permutation $P_0 : L_\beta - \delta \leftrightarrow L_\beta - X$ and $P := P_0 \cup f$ is a β-recursive permutation of L_β.

$(4 \Rightarrow 1)$: Let $Y \in \mathsf{RE}_\beta(X)$ and set $d := P^{-1}[Y]$. Then $d \subseteq \delta < \beta^\star$ is β-r.e. and therefore β-finite by Σ_1 separation. So $L_\beta - Y = P[L_\beta - d]$ is β-r.e. and Y is β-recursive. $\square$

It follows that every $\beta^\star$-finite set is both ℓ-finite (by Σ separation) and thin. The converse holds only for weakly admissible β as demonstrated in the next lemma.

Lemma 3.3 *For any limit ordinal β the following are equivalent:*

1. *β is weakly admissible,*
2. *the ℓ-finite sets in RE_β are exactly the $\beta^\star$-finite sets,*
3. *the thin sets in RE_β are exactly the $\beta^\star$-finite sets.*

Proof. First suppose β is weakly admissible. We must show that ℓ-finite as well as thin sets are $\beta^\star$-finite. By the last lemma ℓ-finite sets are in particular thin. Thus suppose X is thin. Pick an enumeration $f : \delta \leftrightarrow X$ where $\delta := |X|^{\beta,1} < \beta^\star$. But $\beta^\star \leq \kappa$, so one may use Σ collection to show that f and therefore X is $\beta^\star$ -finite. Thus $(1 \Rightarrow 2)$ and $(1 \Rightarrow 3)$. Assume on the other hand that β is strongly inadmissible. Consider the standard Σ_1 cofinality function $q : \kappa \to \beta$ as described in the last section. Set $A := \mathsf{rg}(q) \subseteq \beta$. A is unbounded in β and so clearly not β-finite. However, A is thin as $\kappa < \beta^\star$. Thus $(3 \Rightarrow 1)$. By lemma it suffices to show that A is also β-recursive, for then A is also ℓ-finite and not β-finite as required. Now q is continuous and monotonic, so for $\xi < \beta^\star$ we have $\xi \notin A$ iff $\xi < q(0) \vee \exists \sigma < \kappa(q(\sigma) < \xi < q(\sigma + 1))$. Thus $L_\beta - A$ is β-r.e. , A is ℓ-finite and direction $(2 \Rightarrow 1)$ follows. $\square$

An example like $\beta := \aleph_1 + \omega$ shows that the set A defined in the last proof can actually be $\Delta_0 - L_\beta$, so it is not a particularly good example for a β-infinite set. As a matter of fact, for $\beta = \aleph_1 + \omega$ we have $\beta^\star = \aleph_1$ and $\mathsf{RE}_\beta(\aleph_1)$ and RE_β are isomorphic. But the ℓ-finite subsets of $\aleph_1$ are all countable by lemma 3.2 and thus β-finite. However, for ordinals like $\beta := \aleph_\omega + \omega$ there is no reason why all ℓ-finite subsets of $\beta^\star = \aleph_\omega$ should be β-finite. This is made explicit in the next lemma.

Lemma 3.4 *Let β be strongly inadmissible. Then $\beta^\star$ is Σ_1 regular iff every ℓ-finite subset of $\beta^\star$ is β-finite.*

Proof. First assume $\beta^\star$ is Σ_1 regular and let $X \subseteq \beta^\star$ ℓ-finite. By lemma 3.4 X is thin, so we have an enumeration $f : \delta \leftrightarrow X$ for $\delta := |X|^{\beta,1} < \beta^\star = \sigma 1 \mathsf{cof}(\beta^\star)$. Hence $X = \mathsf{rg}(f)$ must be bounded in $\beta^\star$ and therefore is β-finite by Σ_1 separation. It follows from reflection for L_β that X is actually $\beta^\star$-finite.

For the opposite direction let $\beta^\star$ be Σ_1 irregular, say $\lambda := \sigma 1 \mathsf{cof}^\beta(\beta^\star) < \beta^\star$, $\bar{\lambda} := \mathsf{cof}^\beta(\beta^\star)$ and let $g : \lambda \to \beta^\star$ and $\bar{g} : \bar{\lambda} \to \beta^\star$ be corresponding cofinality functions. So both $\mathsf{rg}(g)$ and $\mathsf{rg}(\bar{g})$ are cofinal in $\beta^\star$, g is β-recursive and $\bar{g}$ is β-finite.

Case 1: $\lambda = \bar{\lambda}$.

Let $\Phi(u,v)$ be a universal Σ_1 formula, $p := p(\beta)$ and $K : \beta^\star \leftrightarrow L_{\beta *}$ a β-finite listing of $L_{\beta *}$. Set $K_\delta := K(\delta)$ and define $C^\sigma := \{\, \langle e, x\rangle \mid e, x < \beta^\star \wedge L_{q(\sigma)} \models \Phi(\langle e, x\rangle, p) \,\}$ and $D := \{\, \langle \delta, \nu, \sigma\rangle \mid \delta < \beta^\star, \nu < \lambda, \sigma < \kappa \wedge K_\delta = C^\sigma \cap g(\nu) \,\}$. Note that D is β-recursive and thin as $|D|^{\beta,1} \leq |\lambda \times \kappa|^\beta < \beta^\star$. So D is ℓ-finite and we only have to show that D is not β-finite. To this end let $C := \bigcup\{C^\sigma \;\big|\; \sigma < \kappa\}$. C is a complete Σ_1 set and so certainly not β-finite. But $C = \bigcup\{K_\delta \;\big|\; \exists \nu < \lambda, \sigma < \kappa(\langle \delta, \nu, \sigma\rangle \in D)\}$ and we are done.

Case 2: $\lambda < \bar{\lambda}$.

Define a map $h : \lambda \to \bar{\lambda}$ by $h(i) := \min(j < \lambda \;\big|\; \bar{g}(j) > g(i))$. h is β-recursive and has range unbounded in $\bar{\lambda}$. Thus h cannot be β-finite and it follows from Σ_1 separation below $\beta^\star$ that $\bar{\lambda} + \beta^\star$. Now define an approximation g^σ to g as follows: $g^\sigma := \{\, (i, x) \mid L_{q(\sigma)} \models g(i) \simeq x \,\} \subseteq \lambda \times \beta^\star$ where $\sigma < \kappa$. $\mathsf{rg}(g^\sigma) \subseteq \beta^\star$ must be bounded as $\bar{\lambda} = \beta^\star$, so we may define $G : \kappa \to \beta^\star$ by $G(\sigma) := \sup(\mathsf{rg}(g^\sigma))$. Since $g = \bigcup g^{\sigma < \kappa} g^\sigma$ the map G is a Σ_1 cofinality function, hence $\lambda \leq \kappa$. But $\lambda < \kappa$ would imply that g is β-finite contradicting $\lambda < \bar{\lambda}$. So really $\lambda = \kappa$. Now define C and C^σ as above and let $D := \{\, \langle \delta, \sigma\rangle \mid \delta < \beta^\star, \sigma < \kappa \wedge K_\delta = C^\sigma \cap g(\sigma) \,\}$. Again D is β-recursive and thin as $|D|^{\beta,1} = \kappa < \beta^\star$. As in case 1 it follows that D is ℓ-finite but fails to be β-finite. □

It follows from the reflection principle that the ℓ-finite subsets of $\beta^\star$ are actually $\beta^\star$-finite whenever $\beta^\star$ is Σ_1^β regular.

Note that as a corollary to the proof of the last lemma we have for strongly inadmissible β such that $\sigma 1 \mathsf{cof}(\beta^\star) < \mathsf{cof}(\beta^\star)$: $\kappa = \sigma 1 \mathsf{cof}(\beta^\star)$ and $\beta^\star = \mathsf{cof}(\beta^\star)$. This fact will be used in the next section.

The set D defined in the proof is a Σ_1 master code in case 1 and a Δ_1 master code in case 2. Thus, from the point of view of fine structure theory, ℓ-finite sets can be rather complicated.

Taking the main lemma 3.1 for granted for the moment we can eliminate the condition of X being β-recursive from lemma 3.2:

Corollary 3.1 *For any β-r.e. set $X \subseteq L_\beta$ the following are equivalent:*

1. *X is ℓ-finite,*
2. *$\mathsf{RE}_\beta(X)$ is a boolean algebra,*
3. *X is thin.*

Proof. As an immediate consequence of lemmata 3.2, 3.4 and the main lemma we have the implications $(1 \Rightarrow 2)$, $(1 \Rightarrow 3)$ and $(3 \Rightarrow 1)$. Hence we only have to show $(2 \Rightarrow 3)$. Note that both $\mathsf{RE}_\beta(\widehat{\beta})$ and $\mathsf{RE}_\beta(\beta^\star)$ fail to be boolean algebras. Thus it suffices to show that, for every non-thin β-r.e. set X, the lattice $\mathsf{RE}_\beta(X)$ is isomorphic to either $\mathsf{RE}_\beta(\widehat{\beta})$ or to $\mathsf{RE}_\beta(\beta^\star)$. To this end let $\delta := |X|^{\beta,1}$, so $\beta^\star \leq \delta \leq \widehat{\beta}$ and $\mathsf{RE}_\beta(X)$ is isomorphic to $\mathsf{RE}_\beta(\delta)$. Suppose $\beta^\star \leq \delta < \widehat{\beta}$. Then β is weakly admissible, i.e., $\widehat{\beta} = \kappa$. Let $f : L_\beta \to \beta^\star$ be a Σ_1 projection. By Σ collection $f \restriction \delta$ is β-finite, which shows that $|\delta|^\beta = \beta^\star$ and we are done. □

Yet another characterization of ℓ-finite sets in terms of their $<_\beta$ order types is given in [10].

Theorem 3.1 *A β-r.e. set $X \subseteq L_\beta$ is ℓ-finite iff X has ordertype less than $\beta^\star$.*

3.2 Proof of the main lemma

We now turn to the proof of the main lemma. For the sake of the argument let us introduce the following terminology.

Definition 3.2 *A non-decreasing sequence $I = \left(I_\nu \;\middle|\; \nu < \lambda\right)$ of simultaneously β-r.e. sets is an* ℓ-cover *for β iff the following holds:*

1. *$I_\nu \subseteq L_\beta$ is ℓ-finite for all $\nu < \lambda$,*
2. *For every thin β-r.e. set X there exists a $\nu < \lambda$ such that $X \subseteq I_\nu$.*

Here λ is a limit ordinal, the length *of the ℓ-cover I.*

To prove the main lemma it clearly suffices to exhibit an ℓ-cover for every limit ordinal β. For then any thin β-r.e. set is a subset of an ℓ-finite set and thus β-recursive by the definition of ℓ-finiteness. This is quite easy as long as β is weakly admissible: let $I = (\, I_\nu \;\middle|\; \nu < \beta^\star \,)$ be a standard enumeration of the $\beta^\star$-finite sets. By corollary 3.1 the $\beta^\star$-finite sets, the ℓ-finite sets and the thin sets all coincide. Thus every thin set actually occurs in the sequence I.

Let us assume from now on that β is strongly inadmissible. Then $\widehat{\beta} = \beta^\star$ and there is a β-recursive bijection $P : L_\beta \leftrightarrow \beta^\star$ which induces an isomorphism between $\mathsf{RE}_\beta(\beta^\star)$ and RE_β that preserves thin sets.

First suppose that $\beta^\star$ is Σ_1 regular. Set $I_\nu := P^{-1}(K_\nu)$ where $(K_\nu \;|\; \nu < \beta^\star)$ is a β-finite listing of the $\beta^\star$-finite subsets of $\beta^\star$. Now let $X \subseteq L_\beta$ be a thin set. Then $P(X) \subseteq \beta^\star$ is thin, thus there is an enumeration $f : \delta \leftrightarrow P(X)$, $\delta := |X|^{\beta,1} < \beta^\star$. As $\beta^\star$ is Σ_1 regular f must have range bounded in $\beta^\star$, say, $P(X) \subseteq \gamma < \beta^\star$. But then $P(X)$ is β-finite by Σ_1 separation and $\beta^\star$ -finite be reflection. Thus for some $\nu < \beta^\star$ we have $P(X) = K_\nu$ and $X = I_\nu$. Hence $I := \left(I_\nu \;\middle|\; \nu < \beta^\star\right)$ is an ℓ-cover and we are through.

Lastly consider the case where $\beta^\star$ is Σ_1 irregular. Friedman showed in [3] that in this situation $\beta^\star$ must be a limit β-cardinal. Let $\left(\alpha_\nu \;\middle|\; \nu < \lambda\right)$ be a β-finite, strictly ascending sequence of β-cardinals cofinal in $\beta^\star$ where $\lambda := \mathsf{cof}^\beta(\beta^\star)$. Note that it is possible that $\beta^\star$ is a regular limit β-cardinal but Σ_1 irregular; see the reference for an example. In this situation we have $\kappa = \sigma 1 \mathsf{cof}(\beta^\star) < \lambda = \beta^\star$ by the remark following lemma 3.4. For the sake of simplicity let us assume that this pathology does not occur. It is easy to see that the arguments given in the sequel can be modified to deal with this case. To summarize, we will assume from now on that:

- β is strongly inadmissible,
- $\sigma 1 \mathsf{cof}(\beta^\star) = \mathsf{cof}^\beta(\beta^\star) = \lambda < \beta^\star$,
- $(\alpha_\nu \;|\; \nu < \lambda)$ is a β-finite strictly ascending sequence of β-cardinals cofinal in $\beta^\star$.

Let $p := p(\beta)$ be the standard parameter and define for all $\nu < \lambda$:

$$H_\nu := SH_1(\alpha_\nu, p; L_\beta) \subseteq L_\beta.$$

We claim that $(H_\nu \mid \nu < \lambda)$ is an ℓ-cover. To see this first note that the sets H_ν are thin: $H_\nu = h_p[D]$ where h_p is the standard Skolem function and $D := \{ (i,x) \in \omega \times \alpha_\nu \mid \exists z(h_p(i,x) \simeq z \wedge z < \alpha_\nu) \}$. D is β-finite by Σ_1 separation and $|D|^\beta \leq \alpha_\nu < \beta^\star$, hence H_ν is also thin. Next suppose $X \subseteq L_\beta$ is thin. Fix an enumeration $f : \delta \leftrightarrow X$ where $\delta := |X|^{\beta,1} < \beta^\star$. Pick $\nu < \lambda$ such that $\delta < \alpha_\nu$ and for some parameter $a < \alpha_\nu$ we have f is Σ_1 definable over L_β with parameters p and a only. Then f is $\Sigma_1{-}H_\nu$ as H_ν is a Σ_1 substructure of L_β and a, $p \in H_\nu$. Now $\mathsf{dom}(f) = \delta \subseteq H_\nu$, so $X = \mathsf{rg}(f) \subseteq H_\nu$.

Hence it remains to show that all the sets H_ν are β-recursive. Note, however, that these sets cannot be β-recursive uniformly in ν. For assume that some $\Sigma_1{-}\emptyset$ formula ϕ defines $L_\beta - H_\nu$ with parameters $a < \beta^\star$ and p such that for all $\nu < \lambda$: $x \notin H_\nu$ iff $L_\beta \models \phi(x, \nu, a, p)$. Pick $\nu < \lambda$ such that a, $\nu < \alpha_\nu$. Then $L_\beta \models \exists x \phi(x, \nu, a, p)$. Since H_ν is a Σ_1 substructure of L_β this implies $H_\nu \models \exists x \phi(x, \nu, a, p)$. Now pick a witness $\bar{x} \in H_\nu$ such that $H_\nu \models \phi(\bar{x}, \nu, a, p)$. But then $L_\beta \models \phi(\bar{x}, \nu, a, p)$, whence $\bar{x} \notin H_\nu$, and we have the desired contradiction.

Since $H_\nu \subseteq H_{\nu'}$ for $\nu \leq \nu' < \lambda$ it suffices to show that H_μ is β-recursive for arbitrarily large $\mu < \lambda$. To this end pick $\bar{\mu} < \lambda$ large enough so as to guarantee that

- κ, $\lambda < \alpha_{\bar{\mu}}$,
- $\beta^\star$, $(K : \beta^\star \leftrightarrow L_\beta *)$ and $(q : \kappa \to \beta)$ are all in $H_{\bar{\mu}}$.

Clearly all μ larger than $\bar{\mu}$ inherit these properties. For the remainder of the argument let μ be arbitrarily large but fixed, $\bar{\mu} \leq \mu < \lambda$. In order to show that H_μ is β-recursive we will use a kind of 1-type restricted to Σ_1 formulae to pin down membership is H_μ. First, the definitions.

Definition 3.3 *Let v be a special variable that we will keep fixed from now on. For $x \in L_\beta$ and $\sigma < \kappa$ define the Σ_1-one-type of x at stage σ (in symbols $T^\sigma(x)$) by*

$$T^\sigma(x) := \{ \Psi(v) \mid \Psi \text{ is a } \Sigma_1{-}(\alpha_\mu \cup \{p\}) \text{ formula and } L_{q(\sigma)} \models \Psi(x) \}.$$

The Σ_1-one-type of x, in symbols $T(x)$, is defined as $T(x) := \bigcup_{\sigma<\kappa} T^\sigma(x)$.

Let $x \notin H_\mu$ and $\sigma < \kappa$. x is discernible at stage σ iff for all $x' \in L_\beta$: $T^\sigma(x) = T^\sigma(x')$ implies $x' \notin H_\mu$. x is discernible iff x is discernible at some stage $\sigma < \kappa$.

The Σ_1 substructure generated by $x \in L_\beta$ at stage $\sigma < \kappa$ (in symbols $X^\sigma(x)$) is defined by

$$X^\sigma(x) := SH_1(\alpha_\mu, (x,p)_s; L_{q(\sigma)}).$$

$X^\sigma(x)$ and $X^\sigma(\bar{x})$ are called strongly isomorphic *(in symbols $X^\sigma(x) \equiv X^\sigma(\bar{x})$) iff there is an epsilon-isomorphism $f : X^\sigma(x) \leftrightarrow X^\sigma(\bar{x})$ s.t. $f \restriction \alpha_\mu = id$, $f(p) = p$ and $f(x) = \bar{x}$.*

The ordinal μ is a hidden parameter in all these definitions. We assume that Σ_1-one-types are coded in some standard fashion; so both $T(x)$ and $T^\sigma(x)$ may be construed as subsets of $\omega \times \alpha_\mu$.

Note that the Σ_1-one-types are β-finite: the satisfaction relation for Σ_1 formulae is Σ_1-L_β, so $T(x) \subseteq \omega \times \alpha_\mu$ is β-finite by Σ_1 separation. The property of Σ_1-one-types crucial for our purposes is that $T(x)$ determines membership in H_μ:

$$x \in H_\mu \text{ iff there exists } i < \omega, e < \alpha_\mu \text{ such that } \big(h_p(i,e) \simeq v\big) \in T(x).$$

In fact any element x in H_μ is uniquely determined by its Σ_1-one-type, i.e., $x \in H_\mu$ and $T(x) = T(x')$ implies $x = x'$. Similarly one can show that for all $\nu < \lambda$ the first β-pseudo stable larger than α_ν is also uniquely determined by its Σ_1-one-type. However, T certainly is far from being injective for simple cardinality reasons. E.g., let $\beta := \aleph_\omega + \omega$ and, say, $\mu = 5$. Then there is a stationary set $E \subseteq \aleph_7$ such that for all $x, x' \in E$: $T(x) = T(x')$. If one interprets stationary as stationary in the sense of L_β this is actually true for all β under consideration here.

The following lemma describes the relationship between the types $T^\sigma(x)$ and the structures $X^\sigma(x)$.

Lemma 3.5 *Let $\sigma < \kappa$, x, $\bar{x} < \beta^\star$. Then $X^\sigma(x)$ and $X^\sigma(\bar{x})$ are strictly isomorphic iff $T^\sigma(x) = T^\sigma(\bar{x})$.*

Proof. If: By the definition of $X^\sigma(x)$ for every $a \in X^\sigma(x)$ there exist $e_a < \alpha_\mu$, $i < \omega$ such that $L_{q(\sigma)} \models (h_{(x,p)}(i, e_a) \simeq a)$. Pick one such e_a and i and set $f(a) := h_{(x,p)}(i, e_a)$. The Skolem function $h_{(x,p)}$ converges on (i, e_a) since $\exists z\big(h_{(v,p)}(i, e_a) \simeq z\big) \in T^\sigma(x) = T^\sigma(\bar{x})$. A similar argument shows that f is a bijection, $f : X^\sigma(x) \leftrightarrow X^\sigma(\bar{x})$. It remains to show that f respects $\in$. So let a, $b \in X^\sigma(x)$, say $a \in b$. Then $\exists u, w\big(h_{(v,p)}(i, e_a) \simeq u \wedge h_{(v,p)}(j, e_b) \simeq w \wedge a \in b\big)$ is a formula in $T^\sigma(x)$ an thus in $T^\sigma(\bar{x})$. Hence $f(a) = h_{(x,p)}(i, e_a) \in h_{(x,p)}(j, e_b) = f(b)$.

Only if: Let $f : X^\sigma(x) \leftrightarrow X^\sigma(\bar{x})$ be the corresponding epsilon-isomorphism. Pick an arbitrary formula $\Psi(v)$ in $T^\sigma(x)$. Then $L_{q(\sigma)} \models \Psi(x)$ and it follows from $X^\sigma(x) \prec_1 L_{q(\sigma)}$ that $X^\sigma(x) \models \Psi(x)$. Applying f we get $X^\sigma(\bar{x}) \models \Psi(\bar{x})$. □

Returning to our argument, we would like to use the Σ_1-one-type $T(x)$ off x to determine membership in $L_\beta - H_\mu$. To this end we may define

$$\begin{aligned} G &:= \{\, t \in L_\beta * \mid t \subseteq \omega \times \alpha_\mu \wedge \forall z \in L_\beta (T(z) = t \Rightarrow z \notin H_\mu) \,\} \\ Z &:= \{\, z \in L_\beta \mid T(z) \in G \,\}. \end{aligned}$$

Then Z is certainly the complement of H_μ in L_β. However, T is not a β-recursive function and so Z need not be β-recursively enumerable. To overcome this difficulty we use the approximation $T^\sigma(x)$ instead of $T(x)$:

$$\begin{aligned} G^\# &:= \{\, t \in L_\beta * \mid t \subseteq \omega \times \alpha_\mu \wedge \forall z \in L_\beta, \sigma < \kappa (T^\sigma(z) = t \Rightarrow z \notin H_\mu) \,\} \\ Z^\# &:= \{\, z \in L_\beta \mid \exists \sigma < \kappa (T^\sigma(z) \in G^\#) \,\}. \end{aligned}$$

We claim that $T^\sigma(x)$ is β-recursive. Note that there is a slight technical difficulty with this. As long as $q(\sigma)$ is a limit ordinal it is obvious that T^σ (and X^σ) is a β-recursive function uniformly in σ. In the case $\beta = \beta_0 + \omega$, however, similar problems occur as in the proof of the reflection principle. One can use the functions $H^{q(\sigma)}_{(x,p)}$ defined in section 2 to overcome these problems. But then $G^\#$ is Π_1-L_β and bounded in $L_{\beta^\star}$, hence β-finite by Σ_1 separation. Therefore $Z^\#$ is indeed

β-recursively enumerable. Also, $Z^{\#} \subseteq L_\beta - H_\mu$ by the definition of $G^{\#}$. However, the problem is now to show that $L_\beta - H_\mu \subseteq Z^{\#}$. For suppose $x \notin H_\mu$. It is conceivable that for all $\sigma < \kappa$ there exists some $\bar{x}$ in H_μ that, at stage σ, has the same Σ_1-one-type as x: $T^\sigma(x) = T^\sigma(\bar{x})$ (though of course $T(x) \neq T(\bar{x})$). Hence none of the types $T^\sigma(x)$ would appear in $G^{\#}$ and x would not be in $Z^{\#}$. This difficulty was addressed in the previous definition: for discernible x there exists a stage σ such that $T^\sigma(x) \in G^{\#}$. Hence every discernible x lies in $Z^{\#}$ as defined above. Thus we only have to verify the following claim:

Claim 1: Every $x \in L_\beta - H_\mu$ is discernible.

Using traces and lemma 2.6 we can immediately reduce claim 1 to showing that all β-pseudo stable ordinals $\xi < \beta^\star$ are discernible:

Claim 2: Let $x \in L_\beta - H_\mu$ with minimal trace $\xi_1, \ldots, \xi_n$. If at least one of the ordinals ξ_i is discernible then x is discernible as well.

Proof. Note that ξ_1 cannot be in H_μ. For otherwise we would have $\xi_1 < \alpha_\mu$ since ξ_1 is β-pseudo stable. But then for all $i \leq n$: $\xi_i \leq \xi_1 < \alpha_\mu \subseteq H_\mu$ and $x \in H_\mu$ by lemma 2.6. For the sake of simplicity let us assume that ξ_1 is discernible at some stage $\tau < \kappa$. By the definition of a minimal trace pick $\tau_1 < \kappa$ such that for some $j < \omega$: $L_{q(\tau)} \models \big(h_p(j, \xi_1, \ldots, \xi_n) \simeq x\big)$. Let σ be the maximum of τ and τ_1.

Subclaim: x is discernible at stage σ.

Assume for the sake of a contradiction that there is some $\bar{x} \in H_\mu$ such that $T^\sigma(x) = T^\sigma(\bar{x})$. By lemma 3.5 this implies that $X^\sigma(x)$ and $X^\sigma(\bar{x})$ are strictly isomorphic; let f be a corresponding isomorphism. Now $x \in X^\sigma(x) \prec_1 L_{q(\sigma)}$ and it follows from the proof of lemma 2.6 that the minimal trace of x lies in $X^\sigma(x)$, in particular $\xi_1 \in X^\sigma(x)$. Let $\bar{\xi}_1 := f(\xi_1)$. $f \restriction X^\sigma(\xi_1)$ shows that $X^\sigma(\xi_1) = X^\sigma(\bar{\xi}_1)$. As $\tau \leq \sigma$ we have $X^\tau(\xi_1) \equiv X^\tau(\bar{\xi}_1)$ and by lemma 3.5 $T^\tau(\xi_1) = T^\tau(\bar{\xi}_1)$. But $\bar{x}$ was in H, so $X^\sigma(\bar{x}) \subseteq H_\mu$ and ξ_1 must be in H_μ. But then ξ_1 is not discernible at stage τ, contradiction. $\square$

The next step is to show that all β-pseudo stable ordinals ξ in $\beta^\star - H_\mu$ are discernible. We will first deal with successor β-pseudo stable ordinals. Note that a successor β-pseudo stable ordinal ξ lies in the complement of H_μ iff $\xi > \alpha_\mu$.

Claim 3: Let $\xi > \alpha_\mu$ be successor β-pseudo stable. Then ξ is discernible at stage 0.

Proof. Let $\eta := |\xi|^\beta$ so that $\eta < \xi < \eta^+$. Set $\gamma := \mathcal{C}(\xi)$, $X := SH_1(\xi, p; L_\beta)$ and let $\pi : X \leftrightarrow L_\gamma$ be the Mostowski collapse. Further let $Y := X^0(\xi)$, so $p \in Y \prec_1 L_{q(0)}$. Then $\xi = \pi(\eta^+)$ and it follows from lemma 2.4 that $\gamma = \mathcal{C}(\xi) = \mathcal{C}'(\xi)$. We conclude that $\gamma \in Y$. Now set $\bar{p} := \pi(p)$ and let $p' := \min\big(z \mid SH_1(\alpha_\mu, z; L_\beta) = L_\gamma\big)$. π is compatible with $<_\beta$, so we have $p' \leq_\beta \bar{p}$. Assume for a contradiction that $p' <_\beta \bar{p}$. Then p has a Σ_1 definition over X and therefore over L_β involving only parameters less than ξ and $\pi^{-1}(p') <_\beta p$. This contradicts the definition of $p = p(\beta)$. Hence $\bar{p} = p'$.

Subclaim: $H_\mu \cap \eta^+ \subseteq Y \cap \xi$.

Proof. To see this let $a \in H_\mu \cap \eta^+$, so $h_p(i, \xi_a) \simeq a$ for some $i < \omega$, $\xi_a < \alpha_\mu$. X is a Σ_1 substructure of L_β, whence $X \models \exists x(h_p(i, \xi_a) \simeq x \wedge x < \eta^+)$. By lemma 2.4 $X \cap L_{\eta^+} = L_\xi$, so there must be

some $\bar{a} < \xi$ such that $X \models \exists x(h_p(i,\xi_a) \simeq \bar{a})$. As $X \prec_1 L_\beta$ this implies $L_\beta \models \exists x(h_p(i,\xi_a) \simeq \bar{a})$. But h_p is a function, so $a = \bar{a} < \xi$ and we can conclude that $H_\mu \cap \eta^+ \subseteq \xi$. It remains to show that $a \in Y$. Note that $\pi \restriction L_\xi = id$ as $X \cap L^+_\eta = L_\xi$ is the transitive part of X. $\pi : X \leftrightarrow L_\gamma$ is an isomorphism, so we have $L_\gamma \models (h_{\bar{p}}(i,\xi_a) \simeq a)$. Since ξ, ξ_a, $p \in Y \prec_1 L_{q(0)}$ we also have $Y \models \exists x(L_\gamma \models h_{\bar{p}}(i,\xi_a) \simeq a \wedge x < \xi)$. Choose a witness $a' \in Y \cap \xi$ such that $Y \models (L_\gamma \models h_{\bar{p}}(i,\xi_a) \simeq a')$. Then $L_\gamma \models (h_{\bar{p}}(i,\xi_a) \simeq a')$ and thus $X \models (h_p(i,\xi_a) \simeq a')$. But this means that $h_p(i,\xi_a) \simeq a'$, so $a = a' \in Y$. This finishes the proof of the subclaim.

Let us now assume for the sake of a contradiction that ξ is not discernible at stage 0. Then there is a $\bar{\xi} \in H_\mu$ such that $T^0(\xi) = T^0(\bar{\xi})$ and, according to lemma 3.5 $X^0(\xi) \equiv X^0(\bar{\xi})$. Set $\bar{Y} := X^0(\bar{\xi})$, so $Y \equiv \bar{Y}$. Let $(H^\sigma_\mu \mid \sigma < \kappa)$ be the enumeration of H_μ defined with respect to the cofinality function q and pick $\tau < \kappa$ such that $\xi \in H^\tau_\mu$. Let ξ' be the least $q(\tau)$-pseudo stable ordinal larger than ξ. An argument similar to the one used in the last claim will show that $H^\tau_\mu \cap \eta^+ \subseteq \xi'$. Therefore we must have $\bar{Y} \cap \bar{\xi} \subseteq H^\tau_\mu \cap \bar{\xi} \subseteq \xi'$. Now consider the two collapse functions $\pi : Y \leftrightarrow L_\delta$ and $\bar{\pi} : \bar{Y} \leftrightarrow L_{\bar{\delta}}$. $Y \equiv \bar{Y}$ implies that $\delta = \bar{\delta}$ and $\pi(\xi) = \bar{\pi}(\bar{\xi})$. But $\bar{\pi}(\bar{\xi}) = \mathsf{otp}(\bar{Y} \cap \bar{\xi}) \leq \mathsf{otp}(H^\tau_\mu \cap \xi')$ and $\pi(\xi) = \mathsf{otp}(Y \cap \xi) \geq \mathsf{otp}(H_\mu \cap \eta^+)$. We are heading for a contradiction, so it would be enough to show that $\xi' \in H$. For then clearly $\mathsf{otp}(H^\tau_\mu \cap \xi') < \mathsf{otp}(H_\mu \cap \eta^+)$ which cannot be the case according to the last inequalities. Recall that $x \in H_\mu$ and $\eta < x < \eta^+$ implies that η, η^+ are in H. Now consider the formula

$$\Phi(z) := \eta < z < \eta^+ \wedge L_{q(\tau)} \models \big(\, z \text{ pseudo stable } \wedge \forall u, \eta < u < z (u \text{ not pseudo stable })\big).$$

Φ can be written as a Σ_1 formula with parameters from $\alpha_\mu \cup \{p\}$: just replace all the objects in Φ by a Σ_1 definition with parameters from $\alpha_\mu \cup \{p\}$ and prefix appropriate existential quantifiers. We have $\Phi(\xi')$ and $\Phi(z)$ implies $z = \xi'$. By our assumption on q we get $SH_1(\alpha_\mu, p; L_{q(\tau+1)}) \models \exists z \Phi(z)$. It follows that ξ' is in $H^{\tau+1}_\mu \subseteq H_\mu$ and we are through. □

Note that the only place where we have used our assumption that ξ is successor β-pseudo stable is in the beginning of the argument to make sure that the collapse $\mathcal{C}(\xi)$ lies in $X^0(\xi)$. Thus the result also holds for all limit β-pseudo stable ξ such that $\mathcal{C}(\xi) \in X^0(\xi)$. However, we have been unable to show that this is true for all β-pseudo stable ξ. In any case, the following argument shows that every β-pseudo stable ordinal is discernible.

Claim 4: Let $\xi > \alpha_\mu$ be an arbitrary β-pseudo stable ordinal. Then ξ is discernible.

Proof. It is convenient to distinguish two cases depending on whether ξ is less than or larger than α^+_μ.

Case 1: $\alpha_\mu < \xi < \alpha^+_\mu$.

We will show that ξ is discernible at stage 0. Assume for the sake of a contradiction that for some $\bar{\xi} \in H_\mu$: $T^0(\xi) = T^0(\bar{\xi})$. $\bar{\xi}$ must be $q(0)$-pseudo stable, i.e., $\bar{\xi}$ is not in $SH_1(\bar{\xi}, z; L_{q(0)})$. For otherwise the Σ_1 formula $\exists z(h_p(i,z) \simeq v \wedge z \in L(v))$ would be in $T^0(\bar{\xi})$ but clearly cannot be in $T^0(\xi)$. Now set $\Theta := X^0(\xi) \cap \alpha^+_\mu$, $\bar{\Theta} := X^0(\bar{\xi}) \cap \alpha^+_\mu$ and let $\pi : X^0(\xi) \leftrightarrow L_\delta$, $\bar{\pi} : X^0(\bar{\xi}) \leftrightarrow L_{\bar{\delta}}$ be the respective collapse functions. By lemma 3.5 the structures $X^0(\xi)$ and $X^0(\bar{\xi})$ are strongly isomorphic, whence $\delta = \bar{\delta}$ and thus $\Theta = (\alpha^+_\mu)^{L_\delta} = (\alpha^+_\mu)^{L_{\bar{\delta}}} = \bar{\Theta}$. Define ξ_1 to be the least β-pseudo stable ordinal larger than α_μ. We have $\xi_1 \leq \xi < \Theta$: for ξ is not a β-cardinal, so there is a β-finite bijection $f : \alpha_\mu \leftrightarrow \xi$. The $<_\beta$-minimal such f must be in $X^0(\xi)$, so $\xi \subseteq X^0(\xi)$ as $\mathsf{dom}(f) = \alpha_\mu \subseteq X^0(\xi)$. Proceeding as

in claim 3 we can show $\bar{\Theta} \leq \xi_1$ and we have the desired contradiction.

Case 2: $\alpha_\mu < \xi$.

Let

$$\Theta := \bigcap \{X(\eta) \mid \eta \text{ } \beta\text{-pseudo stable } \wedge \alpha_\mu^+ < \eta\} \cap \alpha_\mu^+.$$

Let η be the smallest β-pseudo stable larger than Θ. Then $\alpha_\mu < \eta < \alpha_\mu^+$ and indeed $\Theta = X(\eta) \cap \alpha_\mu^+$. Now set $X := SH_1(\alpha_\mu, (p, \eta); L_\beta) = X(\eta) \prec_1 L_\beta$ and $Y := SH_1(\Theta, p; L_\beta) \prec_1 L_\beta$. As before let $\pi : X \leftrightarrow L_\delta$ and $\bar{\pi} : Y \leftrightarrow L_{\bar{\delta}}$ be the corresponding Mostowski collapses. We begin by deriving a series of subclaims.

Subclaim 1: Θ is β-pseudo stable and $\Theta < \alpha_\mu^+$.

It was shown by Friedman that α_μ^+ is Σ_1 regular. But, for any $e \in [\alpha_\mu, \alpha_\mu^+[$, $X(e) \cap \alpha_\mu^+ = \sup(X^\sigma(e) \cap \alpha_\mu^+ \mid \sigma < \kappa)$. So $\Theta < \alpha_\mu^+$. But $[\Theta, \alpha_\mu^+[\cap X = \emptyset$ by the definition of Θ. So $\Theta = \pi(\alpha_\mu^+)$, i.e., $L_\delta \models (\Theta = \alpha_\mu^+)$ and we are done by lemma 2.10.

Subclaim 2: $\sigma 1 \mathsf{cof}^\delta(\eta) = \kappa$.

$X(\eta) = \bigcup \{X^\sigma(\eta) \mid \sigma < \kappa\}$, so $\bar{\kappa} := \sigma 1 \mathsf{cof}^\delta(\eta) \leq \kappa$. Let $g : \bar{\kappa} \to \eta$ be a corresponding $\Sigma_1 - L_\delta$ cofinality function. Now consider the function $f : \kappa \to \Theta$, $f(\sigma) := X^\sigma(\eta) \cap \alpha_\mu^+$. $f(\sigma)$ is an ordinal and as in claim 1 $f(\sigma) < \alpha_\mu^+$. f is also β-recursive; to be more precise, f has a Σ_1 definition over L_β with parameters from $\alpha_\mu \cup \{\eta, p\}$. Also $f \subseteq \kappa \times \Theta$, so f is $\Sigma_1 - X$. L_Θ is a transitive subset of X, hence $f = \pi[f]$ and f is also $\Sigma_1 - L_\delta$. But then f is actually β-finite. Set $f_0 : \bar{\kappa} \to \kappa$, $f_0(i) := \min(\sigma < \kappa \mid g(i) < f(\sigma))$. f_0 is β-finite and $\mathsf{rg}(f_0) \subseteq \kappa$ is unbounded, so $\bar{\kappa} \geq \kappa$.

Subclaim 3: $\delta = \mathcal{C}'(\Theta)$.

Θ is a β-cardinal and therefore $\mathcal{C}'(\Theta) \geq \delta$. It suffices to show that $\sigma 1 p^\delta(\Theta) = \alpha_\mu$. Let f be as in claim 2. For $\nu \in [\alpha_\mu, \Theta[$ choose a δ-finite bijection $c_\nu : \alpha_\mu \leftrightarrow \nu$, say the $<_\beta$-minimal one. Set $g : \kappa \times \alpha_\mu \to \Theta$, $g(\sigma, i) := c_{f(\sigma)}(i)$. The map g is δ-recursive and obviously surjective. Our claim follows from $|\kappa \times \alpha_\mu|^\delta = \alpha_\mu$.

Subclaim 4: $\delta > \gamma$.

We have $\Theta \subseteq X$ and $p \in X$, so $Y \subseteq X$ and $\delta \geq \gamma$. Assume for a contradiction that $\delta = \gamma$. Define $f : X \leftrightarrow Y$ by $f := \bar{\pi}^{-1} \circ \pi$. f is an epsilon-isomorphism and $f \restriction \Theta = id$. We claim that $f(p) = p$. For the sake of simplicity let us only show that $p <_\beta p' := f(p)$ cannot occur; the opposite direction is entirely similar. Set $p'' := f^{-1}(p')$ and assume $p'' \neq p$. By the definition of Y, p' has a Σ_1 definition over Y involving only parameters from $\Theta \cup \{p\}$. Therefore $p = f^{-1}(p')$ has a Σ_1 definition over X involving only parameters from $\Theta \cup \{p''\} = f^{-1}[\Theta \cup \{p\}]$. But$p = p(\beta)$, so $p <_\beta p''$ which implies$p' <_\beta p$, contradiction. Now set $\bar{\eta} := f(\eta)$. η is β-pseudo stable and we have just shown that $f(p) = p$, so $\bar{\eta}$ is also β-pseudo stable. Further $\bar{\eta} > \alpha_\mu^+$ as $\eta > \alpha_\mu^+$. But by claim 1 the only β-pseudo stable ordinals that can occur in Y must be less than Θ and we have a contradiction.

According to subclaims 1, 2 and 4 the ordinal Θ is β-pseudo stable and $\delta = \mathcal{C}'(\eta) > \gamma = \mathcal{C}(\eta)$. So Θ is not successor β-pseudo stable by lemma 2.5. In particular $\Theta > \eta_0$ where η_0 is the least β-pseudo stable ordinal larger than α_μ. Now let $\xi > \alpha_\mu^+$ be an arbitrary β-pseudo stable ordinal. By our choice of Θ we have $\eta_0 < \Theta \leq X(\xi) \cap \alpha_\mu^+$. A similar argument as the one used in subclaim

2 shows that for some $\sigma < \kappa$: $\eta_0 < X^\sigma(\xi) \cap \alpha_\mu^+$. It follows that η_0 is in $X^\sigma(\xi)$. But η_0 is known to be discernible by claim 3, as a matter of fact, η_0 is discernible at stage 0. As in claim 2 one can now show that ξ is discernible at stage σ. This concludes the proof of claim 4.

Summarizing, we have shown in claim 4 that every β-pseudo stable ordinal in the complement of H_μ is discernible. By claim 2 this implies that every element of the complement of H_μ is discernible. This establishes claim 1 and finishes the proof of the main lemma. □

One can use the machinery of Σ_n mastercodes as developed by Jensen to expand the main lemma to Σ_n-L_β sets. For $n \geq 1$ call a Σ_n-L_β set $X \subseteq L_\beta$ n-thin iff $|X|^{\beta,n} < \sigma n \mathsf{prj}(\beta)$. The main lemma then says that every 1-thin set is Δ_1−L_β. Similarly one can show that every n-thin set is Δ_n−L_β. For a proof see [10].

4 Applications: Recursive Subsets, Friedberg Splitting, and Maximal Sets

This section is devoted to applications of our characterization of ℓ-finite sets. One of the main interests of the ideal of finite sets in classical recursion theory is in its use to define notions like simple set, maximal set or major subset. As a first example consider simple sets. An r.e. set S is *simple* iff

$$\mathsf{RE}^*_\omega \models S^* \neq \mathbf{1} \wedge \forall X(X \cap S^* = \mathbf{0} \Rightarrow X = \mathbf{0})$$

where S^* stands for the equivalence class of X in RE^*_ω. Accordingly, one may define a β-r.e. set S to be ℓ-simple iff $\mathsf{RE}^\star_\beta$ is a model of the same formula of lattice theory. Here $\mathsf{RE}^\star_\beta$ is the defined to be the quotient lattice $\mathsf{RE}_\beta/\Im_\beta$ and $\Im_\beta$ denotes the ideal of ℓ-finite sets in RE_β. It was shown by Friedman in [3] that for all β there exists a β-r.e. set S such that $L_\beta - S \subseteq \beta^\star$ that is simple in the following sense: The order type of $\beta^\star - S$ is $\beta^\star$ and for all $X \subseteq \beta^\star$, β-r.e. and unbounded in $\beta^\star$, $X \cap S \neq \emptyset$.

We claim that any such set S is also ℓ-simple. By theorem 3.1 $\beta^\star - S$ cannot be ℓ-finite. So suppose that X is β-r.e. with $X \cap S$ ℓ-finite. Set $X' := X - (X \cap S)$. X' is β-r.e. and trivially disjoint from S. Hence X' must be bounded below $\beta^\star$. But then X' is β-finite by Σ_1 separation and $\beta^\star$-finite by reflection. Thus X' is ℓ-finite and lastly $X = X' \cup (X \cap S)$ is also ℓ-finite as required. Hence ℓ-simple sets exist for arbitrary limit ordinals β.

4.1 Recursive Subsets

As a first application of the machinery developed in the last section we will show how to transfer the following result of classical recursion theory: every infinite recursively enumerable set contains an infinite recursive subset. According to the next lemma this holds for all limit ordinals β if finite is replaced by ℓ-finite. The proof is quite straightforward for weakly admissible β and uses Σ recursion below κ to modify an enumeration for the r.e. set. In the strongly inadmissible case, however, this approach fails and we have to exploit the characterization of ℓ-finite sets developed in section 3.

Lemma 4.1 *Let β be an arbitrary limit ordinal and $A \subseteq L_\beta$ a β-r.e. set that fails to be ℓ-finite. Then A has a β-recursive subset B that also fails to be ℓ-finite.*

Proof. Let us first dispose of the weakly admissible case. We will lift the standard argument from classical recursion theory. Since $|L_\beta|^{\beta,1} = \kappa$ we may safely assume that A is a subset of κ. Fix an enumeration $f : \delta \leftrightarrow A$ where $\delta := |A|^{\beta,1}$. As A fails to be ℓ-finite $\kappa \geq \delta \geq \beta^\star$. Using Σ recursion one may define a strictly increasing function f' with domain δ and range a subset of A. Lastly, set $B := \mathsf{rg}(f')$. Then B is clearly β-recursive. Also, B cannot be ℓ-finite since $|B|^{\beta,1} = \delta \geq \beta^\star$.

Now consider a strongly inadmissible ordinal β. Since $|L_\beta|^{\beta,1} = \beta^\star$ we may safely assume that A is a subset of $\beta^\star$. Define $A^\sigma := \{\, \xi < \beta^\star \mid L_{q(\sigma)} \models (\xi \in A) \,\}$ and set $\Theta^\sigma := |A^\sigma|^\beta$. If there exists a stage $\sigma < \kappa$ such that $\Theta^\sigma = \beta^\star$ we may simply define $B := A^\sigma$. For then B is β-finite and has β-cardinality $\beta^\star$, thus B is β-recursive and fails to be ℓ-finite as required.

So let us assume from now on that for all $\sigma < \kappa$: $\Theta^\sigma < \beta^\star$.

Claim 1: $\beta^\star$ is a Σ_1 irregular β-cardinal.

Suppose for the sake of a contradiction that $\beta^\star$ is Σ_1 regular. Then $A^\sigma \subseteq \beta^\star$ must be bounded below $\beta^\star$ for all $\sigma < \kappa$ as $\Theta^\sigma < \beta^\star$. But then the function $\kappa \to \beta^\star$, $\sigma \mapsto \sup(A^\sigma)$ must have range bounded below $\beta^\star$ as $\beta^\star$ is Σ_1 regular. Then A is bounded below $\beta^\star$ and thus $\beta^\star$-finite by Σ_1 separation. But then A is ℓ-finite, contradiction. By the remark following lemma 3.4 this implies that $\kappa = \sigma 1 \mathsf{cof}(\beta^\star)$.

Fix a strictly increasing β-r.e. sequence $(\alpha_\nu)_{\nu<\kappa}$ of β-cardinals less than $\beta^\star$ with supremum $\beta^\star$. Define two functions $g, h : \kappa \to \kappa$ by

$$h(\nu) := \min\big(\, \sigma < \kappa \mid \exists \mu < \kappa (|A^\sigma \cap \alpha_\mu|^\beta \geq \alpha_\nu \,\big)$$
$$g(\nu) := \min\big(\, \mu < \kappa \mid |A^{h(\nu)} \cap \alpha_\mu|^\beta \geq \alpha_\nu \,\big).$$

Both h and g are β-recursive functions and thus β-finite by Σ_1 separation. Note that $g(\nu) \geq \alpha_\nu$. Now define a β-r.e. sequence $(\sigma_i)_{i<\kappa}$ of ordinals by induction on $i < \kappa$ as follows: $\sigma_0 := 0$, $\sigma_{i+1} := g(\sigma_i) + 1$ and, for limit ordinals λ, $\sigma_\lambda := \sup(\sigma_i \;\big|\; i < \lambda)$. Since κ is a regular β-cardinal this is well defined for all $i < \kappa$ and $\sup(\sigma_i \;\big|\; i < \kappa) = \kappa$. Finally define a β-r.e. set $B = \bigcup_{i<\kappa} B^i$ by

$$B^i := A^{h(\sigma_{i+1})} \cap \big]\alpha_{g(\sigma_i)}, \alpha_{g(\sigma_{i+1})}\big[\,.$$

Observe that $\xi \notin B$ iff $\exists i < \kappa (\alpha_{g(\sigma_i)} < \xi \leq \alpha_{g(\sigma_{i+1})} \wedge \xi \notin h(\sigma_{i+1}))$. Hence B is a β-recursive subset of A. It remains to verify the following claim.

Claim 2: B is not ℓ-finite.

Assume otherwise. Then B has an enumeration $f : \delta \leftrightarrow B$ for some β-cardinal $\delta < \beta^\star$. Suppose $\rho < \beta^\star$ is a β-cardinal such that

(1) B is Σ_1 definable over L_β with parameters $a < \rho$ and $p := p(\beta)$ only,

(2) $B \cap \rho$ has β-cardinality larger than δ.

The first condition is trivially satisfied for all sufficiently large ρ. For the second condition note that by the definition of g, h and $(\sigma_i)_{i<\kappa}$ we have

$$|A^{h(\sigma_{i+1})} \cap \alpha_{g(\sigma_{i+1})}|^\beta \geq \alpha_{\sigma_{i+1}} = \alpha_{g(\sigma_{i+1})} > \alpha_{g(\sigma_i)} \geq |A^{h(\sigma_{i+1})} \cap \alpha_{g(\sigma_i)}|^\beta.$$

Hence $|B^i|^\beta \geq \alpha_{\sigma_{i+1}}$ and the second condition is also satisfied for all sufficiently large $\rho < \beta^\star$. Now let η be the least β-pseudo stable ordinal larger than ρ and set $X := SH_1(\eta, p; L_\beta)$. Let $\pi : X \leftrightarrow L_\gamma$ be the Mostowski collapse where $\gamma = \mathcal{C}(\eta)$. Set $B' := \{\, \xi < \gamma^* \mid L_\gamma \models (\xi \in B) \,\}$ and $f' := \{\, (x, y) \in \gamma^* \times \gamma^* \mid L_\gamma \models (f(x) \simeq y) \,\}$. Note that B' and f' are β-finite and $B \cap \rho = B' \cap \rho$. But f' is a bijection, $f' : \delta \leftrightarrow B'$. Hence $|B'|^\beta \leq \delta < \rho$, contradiction. This finishes the proof of the lemma. □

4.2 Friedberg Splitting

Another one of the basic structure theorems about the lattice of r.e. sets is Friedberg Splitting: every non-recursive set can be split into two disjoint r.e. sets that both fail to be recursive. See for example [9] for a presentation of this result in classical recursion theory. Friedberg Splitting was generalized by Machtey to admissible ordinals α such that $\alpha^* = \omega$ and finally by Lerman to all admissible ordinals. Lerman used a short indexing of length α^* in order to make sure that the injuries that occur during the construction are α-finite. To lift this result to weakly admissible β one can exploit the admissible collapse $\mathfrak{A}_\beta$ of L_β.

The admissible collapse was originally introduced by Maass to facilitate the study of β-r.e. degrees, see [7]. In short, the admissible collapse of L_β is the amenable structure $\mathfrak{A}_\beta = \langle L_\kappa, D \rangle$ where D is a Δ_1 master code. The crucial property of $\mathfrak{A}_\beta$ is that Σ_1 definability over L_β is preserved in the collapse: a set $X \subseteq L_\kappa$ is Σ_1–L_β iff X is Σ_1–$\mathfrak{A}_\beta$. Consequently, the Σ_1 cofinality of κ in $\mathfrak{A}_\beta$ is κ. Hence $\mathfrak{A}_\beta$ is an admissible structure and many arguments from α-recursion theory carry over to $\mathfrak{A}_\beta$. With respect to the lattice of β-r.e. sets note that RE_β is isomorphic to $\mathsf{RE}_\beta(L_\kappa)$ for weakly admissible β: there exists a β-recursive bijection $f : L_\beta \leftrightarrow L_\kappa$ which clearly induces an isomorphism as desired. But $\mathsf{RE}_\beta(L_\kappa)$ can also be construed as the lattice of Σ_1–$\mathfrak{A}_\beta$ sets. Thus in order to establish Friedberg Splitting for all weakly admissible β it completely suffices to show that it holds for all the structures $\mathfrak{A}_\beta$. It is quite straightforward - though somewhat tedious - to check that Lerman's proof carries over to $\mathfrak{A}_\beta$.

For strongly inadmissible β, however, a direct construction is needed. It is easy to see that the injury sets that occur in the standard construction in general fail to be β-finite. Fortunately, they are always thin and therefore ℓ-finite by the main lemma. This is the key to the following argument.

Theorem 4.1 (Friedberg Splitting) *Let β be an arbitrary limit ordinal and $A \subseteq L_\beta$ a β-r.e. set that fails to be β-recursive. Then there are β-r.e. sets A_0 and A_1 such that $A = A_0 \cup A_1$, $A_0 \cap A_1 = \emptyset$ and both A_0 and A_1 fail to be β-recursive.*

Proof. According to the preceding remark we may safely assume that β is strongly inadmissible. Then $\widehat{\beta} = \beta^\star > \kappa$ and RE_β is isomorphic to $\mathsf{RE}_\beta(\beta^\star)$. So we may further assume that $A \subseteq \beta^\star$. We will construct the sets A_0 andA_1 in κ stages, each stage consisting of at most $\beta^\star$ steps. Recall that we always assume $q : \kappa \to \beta$ to be a strictly monotonic, continuous Σ_1 cofinality function. Let $(A^\sigma \mid \sigma < \kappa)$ be an enumeration of A where $A^\sigma := \{\, x \in \beta^\star \mid L_{q(\sigma)} \models (x \in A) \,\}$. Set $\Theta_\sigma := \mathsf{otp}(A^{\sigma+1} - A^\sigma)$ and let $a^\sigma(j) :=$the j-th element of $A^{\sigma+1} - A^\sigma$ for $j < \Theta_\sigma$. Hence $A = \{\, a^\sigma(j) \mid \sigma < \kappa \wedge j < \Theta_\sigma \,\}$. At stage $\sigma < \kappa$ we will perform exactly Θ_σ steps j. Let $A_i^{<\sigma,j}$ be the part of A_i constructed prior to step j of stage σ, i.e., either at some stage $\sigma' < \sigma$ or at stage σ but during some step $j' < j$. At stage σ, step j element $a^\sigma(j)$ will be put either into A_0 or into A_1. This guarantees that $A_0 \cup A_1 = A$

and $A_0 \cap A_1 = \emptyset$. Now let $(W_\xi \mid \xi < \beta^\star)$ be a standard simultaneous enumeration of all β-r.e. subsets of $\beta^\star$ and set $W_\xi^\sigma := \{ x \in \beta^\star \mid L_{q(\sigma)} \models (x \in W_\xi) \}$. In order to make sure that A_i is not β-recursive we use the standard simplicity requirements of the form:

$$R_{i,\xi}: \quad W_\xi \cap A_i \neq \emptyset.$$

Here $i = 0, 1$ and $\xi < \beta^\star$. As usual we say that requirement $R_{i,\xi}$ has higher priority than $R_{k,\eta}$ iff $\xi < \eta$ or ($\xi = \eta$ and $i < k$), in symbols $R_{i,\xi} \lhd R_{k,\eta}$. $R_{i,\xi}$ requires attention at stage σ, step j, iff $a^\sigma(j) \in W_\xi^\sigma$ and $A_i^{<\sigma,j} \cap W_\xi^\sigma = \emptyset$.

The Construction:
Initially set $A_i^{<0,0} := \emptyset$.

Now consider stage $\sigma < \kappa$, step $j < \Theta_\sigma$. Let $R_{i,\xi}$ be the requirement of highest priority that requires attention at stage σ, step j. Put $a^\sigma(j)$ into A_i. If no such requirement exists put $a^\sigma(j)$ into A_0.

Let $A_i := \bigcup_{\sigma<\kappa, j<\Theta_\sigma} A_i^{<\sigma,j}$. We will say that $R_{i,\xi}$ receives attention at stage σ, step j, iff $a^\sigma(j)$ is put into A_i because of requirement $R_{i,\xi}$ at stage σ, step j. Each requirement can receive attention at most once throughout the whole construction. Therefore there is a partial β-recursive function $r : \{0,1\} \times \beta^\star \to \beta^\star$ such that $r(i,\xi) \simeq x$ iff $R_{i,\xi}$ receives attention at stage σ, step j, and $a^\sigma(j) = x$. The only possible conflict between our requirements is that, at some stage σ, step j, $R_{i,\xi}$ requires attention but does not receive it because of some requirement $R_{k,\eta}$ of higher priority. We will say that requirement $R_{i,\xi}$ is foiled at stage σ, step j. This leads to the definition of the following injury set:

$$I_{i,\xi} := \{ x < \beta^\star \mid \exists \sigma < \kappa, j < \Theta_\sigma (x = a^\sigma(j) \wedge R_{i,\xi} \text{ is foiled at } \sigma, j) \}.$$

We claim that $I_{i,\xi}$ is thin. To see this let

$$D_{i,\xi} := \{ (k,\eta) \in \{0,1\} \times \beta^\star \mid \exists \sigma < \kappa, j < \Theta_\sigma (R_{k,\eta} \lhd R_{i,\xi} \wedge R_{k,\eta} \text{ receives attn. at } \sigma, j) \}.$$

$D_{i,\xi}$ is β-r.e. and bounded in $L_\beta^\star$, hence $\beta^\star$-finite by Σ separation. But $I_{i,\xi}$ clearly is a β-r.e. subset of $r[D_{i,\xi}]$ and therefore also thin.

Now assume for the sake of a contradiction that A_i is β-recursive. Then for some $\xi < \beta^\star$: $W_\xi = \beta^\star - A_i$. Define a β-recursive function $st : W_\xi \to \kappa$ by $st(x) := \min(\sigma < \kappa \mid x \in W_\xi^\sigma)$.

Claim: If $x \in W_\xi \cap A$ and $x \notin I_{i,\xi}$ then $x \in A^{st(x)}$.

For assume otherwise, say $x \in A^{\sigma+1} - A^\sigma$ for some $\sigma \geq st(x)$, $x = a^\sigma(j)$. But then $x \in W_\xi^{st(x)} \subseteq W_\xi^\sigma$, so $R_{i,\xi}$ requires attention at stage σ, step j. However, $R_{i,\xi}$ never receives attention, so some requirement of higher priority grabs x. Thus x lies in the injury set $I_{i,\xi}$, contradiction.

According to the claim we have for all $x < \beta^\star$:

$$x \in A \text{ iff } x \in A_i \cup (W_\xi \cap I_{i,\xi}) \cup ((W_\xi - I_{i,\xi}) \cap A^{st(x)}).$$

The sets A_i and $W_\xi = \beta^\star - A_i$ are β-recursive by our assumption and $I_{i,\xi}$ is β-recursive by the main lemma, hence A is also β-recursive. This contradicts our hypothesis and we are done. $\square$

Note that the sets A_0 and A_1 constructed in the last proof are actually recursively inseparable. For assume that there is a β-recursive set R such that $A_0 \subseteq R$ but $A_1 \cap R = \emptyset$, say $R = W_\xi$. Then in

the notation from the last proof $x \in A_0$ iff $x \in (A_0 \cap I_{1,\xi}) \cup ((W_\xi - I_{1,\xi}) \cap A^{st(x)})$. This implies that A_0 is β-recursive and we obtain a contradiction.

4.3 Maximal Sets

As in the example at the beginning of this section $\mathsf{RE}^\star_\beta$ provides the following natural generalization of maximal sets to β-recursion theory. The reader should consult [6] for an extensive discussion of various notions of maximal set in α-recursion theory.

Definition 4.1 *A β-r.e. set $X \subseteq L_\beta$ is called ℓ-maximal iff X^* is a co-atom in $\mathsf{RE}^\star_\beta$.*

For $X \subseteq L_\beta$ let $\bar{X} := L_\beta - X$ denote the complement of X. Then X is ℓ-maximal iff $\bar{X}$ fails to be ℓ-finite but for all Y in RE_β either $\bar{X} \cap Y$ or $\bar{X} - Y$ is ℓ-finite. In the remainder of this section we will provide various existence and non-existence results for ℓ-maximal sets. We begin with a brief comment on the weakly admissible situation. The admissible ordinals α for which ℓ-maximal sets exist were characterized by Lerman in [6] in terms of S^α_3 projections. A function $f : \delta \to \gamma$, $\delta, \gamma \leq \alpha$, is S^α_3 iff there exists a α-rec function $f : \alpha \times \alpha \times \delta \to \gamma$ such that for all $x < \delta$: $f(x) = \lim_{\xi \to \alpha} \lim_{\eta \to \alpha} f(\xi, \eta, x)$. Here the limits are taken with respect to the discrete topology. The S^α_3 projectum of δ is defined by

$$s3p^\alpha(\delta) := \min\big(\, \gamma \mid \exists\, f : \delta \to \gamma\, S^\alpha_3(\, f \text{ injective })\,\big).$$

Similarly one may define the $S_3 - \mathfrak{A}$ projectum for an amenable structure $\mathfrak{A}$. Lerman showed that for an admissible ordinal α the lattice RE_α contains an ℓ-maximal set iff $s3p^\alpha(\alpha) = \omega$. Again it is quite straightforward to check that Lerman's argument can be carried out in $\mathfrak{A}_\beta$ as well. Hence for weakly admissible β there exist ℓ-maximal sets iff the $S_3 - \mathfrak{A}$ projectum of κ is ω. But the $S_3 - \mathfrak{A}$ projectum of κ is $s3p^\beta(\beta)$, hence we have the following corollary to Lerman's result.

Theorem 4.2 (Lerman) *Let β be weakly admissible. Then ℓ-maximal sets exist in RE_β iff $s3p^\beta(\beta) = \omega$.*

We now turn to strongly inadmissible ordinals. Note that $s3p^\beta(\beta) = \omega$ implies that β is weakly admissible, so S_3-projections are not useful in this context. However, one can modify an argument first used by Sacks for $\alpha = \aleph_1$ to show that ℓ-maximal sets fail to exist if $\beta^\star$ is sufficiently regular.

Lemma 4.2 *Let β be strongly inadmissible and assume $\beta^\star$ is a Σ^β_3 regular successor β-cardinal. Then there are no ℓ-maximal sets in RE_β.*

Proof. Assume for the sake of a contradiction that $M \subseteq L_\beta$ is ℓ-maximal. Let $\mathcal{F} : L_\beta \leftrightarrow \beta^\star$ be a β-recursive bijection. Then $\mathcal{F}[M] \cup (L_\beta - \beta^\star)$ is easily seen to be ℓ-maximal: ℓ-finite sets are the same as thin sets and thinness is preserved by $\mathcal{F}$. Thus we may assume without loss of generality that $\bar{M} := L_\beta - M$ is a subset of $\beta^\star$. $\beta^\star$ is a successor β-cardinal, so let $\beta^\star = \rho^+$. Fix a β-finite enumeration $\mathcal{P} = (S_\nu \mid \nu < \beta^\star)$ of the L_β-powerset of ρ such that every β-finite subset of ρ occurs exactly once as one of the sets S_ν. Now define for all $\xi < \rho$: $C_\xi := \{\, \nu < \beta^\star \mid \xi \in S_\nu \,\}$. Clearly $C := (C_\xi \mid \xi < \rho)$ is again β-finite. The sets $C_\xi \subseteq \beta^\star$ are also β-finite and therefore cannot split $\bar{M}$ into two ℓ-infinite parts: either $\bar{M} \cap C_\xi \subseteq \beta^\star$ or $\bar{M} - C_\xi \subseteq \beta^\star$ must be ℓ-finite. $\beta^\star$ is assumed

to be Σ_3 regular, hence the ℓ-finite subsets of $\beta^\star$ are all $\beta^\star$ -finite and in particular bounded below $\beta^\star$. Therefore for all $\xi < \rho$ there exists a $\gamma < \beta^\star$ such that $\bar{M} \cap C_\xi \subseteq \gamma$ or $\bar{M} - C_\xi \subseteq \gamma$. The last expression can be rewritten as a Σ_3 formula, thus by Jensen's uniformization theorem there exists a $\Sigma_3 - L_\beta$ function $f : \rho \to \beta^\star$ such that $\forall \xi < \rho (\bar{M} \cap C_\xi \subseteq f(\xi) \vee \bar{M} - C_\xi \subseteq f(\xi))$. $\beta^\star$ is Σ_3 regular, so f must have range bounded below $\beta^\star$, say, $\mathsf{rg}(f) \subseteq \Gamma < \beta^\star$. The complement of a ℓ-maximal set $\bar{M}$ fails to be ℓ-finite, hence $\bar{M} \subseteq \beta^\star$ is unbounded in $\beta^\star$ (otherwise $\bar{M}$ would be $\beta^\star$-finite and thus ℓ-finite by Σ separation). Therefore one can find two ordinals ν and μ in $\bar{M}$ such that $\Gamma < \nu < \mu < \beta^\star$. But then for all $\xi < \rho$ either both ν and μ lie in C_ξ or both fail to lie in C_ξ. By the definition of C_ξ this means that $S_\nu = S_\mu$ and we have the desired contradiction. □

We now turn to the case where $\beta^\star$ is Σ_1 irregular. Recall from the proof of the main lemma that for any limit ordinal β there exists an ℓ-cover $I = \big(I_\nu \bigm| \nu < \lambda\big)$ where $\lambda = \sigma 1 \mathsf{cof}^\beta(\beta^\star)$. It is not hard to see that for every set X in the boolean algebra generated by RE_β we have: X is ℓ-finite iff there exists a $\nu < \lambda$ such that $X \subseteq I_\nu$. This fact can be used to rule out the existence of ℓ-maximal sets for strongly inadmissible β such that $\sigma 1 \mathsf{cof}(\beta^\star) < \sigma 2p(\beta)$.

Theorem 4.3 *Let β be strongly inadmissible and assume $\sigma 1 \mathsf{cof}(\beta^\star) < \sigma 2p(\beta)$. Then there are no ℓ-maximal sets in RE_β.*

Proof. Assume for the sake of a contradiction that $M \subseteq L_\beta$ is ℓ-maximal. We will use an ℓ-cover of length $\lambda := \sigma 1 \mathsf{cof}^\beta(\beta^\star)$ to split the complement of M. As in the proof of lemma 4.2 we may assume that $\bar{M} := L_\beta - M$ is a subset of $\beta^\star$. Note that $\beta^\star$ cannot be a successor β-cardinal. Thus there exists a strictly increasing β-r.e. sequence $(\alpha_\nu \bigm| \nu < \lambda)$ of β-cardinals with supremum $\beta^\star$.

Let $(I_\nu \bigm| \nu < \lambda)$ be an ℓ-cover. Now define a binary relation $R \subseteq \lambda \times \lambda$ by:

$$R(\nu, \mu) \iff \exists \xi < \beta^\star \big(\alpha_\mu < \xi < \alpha_{\mu+1} \wedge \xi \notin (M \cup I_\nu)\big).$$

The sets I_ν are simultaneously β-r.e. , whence R is $\Sigma_2 - L_\beta$. By our assumption $\lambda < \sigma 2p(\beta)$, so R is β-finite by Σ_2 separation. Also observe that domain of R is λ. For otherwise there would be some ordinal $\nu < \lambda$ such that $\beta^\star \subseteq M \cup H_\nu$, or, in other words, $\bar{M} \subseteq H_\nu$. But then $\bar{M}$ is ℓ-finite, contradicting our assumption that M is ℓ-maximal. Now define a β-finite function $f : \lambda \to \lambda$, $f(\nu) := \min\big(\mu < \lambda \mid R(\nu, \mu)\big)$. Let T denote the range of f. T is β-finite and must be unbounded in λ since for all $\mu < \lambda$ there exists a $\nu_0 < \lambda$ such that for all $\nu \geq \nu_0$: $\alpha_\mu \subseteq I_\nu$. But λ is a regular β-cardinal by its very definition, hence T must have ordertype λ. Let $T = \{\, \mu_i \mid i < \lambda \,\}$ where $\mu_i < \mu_j$ for all $i < j < \lambda$. We can now define a β-r.e. set X that splits $\bar{M}$ as follows:

$$X := \bigcup_{i<\lambda} \big[\alpha_{\mu_{2i}}, \alpha_{\mu_{2i+1}}\big[\,.$$

X is clearly β-finite.

Claim: $\bar{M} \cap X$ fails to be ℓ-finite.

Let us suppose for the sake of a contradiction that $\bar{M} \cap X$ is ℓ-finite. Since $(I_\nu \bigm| \nu < \lambda)$ is an ℓ-cover there is some $\nu < \lambda$ such that $\bar{M} \cap X \subseteq I_\nu$. Now pick $\bar{\nu} \geq \nu$ such that $f(\bar{\nu}) = \mu_{2i}$. It follows from $I_\nu \subseteq I_{\bar{\nu}}$ that $\bar{M} \cap X$ and $I_{\bar{\nu}}$ are disjoint. On the other hand by the definition of f and since $\alpha_{\mu_{2i+1}} \geq \alpha_{\mu_{2i}+1}$ we have $\bar{M} \cap X \cap \big[\alpha_{\mu_{2i}}, \alpha_{\mu_{2i+1}}\big[\neq \emptyset$, contradiction.

An entirely similar argument shows that $\bar{M} - X$ also fails to be ℓ-finite. Hence M is not ℓ-maximal and we have the desired contradiction. □

Corollary 4.1 *Let β be a limit ordinal such that $\rho = \beta^\star$ is an uncountable cardinal in L. There are no ℓ-maximal sets in RE_β provided one of the following conditions is satisfied:*

- *ρ is a successor cardinal in L,*
- *ρ is Σ_1 irregular in L_β.*

Proof. If β is weakly admissible then clearly $s3p^\beta(\beta) \geq \rho > \omega$ and our claim follows from theorem 4.2. If on the other hand β is strongly inadmissible and ρ is a successor cardinal in L we are done by lemma 4.2. Lastly suppose β is strongly inadmissible and ρ is Σ_1 irregular in L_β. Since ρ is a L-cardinal, $\sigma 2p^\beta(\rho) = \rho > \sigma 1\mathsf{cof}^\beta(\rho)$ and the non-existence of ℓ-maximal sets follows from theorem 4.3. □

References

[1] K. J. Devlin. *Aspects of Constructibility.* Number 354 in Lecture Notes in Math. Springer Verlag, 1973.

[2] K. J. Devlin. *Handbook of Math. Logic*, chapter Constructibility. North-Holland, 1977.

[3] S. D. Friedman. β-recursion theory. *Trans. Am. Math. Soc.*, 255:173–200, 1979.

[4] S. D. Friedman. Post's problem without admissibility. *Adv. in Mathematics*, 35:30–49, 1980.

[5] S. D. Friedman. Negative solutions to post's problem. *Annals of Math.*, 113:25–43, 1981.

[6] M. Lerman. Maximal α-r.e. sets. *Trans. Amer. Math. Soc.*, (188):341–386, 1974.

[7] W. Maass. Inadmissibility, tame r.e. sets and the admissible collapse. *Ann. Math. Logic*, 13:149–170, 1978.

[8] W. Maass. Recursively invariant β-recursion theory. *Ann. Math. Logic*, 21:27–73, 1981.

[9] R. I. Soare. *Recursively Enumerable Sets and Degrees.* Perspectives in Mathematical Logic. Springer, Berlin, 1987.

[10] K. Sutner. The ordertype of β-r.e. sets. *J. Symbolic Logic*, 55(2):573–576, 1990.