

## A PROBLEM OF MINIMUM IN GRAPH THEORY

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**Definition1** We say that the pair  $(X, Y)$ , where  $X \neq Y$  are two vertices of a graph  $G$ , **has the property P** if  $G$  has a vertex  $Z$  ( $Z \neq X$  and  $Z \neq Y$ ) which is adjacent with both  $X$  and  $Y$ . We say that a graph  $G$  **is a P-graph** if for every  $X$  and every  $Y$  ( $X \neq Y$ ) vertices of  $G$ , the pair  $(X, Y)$  has the property  $P$ .

Let  $a(n)$  be the minimum number of edges that a  $P$ -graph with  $n$  vertices ( $n \geq 3$ ) may have. We want to show that  $a(n) = n - 1 + [\frac{n}{2}]$ , where  $[m]$  is the greatest integer less than or equal to  $m$ .

Problems 8.9 and 8.20 from [2] are somehow related to this problem. Some similar results are also proved in [1].

**LEMMA**  $a(n) \leq n - 1 + [\frac{n}{2}]$ .

**Proof** We have two possibilities:

a)  $n$  is odd ;

b)  $n$  is even ;

a)  $n$  is of the form  $n = 2k + 1$  where  $k$  is a natural number. Let an  $A$ -graph be a graph with the following configuration:

$A$ - graph

It is easy to see that the  $A$ -graph is a  $P$ -graph. The number of edges of an  $A$ -graph is  $3k$ . (there are  $k$  figures  $\triangle^w$  with a commun vertex.)

We also have  $n-1+\lceil\frac{n}{2}\rceil = 2k+k = 3k$ . So, the number of edges of an  $A$ -graph is  $n-1+\lceil\frac{n}{2}\rceil$ . We don't know whether this is the minimum number of edges or not. Thus we have

$$a(n) \leq n-1+\lceil\frac{n}{2}\rceil.$$

b)  $n$  is of the form  $n = 2k$  where  $k$  is a natural number. Let a  $B$ -graph be a graph with the following configuration:

$B$ -graph

It is easy to notice that the  $B$ -graph is a P-graph. The number of edges of a  $B$ -graph is  $3(k-2)+5 = 3k-1$  (there are  $k-2$  figures  $\Delta^{\mathcal{W}}$  and one figure

But we also have  $n-1+\lceil\frac{n}{2}\rceil = 2k-1+k = 3k-1$ . So the number of edges of a  $B$ -graph is  $n-1+\lceil\frac{n}{2}\rceil$ . Thus we have:

$$a(n) \leq n-1+\lceil\frac{n}{2}\rceil.$$

From a) and b) we infer that  $a(n) \leq n-1+\lceil\frac{n}{2}\rceil$  for every  $n \geq 3$ . □

**Definition2** We say that a vertex of an  $A$ -graph or a  $B$ -graph is **the center** of that graph if it is adjacent with all the remaining vertices.

**Definition3** We denote the vertices of a graph  $G$  with  $n$  vertices by  $1, 2, \dots, n$ . We denote by  $d_G(X)$  the number of vertices of  $G$  which are adjacent with the vertex  $X$ .

**Observation1** : In a P-graph  $G$  it is not possible to have a vertex  $X$  with  $d_G(X) = 1$ . If it were possible then let  $Y$  be the only vertex adjacent to  $X$ .

Then,  $(X,Y)$  doesn't have the property P and this means that G is not a P-graph. We cannot have a vertex X with  $d_G(X) = 0$ . If it were possible then let Y be a vertex different from X. Then,  $(X,Y)$  doesn't have the property P.

**THEOREM**  $a(n) = n - 1 + \lceil \frac{n}{2} \rceil$  and every P-graph with n vertices and  $a(n)$  edges is either an A-graph (if n is odd) or a B-graph (if n is even).

**Proof** First let us see what happens for  $n = 3$  and  $n = 4$ .

$n = 3$ . The only possible configuration is that of a triangle

Thus  $a(3) = 3 = 3 - 1 + \lceil \frac{3}{2} \rceil$  and the only P-graph with 3 vertices is an A-graph.

$n = 4$ . We will prove that four edges are not enough for the graph to have the desired property ( obviously, if four edges are not enough, we cannot have less than five). We have two possibilities:

1. two edges (from six possible) with no common endpoint are missing

Then  $(A,B)$  doesn't have the property P.

2. two edges (from six possible) with one common endpoint are missing

Then  $(A,B)$  doesn't have the property P.

We have proved that four edges are not enough for a P-graph with four vertices. Here follows an example of a P-graph with four vertices and five edges:

Thus  $a(4) = 5 = 4 - 1 + \lceil \frac{4}{2} \rceil$ . The graph above is a B-graph. Because all graphs with four vertices and five edges are isomorphic, we get that a P-graph with four vertices and  $a(4)$  edges must be a B-graph.

We shall use induction from n to n+2. Let us suppose that  $a(t) = t - 1 + \lceil \frac{t}{2} \rceil$  and that a P-graph with t vertices and  $a(t)$  edges is either an

A-graph or a B-graph for all  $t \leq n$ . We shall prove that  $a(n+2) = a(n) + 3 = (n+2) - 1 + \lceil \frac{n+2}{2} \rceil$ . We shall also prove that a P-graph with  $a(n+2)$  edges must be either an A-graph or a B-graph.

Let  $G$  be a minimal P-graph with  $n+2$  vertices (a P-graph having  $n+2$  vertices and  $a(n+2)$  edges). It is known that  $\sum_{X=1}^{n+2} d_G(X) = 2a(n+2)$ . From Lemma, we have  $a(n+2) \leq n+1 + \lceil \frac{n+2}{2} \rceil$ . Thus, we obtain :

$$\sum_{X=1}^{n+2} d_G(X) \leq 2(n+1 + \lceil \frac{n+2}{2} \rceil) = 2n+2 + 2\lceil \frac{n+2}{2} \rceil \leq 3n+4. \quad (1)$$

From Observation1,  $d_G(X) \geq 2$ , where  $X$  is a vertex of  $G$ . Then, for (1) to take place, we must have at least two vertices  $X$  and  $Y$  with  $d(X) = 2$  and  $d(Y) = 2$ .

Let  $X_1, X_2$  be the vertices adjacent to  $X$  and  $Y_1, Y_2$  be the vertices adjacent to  $Y$ .

There are two possibilities:

1.  $X$  is adjacent to  $Y$ .

There exists one vertex  $Z$  adjacent to both  $X$  and  $Y$ . We get  $X = Y_2, Y = X_2, Z = X_1 = Y_1$ . Let  $G_1$  be the graph with  $n$  vertices obtained from  $G$  by taking out the vertices  $X, Y$  and the edges  $XZ, YZ$  and  $XY$ . It is easy to see that  $G_1$  is a P-graph with  $n$  vertices. Thus  $G_1$  has at least  $a(n)$  edges. We obtain  $a(n+2) \geq a(n) + 3 = (n+2) - 1 + \lceil \frac{n+2}{2} \rceil$ . From Lemma, we get  $a(n+2) \leq (n+2) - 1 + \lceil \frac{n+2}{2} \rceil$ . Thus, we must have  $a(n+2) = (n+2) - 1 + \lceil \frac{n+2}{2} \rceil$ . This also implies that  $G_1$  has  $a(n)$  edges. From the induction hypothesis,  $G_1$  must be either an A-graph, or a B-graph. Let  $T$  be a vertex of  $G$  ( $T \neq X, T \neq Y, T \neq Z$ ). The pair  $(T, X)$  has the property P. Because  $d_G(X) = d_G(Y) = 2, T \neq Z$  and  $T \neq Y$ ,  $T$  must be adjacent to  $Z$ . That means  $Z$  is the center of  $G_1$ . Thus,  $G$  is either an A-graph or a B-graph (because we add to  $G_1$  a triangle with one vertex in the center of  $G_1$ ).

2.  $X$  is not adjacent to  $Y$  .

a)  $\{X_1, X_2\} \cap \{Y_1, Y_2\} = \emptyset$ .

This is a contradiction because  $(X, Y)$  doesn't have the property P.

b)  $|\{X_1, X_2\} \cap \{Y_1, Y_2\}| = 1$ .

Let  $Z$  denote the only vertex adjacent to both  $X$  and  $Y$ . We may consider  $Z = X_2 = Y_2$ . Because  $(X, X_1)$  and  $(Y, Y_1)$  have the property P, we get that  $ZX_1$  and  $ZY_1$  are edges of  $G$ .

Let  $G_2$  be the graph obtained from  $G$  by taking out the vertices  $X$  and  $Y$  and the edges  $XX_1, XZ, YZ$  and  $YY_1$  and add the edge  $X_1Y_1$  if this edge wasn't from the beginning. It is easy to see that  $G_2$  is a P-graph with  $n$  vertices. Thus  $G_2$  has at least  $a(n)$  edges. We obtain  $a(n+2) \geq a(n)+3$  if  $X_1Y_1$  is not an edge in  $G$  or  $a(n+2) \geq a(n)+4$  if  $X_1Y_1$  is an edge in  $G$ . Using the induction hypothesis, in the second case, we get  $a(n+2) > (n+2) - 1 + \lceil \frac{n+2}{2} \rceil$  which contradicts Lemma.

In the first case, using the induction hypothesis, we get  $a(n+2) \geq a(n)+3 = (n+2) - 1 + \lceil \frac{n+2}{2} \rceil$ . From Lemma, we get  $a(n+2) \leq (n+2) - 1 + \lceil \frac{n+2}{2} \rceil$ . Thus, we must have  $a(n+2) = (n+2) - 1 + \lceil \frac{n+2}{2} \rceil$ . This also implies that  $G_2$  has  $a(n)$  edges. From the induction hypothesis,  $G_2$  must be either an A-graph, or a B-graph.

**Observation2 :**  $X_1$  is not adjacent to  $Y_1$  in  $G$  (as proved above) but  $X_1Y_1$  is an edge in  $G_2$  (from the construction of  $G_2$ ).

Because  $G_2$  is either an A-graph or a B-graph and because  $X_1Y_1, X_1Z, Y_1Z$  are edges in  $G_2$ , we get that one of  $X_1, Y_1$  and  $Z$  is the center of  $G_2$  (in every A-graph and every B-graph, every 'triangle' has a vertex which is the center of that graph).

If  $X_1$  is the center of  $G_2$  and  $n > 4$ , then in  $G_2$  there exists a vertex  $T$  which is not adjacent to  $Z$  and is not adjacent to  $Y_1$ . Then  $(Y, T)$  doesn't

have the property P in G. If  $X_1$  is the center of  $G_2$  and  $n = 3$  we get that the only possible configuration for G is that of an A-graph. If  $X_1$  is the center of  $G_2$  and  $n = 4$  we get that the only possible configuration for G is that of a B-graph. The case when  $Y_1$  is the center of  $G_2$  can be treated in a similar way.

If Z is the center of  $G_2$  and  $d_{G_2}(X_1) = d_{G_2}(Y_1) = 2$  then, because we know from Observation2 that  $X_1$  is adjacent to  $Y_1$  in  $G_2$ , we get that G is either an A-graph or a B-graph (we remove from  $G_2$  the edge  $X_1Y_1$  that doesn't exist in G, as we know from Observation2, and we add the four edges which have one endpoint in X or in Y). If Z is the center of  $G_2$  and  $d_{G_2}(X_1) = 3$  or  $d_{G_2}(Y_1) = 3$  then G is a B-graph (we remove from  $G_2$  the edge  $X_1Y_1$  and we add the four edges which have one endpoint in X or in Y).

$$\text{c) } \left| \{X_1, X_2\} \cap \{Y_1, Y_2\} \right| = 2.$$

Let  $Z$  and  $T$  be the only two vertices adjacent to both  $X$  and  $Y$ . We may consider  $Z = X_1 = Y_1, T = X_2 = Y_2$ . Let  $U$  be a vertex of G, different from  $X, Y, Z, T$ .

**Observation3 :**  $U$  must be adjacent to either  $Z$  or  $T$  (otherwise the pair  $(X,U)$  does not have the propriety P).

Because of the simetry, we may consider that  $U$  is adjacent to  $T$ . Let  $G_3$  be the graph obtained from  $G$  by taking out the vertices  $X$  and  $Y$  and the edges  $XT, XZ, ZY$  and  $YT$  and, eventually, add the edge  $ZU$  if this edge wasn't from the beginning. It is easy to see that  $G_3$  is a P-graph with  $n$  vertices. Thus,  $G_3$  has at least  $a(n)$  edges. We obtain  $a(n+2) \geq a(n) + 3$  (if  $Z$  is not adjacent to  $U$  in G) or  $a(n+2) \geq a(n) + 4$  (if  $Z$  is adjacent to  $U$  in G). We proved when we analyzed case  $b$ ) that the second case was not possible because, using the induction hypothesis, we reached a contradiction to Lemma.

In the first case, using the induction hypothesis, we get  $a(n+2) \geq a(n) + 3 = (n+2) - 1 + \lceil \frac{n+2}{2} \rceil$ . From Lemma, we get  $a(n+2) \leq (n+2) - 1 + \lceil \frac{n+2}{2} \rceil$ . Thus, we must have  $a(n+2) = (n+2) - 1 + \lceil \frac{n+2}{2} \rceil$ . This also implies that  $G_3$  has  $a(n)$  edges. From the induction hypothesis,  $G_3$  must be either an

A-graph, or a B-graph.

**Observation4 :**  $(X,Z)$  has the property P. We infer that Z must be adjacent to T.

If the center of  $G_3$  is different from Z and T, and  $n > 4$  then in  $G_3$  there exists a vertex which is not adjacent to Z and is not adjacent to T. This contradicts Observation3. If  $n = 3$  then  $d_G(U) = 1$  and this is a contradiction to Observation1. If  $n = 4$  then Z or T must be a center for  $G_3$  (because Z is adjacent to T and, in this case and only in this case,  $G_3$  has two centers). If Z (or T) is a center for  $G_3$ , then  $d_{G_3}(U) \geq 3$ . Indeed, if  $d_{G_3}(U) = 2$ , we get  $d_G(U) = 1$  and this is a contradiction to Observation1. Because Z (or T) is a center for  $G_3$ , the only possibility is  $d_{G_3}(U) = 3$  and this happens only when  $G_3$  is a B-graph. Let V be the third vertex adjacent to U in  $G_3$  (and also in G).  $(U,V)$  has the property P in G. There exists a vertex in G which is adjacent with both U and V. U is adjacent only to V and T in G (we proved before that U is not adjacent to Z in G). Then, the only possibility is to have V adjacent to T in G (and also in  $G_3$  from the construction of  $G_3$ ). We obtain that  $d_G(T) = d_{G_3}(T) \geq 3$ . Because a B-graph cannot have more than two vertices (one is T and one is U) of degree greater than 2, we infer that  $d_{G_3}(Z) = 2$ . That means Z cannot be a center for  $G_3$  (because for  $n = 3$  we reached a contradiction to Observation1). Thus, T is a center for  $G_3$ . Then  $G_3$  is a B-graph with the following configuration :

Removing the edge ZU from  $G_3$  and adding the vertices X, Y and the edges

XZ, YZ, XT, YT, we find that G is a B-graph of the following form :

Thus, in all cases we get  $a(n+2) = (n+2) - 1 + \lfloor \frac{n+2}{2} \rfloor$ . We also obtain that a P-graph with  $n+2$  vertices and  $a(n+2)$  edges must be either an A-graph or a B-graph. The proof of the induction step is now complete.

We have  $a(n) = n - 1 + \lfloor \frac{n}{2} \rfloor$ . for every  $n \geq 3$

Because an A-graph has an odd number of vertices and a B-graph has an even number of vertices, it can be inferred that a P-graph with  $n$  vertices and  $a(n)$  edges must be an A-graph if  $n$  is odd, or a B-graph if  $n$  is even. Our theorem is completely proved.  $\square$

The following result, connected to the Theorem above, was proved in [1] :  
"If  $G_n$  is a graph with  $n$  vertices in which any two vertices are connected by a path of length two and which does not contain any cycle of length four, then  $n = 2k + 1$  and  $G_n$  consists of  $k$  triangles which have one common vertex." This is equivalent to : "A P-graph which does not contain any cycle of length four must be an A-graph."

## References

- [1] **P. Erdos, A. Renyi, V. Sos**, *On a problem of graph theory*, Studia Sci. Math., 1(1966).
- [2] **Ioan Tomescu**, *Probleme de combinatorica si teoria grafurilor*, Editura Didactica si Pedagogica, Bucharest, 1981.
- [3] **Ioan Tomescu**, *Introduction to combinatorics*, Collet's (Publishers) Ltd., London and Wellingborough, 1975.