SOLITAIRE ARMY AND RELATED GAMES

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We present some variations of the Solitaire Army game. This game is part of a broader category of one person games, namely, Peg Solitaire games. While Central Solitaire is a relatively well-known and studied game, which is played on a cross-shaped board, little research has been done on Solitaire Army game and its variants. The purpose of this paper is to present Solitaire Army game, its variants that have already been studied and, in addition, a number of new variations of this game: on different boards (hexagonal, graph boards) and using different types of moves (diagonal and longer jumps). At the end of the paper we present REPULSION, a new one-person game which is related to Solitaire Army, but in which the pegs are not removed from the board.

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1. INTRODUCTION

Peg Solitaire games constitute probably one of the most important classes of one-person games. These are in fact puzzles but, as argued in [7], they are called *games* because often people feel like playing with an invisible opponent. The most played game in this class is *HI-Q*, also known as *Central Solitaire*. In this game, the board is cross-shaped. Initially, all positions are occupied except for the central position. A move is allowed when a peg can jump horizontally or vertically over another peg into a previously empty position. In this case, the second peg is removed from the board. The goal is to end the game with only one peg at the center of the board. The initial and final boards are shown in Figure 1.

A detailed analysis of this game can be found in [7]. The most important issue about this game is to prove that it is possible to get from the initial position to the final position. The key to solving this problem is dividing the board into *packages*. A *package* is a configuration of pegs that can be cancelled, provided that there exist an additional peg in a specified position (called *the catalyst*) and a specified empty position. After some moves involving the pegs and positions in the initial package, the catalyst and the empty position, the package disappears, the catalyst returns to its initial position and the initial

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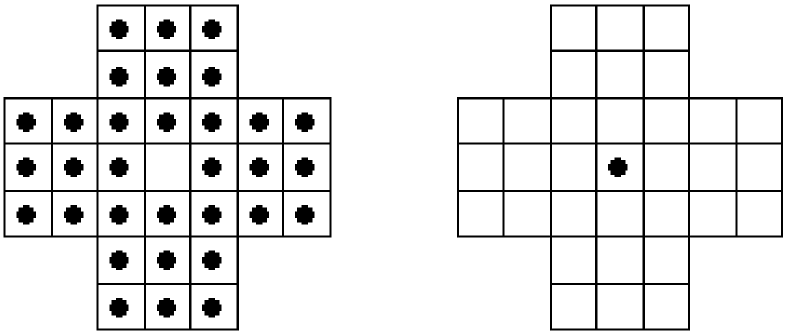

Fig. 1. Central Solitaire board, the initial and final positions.

empty position becomes again empty. There are many types of packages and they can be found in [7].

Another variation of Central Solitaire is one-dimensional Peg Solitaire. The player starts from a configuration of pegs on a line and his goal is to reach a position with only one peg on the board. The moves are identical with those in Central Solitaire, with the only difference that they are performed only on the same line. While in Central Solitaire it is known that the game can end with one peg on the board, in one-dimensional Peg Solitaire this is not always possible. In [13] one can find the set of the initial boards such that the goal can be achieved. It is interesting that this set can be described in terms of a regular language.

The focus of this paper is Solitaire Army, a less known Peg Solitaire person game, which is played on an infinite two dimensional grid. Each cell of the grid corresponds to a hole. We can imagine playing either with pegs in the cells of the grid or at the nodes of the grid. The initial position consists of a finite number of pegs, all of them placed in holes below a given horizontal line. Only one peg can occupy a hole (position). The moves are the same as in Central Solitaire, but the goal is to send a peg as far as possible (measured as the distance to the given line) into the upper half plane. It is proved in [7] that it is impossible to send a peg forward five holes above the given line. The main goal of this paper is to examine some natural extensions of this game. In the next sections we will cover the following variations of the game:

1. Two-dimensional Solitaire Army where one peg is allowed to jump over any number of pegs, these pegs being removed from the board.

2. The one-dimensional variant of the above game.

3. Two-dimensional Solitaire Army where only diagonal moves are allowed.

4. Two-dimensional Solitaire Army where horizontal, vertical and diagonal moves are allowed.

5. Solitaire Army on a board in a space with n dimensions.

6. Solitaire Army on triangular and hexagonal boards.

7. A more general framework for analyzing games related to Solitaire Army on graph boards.

The paper proceeds as follows. Section 2 talks about the original Solitaire Army game as described in [7]. In Sections 3 and 4 we present several extensions of the Solitaire Army game, involving different types of moves and boards. While some of these variations have already been studied, many constitute completely new puzzles. Most of the proofs in these sections are done using potential functions. Other potential functions were tried and they are discussed in Section 5, along with their disadvantages. Section 6 describes a new one person game, called REPULSION, which mimics the movement of positively charged particles. This game is similar to Solitaire Army, but instead of removing pegs from the board, the pegs involved in one step move to new positions. Here the goal is to be able to make as many moves as possible. Future work directions are suggested in Section 7 and conclusions are summarized in Section 8.

For the above problems we will investigate both lower and upper bounds of the maximum distance from the boundary to a position that one can reach. We will analyze the use of other potential functions and argue which one is better and why.

2. CONWAY'S SOLITAIRE ARMY

The Solitaire Army game was described in Section 1. Following [7, pp. 697–734], we show that one cannot send a peg five places above the initial line in this one-person game. The potential functions for analyzing moving strategies will also be used in the next sections.

THEOREM 1. *In Solitaire Army, the fifth line above the initial line is unreachable using a sequence of legal moves.*

Proof. Let us assume by contradiction that one can send a peg five places above the initial line. Consider the first move when this happens. We infer that all other pegs are at most four places above the initial line. As in many proofs by contradiction, we will use a potential function ϕ with the following properties:

1. ϕ decreases or stays constant with every move;

2. $\phi_i < \phi_f$, where $\phi_i = \phi_{initial}$ and $\phi_f = \phi_{final}$.

Note that properties 1. and 2. will obviously give a contradiction and this will end the proof. Therefore, the only thing that has to be done is to construct this function ϕ.

Consider that the reachable position five places above the initial line is the origin of an orthogonal coordinate system. Each position (i, j) can be assigned a potential (label) $\phi(i, j) = \omega^{|i|+|j|}$. The value of ω will be determined later in the proof. The potential ϕ of the current board is now given by the formula $\phi = \sum_{(i,j)} \phi(i, j)$. Figure 2 shows how the potential varies for a subset of the board.

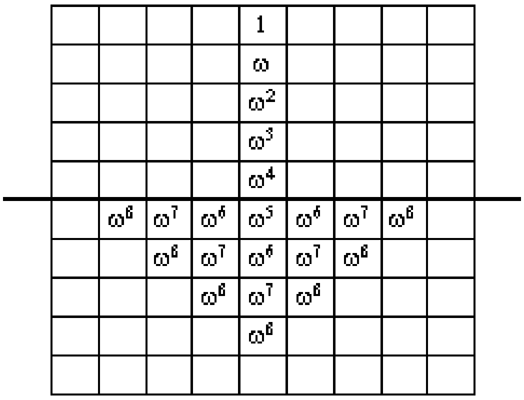

Fig. 2. The potential for a subset of a Solitaire Army board.

Let us find values for ω such that property 1. above is verified. Let us see what happens when one moves. Given the symmetry of the labelling of the board, we can analyze only horizontal moves. We can have three possible types of moves:

a) The peg at position (i, j) jumps over the peg at position $(i + 1, j)$ and gets to the empty position $(i + 2, j)$ where $i \geq 0$. Then the difference between the potential after the move and the potential before the move is $\omega^{|i|+|j|} \cdot (\omega^2 - \omega - 1)$. This move has also a mirror move, when the peg at position (i, j) jumps over the peg at position $(i - 1, j)$ and gets to the empty position $(i - 2, j)$ where $i \leq 0$. The difference in potential remains the same.

b) The peg at position $(-1, j)$ jumps over the peg at position $(0, j)$ and gets to the empty position $(1, j)$. Then the difference between the potential after the move and the potential before the move is $-\omega^{|j|}$. This move has a mirror move, when the peg at position $(1, j)$ jumps over the peg at position $(0, j)$ and gets to the empty position $(-1, j)$. The difference in potential remains the same.

c) The peg at position $(i + 2, j)$ jumps over the peg at position $(i + 1, j)$ and gets to the empty position (i, j), where $i \geq 0$. Then the difference between the potential after the move and the potential before the move is

$\omega^{|i|+|j|} \cdot (1-\omega-\omega^2)$. This move has also a mirror move, when the peg at position $(i-2,j)$ jumps over the peg at position $(i-1,j)$ and gets to the empty position (i,j) where $i \leq 0$. The difference in potential remains the same.

Therefore, for condition 1. to hold it is enough that ω verifies the inequalities $0<\omega$, $\omega^2-\omega-1 \leq 0$ and $1-\omega-\omega^2 \leq 0$. Consider ω to be the positive root of the equation $\omega^2+\omega=1$. It is easy to verify the above three inequalities and therefore ϕ verifies property 1.

Now, let us compute the potential of the initial board. In fact, we give just an upper bound on this potential. Since $\omega > 0$, the initial potential is bounded by the potential of the full initial board. The nth line above the origin of the initial full board is of the form $\ldots\ \omega^{n+2}\ \omega^{n+1}\ \omega^n\ \omega^{n+1}\ \omega^{n+2} \ldots$ and therefore the potential of this line is

$$\omega^n + 2 \cdot \sum_{i \geq n+1} \omega^i = w^n + \frac{2 \cdot \omega^{n+1}}{1-w} = \omega^n \cdot \frac{1+\omega}{1-\omega} = w^n \cdot \frac{\frac{1}{\omega}}{\omega^2} = \omega^{n-3}.$$

We sum up the above quantity for $n \geq 5$ and obtain the inequality

$$\phi_i \leq \sum_{n \geq 5} \omega^{n-3} = \frac{\omega^2}{1-\omega} = \frac{\omega^2}{\omega^2} = 1.$$

Since the initial board has only a finite number of pegs, we infer that $\phi_i < 1$. However, $\phi_f \geq \phi(0,0) = 1$ and therefore property 2. is also verified. Given that both properties of ϕ hold, we have an obvious contradiction. This concludes the proof of the fact that one cannot send a peg five positions above the initial line. □

Note also that the condition that the peg which jumps should land at an empty position is somehow artificial. If we allow a peg to land at an occupied position and remove what was already there, the above argument also holds. Moreover, suppose that we allow more pegs to share the same position (i.e. one peg can land at an occupied position without removing what was already there), then it is not possible to send a peg five levels above the initial line provided that initially at most one peg was at every position.

It is easy to see that one can send a peg one place above the initial line using an initial board with two pegs. To send a peg two positions above the line one needs four pegs and for sending it three positions above, one needs eight pegs. Although it is easy to construct the above initial boards, it is not at all obvious how to send a peg four positions above the line. Figure 3 presents one of the two initial boards with twenty pegs from which a peg can be sent four positions above the line

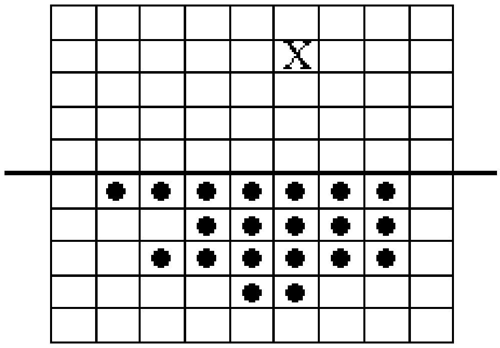

Fig. 3. One peg can reach the 4th level given this initial board.

3. **EXTENSIONS OF SOLITAIRE ARMY GAMES**

In this section we present and analyze some extensions of Conway's Solitaire Army. These extensions include allowing different types of moves or playing on different types of boards.

3.1. SOLITAIRE ARMY ALLOWING LONGER JUMPS

In this subsection we analyze the effect of allowing longer jumps. That means a peg (peg) can jump horizontally or vertically over any number of pegs provided that it lands at an empty position. Consider the potential function for Conway's Solitaire Army problem. If we can prove that this potential cannot increase after a move, then no peg can reach the fifth level (remember that $\phi_i < 1 \leq \phi_f$ holds no matter what moves one makes). We have two types of possible jumps over n pegs:

a) The $n+2$ positions involved (if all non-empty) have potential $\omega^i, \ldots, \omega^{i+n+1}$. Then the decrease in potential is at least 0 (0 can be reached only when $n = 1$).
b) The $n+2$ positions involved (if all non-empty) have potentials $\omega^{i+k}, \ldots, \omega^{i+1}$, $\omega^i, \omega^{i+1}, \ldots, \omega^{i+n-k+1}$, where $k < n+1$. Then the decrease in potential is at least $\omega^i - \max\{\omega^{i+k}, \omega^{i+n-k+1}\} \geq 0$.

This proves that ϕ decreases or stays constant after one moves. However, this contradicts the fact that $\phi_i < \phi_f$. Therefore, it is not possible to send a peg five levels above the initial line. Note that this argument will also apply when we analyze n-dimensional Solitaire Army.

3.2. ONE-DIMENSIONAL SOLITAIRE ARMY

In this version of Solitaire Army, all pegs are placed initially to the right of a given line and the goal is to send a peg as far as possible to the left of that line. First note that one can send a peg one position to the left by using an initial board with only two pegs. We claim that it is impossible to send a peg two positions to the left of the given line. Even though a proof by induction on the number of pegs is also available, we will only present a proof based on potential functions in order to be consistent with the proof of Conway's Solitaire Army.

THEOREM 2. *In one-dimensional Solitaire Army, a peg cannot be sent two positions to the left of the starting line using a sequence of legal moves.*

Proof. Let us assume by contradiction that one can send a peg two places to the left of the initial line. Consider the first move when this happens. We infer that all other pegs are at most one place to the left of the initial line. Consider each position on the line labelled with a number i such that the peg which lies two positions to the left of the initial line is labelled with 0 and, for each position P, i represents the distance to position 0. Given the board at a certain moment in time, let $\phi(i) = \omega^i$ be the potential of position i. The potential ϕ of the current line is now given by $\phi = \sum_i \phi(i)$. A legal board in this game is shown in Figure 4.

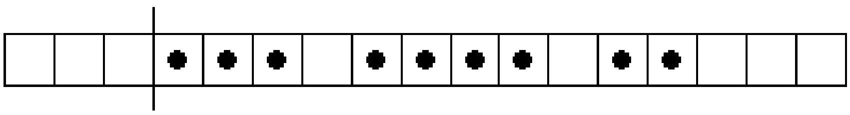

Fig. 4. A legal one-dimensional Solitaire Army board.

If we consider the same ω as in the previous proof, it is obvious that every move either decreases or keeps constant the potential of the board. That happens because we can look at our board as a line from the two dimensional version, where we know that the property holds (by the previous proof). The potential of the initial board is bounded by the potential of the full initial board:

$$\phi_i \leq \sum_{n \geq 2} \omega^i = \frac{\omega^2}{1-\omega} = \frac{\omega^2}{\omega^2} = 1.$$

Since the initial board has only a finite number of pegs, we infer that $\phi_i < 1$. However, $\phi_f \geq \phi(0) = 1$ and therefore $\phi_i < \phi_f$. Given that we proved that ϕ is decreasing or constant, we get a contradiction. Therefore, one cannot send a peg two positions to the left of the initial line. $\square$

3.3. SOLITAIRE ARMY IN A SPACE WITH n DIMENSIONS

In a space with n dimensions all pegs are initially placed in the lower halfspace bordered by an $(n-1)$ dimensional hyperplane S orthogonal to one of the axis d (we consider S included in the halfspace). The moves are similar to those in Solitaire Army: the jumps are made in one of the n possible directions parallel with the space axis. Denote by $l(n)$ the maximum level above S that can be reached in this game.

THEOREM 3. *For $n \geq 2$, the inequalities $2n \leq l(n) \leq 3n-2$ do hold.*

Proof. First, let us prove the first inequality. For $n = 2$ we know that $l(2) = 4$ (that is, just Conway's Solitaire Army). Now, assume that $2(n-1) \leq l(n-1)$. We may look at a space with n dimensions as a collection of parallel spaces with $n-1$ dimensions. Consider Figure 5 below, where each column represents a line in such an $(n-1)$-dimensional space. We know that we can get the configuration with a peg on level $l(n-1)$ in each of these columns by using only the pegs in the corresponding $(n-1)$-dimensional space. Therefore, it is also possible to reach level $l(n-1)-1$. Note also that we can reach this position in a way such that the pegs in different parallel $(n-1)$-dimensional spaces do not interact with each other. By translations of the initial $(n-1)$-dimensional hyperplanes, we can align the pegs so that they form the configuration in Figure 5 in the same two-dimensional plane. Now, it is easy to see that in this configuration we can send a peg two levels above level $l(n-1)$. This proves that $l(n) \geq l(n-1) + 2 \geq 2n$. Therefore, the first inequality is proved.

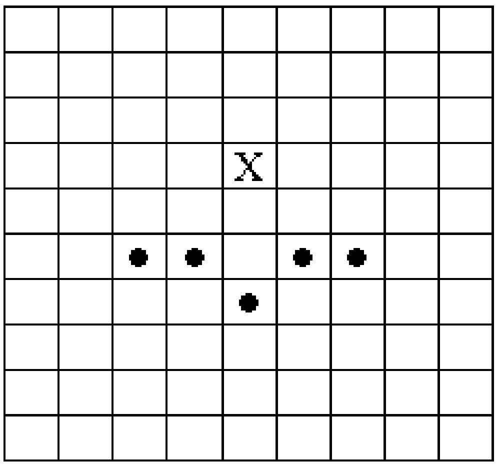

Fig. 5. The induction step. Each column represents an $(n-1)$-dimensional space.

Now, assume by contradiction $l(n) \geq 3n-1$. Consider the first position that is reached on level $l(n)$ to be the origin of the coordinates system. Consider the potential $\phi(i_1,\ldots,i_n) = \omega^{\sum_{k=1}^{n}|i_k|}$ of a position. Remember that ω is the positive root of the equation $\omega^2+\omega = 1$. The potential of the whole board is given by $\phi = \sum_{(i_1,\ldots,i_n)} \phi(i_1,\ldots,i_n)$. By an argument similar to that for Solitaire Army we get that ϕ decreases or stays constant when one moves. We want to be able to bound the initial potential. In order to do this, consider the quantity

$$S(n) = \sum_{i_1,\ldots,i_n} \omega^{\sum_{k=1}^{n}|i_k|} = S(n-1)\cdot\sum_{i_n}\omega^{|i_n|} = S(n-1)\cdot\left(1+\frac{2\omega}{1-\omega}\right) =$$
$$= S(n-1)\cdot\frac{1+\omega}{1-\omega} = S(n-1)\cdot\frac{\frac{1}{\omega}}{\omega^2} = S(n-1)\cdot\omega^{-3}.$$

Because $S(1) = \frac{1+\omega}{1-\omega} = \omega^{-3}$, we infer that $S(k) = \omega^{-3k}$ for all $k \geq 1$. Now, the half space determined by S contains levels at distance at least $l(n)$ from the origin, each such level being an $(n-1)$-dimensional space. Therefore, we have

$$\phi_i \leq \sum_{k\geq l(n)} S(n-1)\cdot\omega^k = S(n-1)\cdot\omega^{l(n)}\cdot\frac{1}{1-\omega} =$$
$$= \omega^{l(n)-3(n-1)-2} \leq \omega^{3n-1-3(n-1)-2} = 1.$$

Since we have only a finite number of pegs in the initial configuration, we infer that $\phi_i < 1 = p(0,\ldots,0) \leq \phi_f$. However, we proved that ϕ cannot increase, so we just reached a contradiction. That proves that our assumption that $l(n) \geq 3n-1$ is false. That gives $l(n) \leq 3n-2$. This concludes the proof of the theorem. □

In particular, we know that $6 \leq l(3) \leq 7$. From the above proof, we also know how to construct a configuration to reach the sixth level. However, in order to reach the seventh level, we have to use an initial board with at least 193 pegs. This can be easily proved by showing that the largest potential of 192 different initial positions is less than 1.

3.4. SOLITAIRE ARMY ALLOWING DIAGONAL MOVES

This game is similar to Solitaire Army in two dimensions. Now, one is allowed to make horizontal, vertical or diagonal jumps over one peg provided that the peg which jumps lands at an empty position. This game can be also generalized to a board in a space with n dimensions, but this is beyond the scope of this paper. In this section we prove a theorem that bounds the level (above the initial line) that one can reach in this game.

THEOREM 4. *It is impossible to send a peg nine levels above the base line.*

Proof. Suppose by contradiction that this is possible. Now, let P be the first position on level nine that one reaches. Consider the following labelling of the board: label P with 1, then label all its neighbors with ω, then label all the unlabelled positions adjacent to these ones with ω^2 and so on. Figure 6 shows how this labelling was done.

				ω^4				
	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	
	ω^3	ω^2	ω^2	ω^2	ω^2	ω^2	ω^3	
	ω^3	ω^2	ω	ω	ω	ω^2	ω^3	
ω^4	ω^3	ω^2	ω	1	ω	ω^2	ω^3	ω^4
	ω^3	ω^2	ω	ω	ω	ω^2	ω^3	
	ω^3	ω^2	ω^2	ω^2	ω^2	ω^2	ω^3	
	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	ω^3	
				ω^4				

Fig. 6. The potential of a subset of the board when we allow diagonal moves.

As in the previous proofs, the potential of a board is the sum of the potentials (labels) of the positions that hold a peg on them. To prove that the potential never increases, we have to analyze the four types of moves below.

a) The positions involved (if all non-empty) have potential $\omega^i, \omega^{i+1}, \omega^{i+2}$. Then the decrease in potential is at least 0.

b) The positions involved (if all non-empty) have potentials $\omega^i, \omega^i, \omega^{i+1}$. Then the decrease in potential is at least ω^{i+1}.

c) The positions involved (if all non-empty) have potentials $\omega^i, \omega^i, \omega^i$. Then the decrease in potential is exactly ω^i.

d) The positions involved (if all non-empty) have potentials $\omega^{i+1}, \omega^i, \omega^{i+1}$. Then the decrease in potential is exactly ω^i.

To bound ϕ_i, note that the nth line (P is on line 0) has the form $\ldots, \omega^{n+2}, \omega^{n+1}, 2n+1$ times $\omega^n, \omega^{n+1}, \omega^{n+2}, \ldots$. We thus get

$$\phi_i \leq \sum_{n\geq 9}\left((2n+1)\cdot\omega^n + \frac{2\omega^{n+1}}{1-\omega}\right) =$$

$$= 2\cdot\frac{9\cdot\omega^9\cdot(1-\omega)+\omega^{10}}{(1-w)^2} + \frac{\omega^9}{1-\omega} + \frac{2\omega^{10}}{(1-\omega)^2} =$$

$$= 19\cdot\omega^7 + 4\cdot\omega^6 = 0.877 < 1.$$

We infer that $\phi_i < 1 = \phi(P) \le \phi_f$ which is obviously a contradiction to the fact that ϕ never increases. $\square$

Allowing longer jumps does not modify the conclusion of the above theorem. The proof is just a case analysis and it is similar to the analysis of longer jumps in Solitaire Army. We will give a rigorous proof of this fact when we analyze Solitaire Army on arbitrary graphs.

3.5. SOLITAIRE ARMY ALLOWING ONLY DIAGONAL MOVES

The game is similar to the previous one, but only diagonal moves are allowed.

THEOREM 5. *It is impossible to send a peg seven levels above the base line.*

Proof. Suppose by contradiction that this is possible. Now, let P be the first position on level seven that one reaches. We use the same potential as in the previous problem. We also proved in the previous subsection that the potential never increases (we proved for all types of moves, in particular for diagonal moves).

Note that if we color the board as a chessboard, then a diagonal move preserves the color and therefore we have two different, independent games (one on white positions and one on black positions). Therefore, it makes sense to consider that P is a white position and to only analyze what happens on the white positions of the board. The white positions on line n (line 0 is the line on which P lies) have the labels $\ldots, \omega^{n+4}, \omega^{n+2}, n+1$ times ω^n, $\omega^{n+2}, \omega^{n+4}, \ldots$. We thus get

$$\begin{aligned}\phi_i &\le \sum_{n\ge 7}\left((n+1)\cdot\omega^n + \frac{2\omega^{n+2}}{1-\omega^2}\right) = \\ &= \frac{8\cdot\omega^7\cdot(1-\omega)+\omega^8}{(1-w)^2} + \frac{2\omega^9}{(1-\omega^2)\cdot(1-\omega)} = \\ &= 2\cdot\omega^6 + 8\cdot\omega^5 + \omega^4 = 0.978 < 1.\end{aligned}$$

Therefore, $\phi_i < 1 = \phi(P) \le \phi_f$ which is obviously a contradiction to the fact that ϕ never increases. This completes the proof. $\square$

3.6. SOLITAIRE ARMY ON TRIANGULAR BOARDS

A triangular board is represented by a triangle split in smaller triangles similar with the initial one. The pegs stay at the nodes of the board. For the sake of simplicity consider the base of the triangle to be horizontal. We can label each node on this grid with the level it belongs to, i.e., label the top vertex with 0, then label the two vertices on the next level with one, and so on. Initially, all pegs are placed at nodes on level l and below. A move is allowed when a peg can jump over a neighbor and land at an empty position. In this case, the second peg is removed from the board.

THEOREM 6. *If $l \geq 7$, it is not possible to make a peg reach the top of the big triangle.*

Proof. Let us assume by contradiction that one can send a peg to the top of the big triangle. We construct a potential function that will help us reach a contradiction. For a node P, the potential of that node is defined as $\phi(P) = \omega^{level(P)}$ if P has a peg on it and $\phi(P) = 0$ otherwise. The total potential ϕ of the current board is the sum of the potentials of the occupied nodes ($\phi = \sum_p \phi(P)$). Figure 7 shows the potential of a subset of the board, as described.

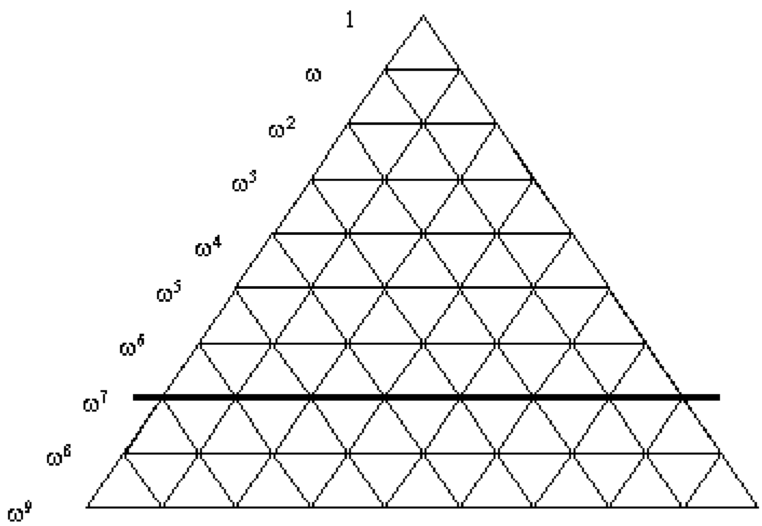

Fig. 7. The potential of some positions on a triangular board.

Consider again ω to be the positive root of $\omega^2 + \omega = 1$. We want to prove that the potential does not increase when one moves. We can have four possible types of moves:

a) The three positions involved all have potential ω^i. Then the decrease in potential is just $\omega^i > 0$.
b) The three positions involved have potentials ω^i, ω^{i+1} and ω^{i+2}. Then the decrease in potential is at least 0.

c) The three positions involved have potentials ω^i, ω^{i+1} and ω^{i+1}. Then the decrease in potential is at least $\omega^i(2w-1) > 0$.
d) The three positions involved have potentials ω^i, ω^i and ω^{i+1}. Then the decrease in potential is at least $\omega^{i+1} > 0$.

Now that we proved that the potential cannot increase after one move, let us compute the potential of the initial board. The initial potential is bounded by the potential of the full initial board. The nth level has total potential $(n+1)\cdot\omega^n$. We sum up the above quantity for $n \geq l$ and obtain the inequality

$$\phi_i \leq \sum_{n\geq l}(n+1)\cdot\omega^n = \frac{(l+1)\cdot\omega^l\cdot(1-\omega)+\omega^{l+1}}{(1-w)^2} = (l+1)\cdot\omega^{l-2}+\omega^{l-3}.$$

It is easy to see that the above quantity is a decreasing function of l (for $l \geq 7$ the derivative is negative) and for $l = 7$ this quantity is less than 1 and therefore $\phi_i < 1$ for all $l \geq 7$. However, $\phi_f \geq \phi(\text{Top}) = 1$ and therefore $\phi_i < \phi_f$. This is obviously a contradiction because we proved before that ϕ never increases when one moves. This concludes the proof of the fact that one cannot send a peg to the top vertex of the big triangle. □

For any $l \leq 6$ it is possible to send a peg to the top of the board. It is easy to see that for $l \in \{0, 1, 2, 3\}$. For $l = 5$ and $l = 6$ the configurations are quite complicated (they can be found in [12], along with a proof of the above theorem). Figure 8 shows an initial configuration for $l = 4$ such that one peg can reach the top

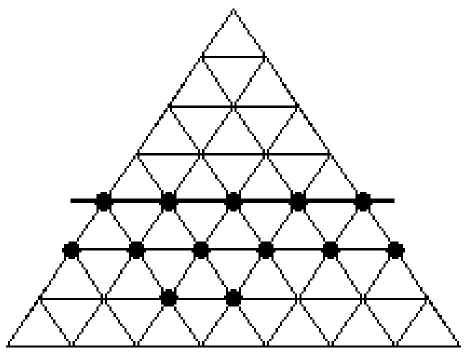

Fig. 8. Initial configuration allowing one peg to reach the top.

3.7. SOLITAIRE ARMY ON HEXAGONAL BOARDS

On a hexagonal board the pegs are placed at the nodes of the board and each move is a jump of a peg over another peg, the last one being removed from the board. Note that, after a move is made, a peg can land at six different positions in this game. For this game we know that the maximum reachable level above the initial line is four. In order to see this, we first have to prove the following result.

THEOREM 7. *It is not possible to send a peg five levels above the base line.*

Proof. Let us assume by contradiction that one can send a peg to the fifth level. Denote by P the first position reached on level five above the base line. Now, label the nodes as follows: label P with 1, then label the node below it with ω and the node below this one with ω^2, and so on. Now, on line n (line 0 is the line that contains P) we have a position labelled with ω^n. On line n we label the positions to the left of ω^n with $\omega^{n+1}, \omega^{n+2}, \ldots$ in this order (from right to left). Then we proceed similarly with the positions to the right of ω^n. The labelling is best seen in Figure 9 below.

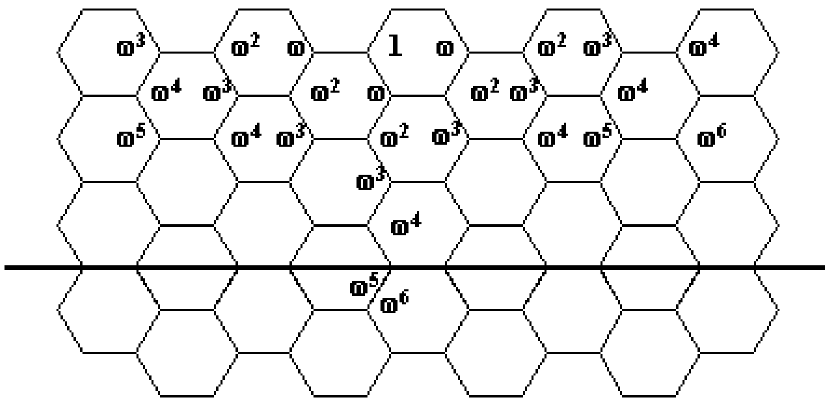

Fig. 9. The potential of a subset of a hexagonal board.

Remember that ω is the positive root of $\omega^2 + \omega = 1$. The potential ϕ of a board is defined as the sum of the labels of the occupied nodes. We want to prove that the potential does not increase when one moves. We can have three possible types of moves:

a) The three positions involved have potentials ω^i, ω^{i+1} and ω^{i+2}. Then the decrease in potential is at least 0.
b) The three positions involved have potentials ω^i, ω^{i+1} and ω^i. Then the decrease in potential is exactly $\omega^{i+1} > 0$.

c) The three positions involved have potentials ω^{i+1}, ω^i and ω^{i+1}. Then the decrease in potential is exactly $\omega^i > 0$.

Now that we proved that ϕ cannot increase, we have to bound the initial potential. On line n the sum of the labels is $\omega^n + \frac{2\cdot\omega^{n+1}}{1-w} = \omega^n \cdot \frac{1+\omega}{1-\omega} = \omega^n \frac{\frac{1}{\omega}}{\omega^2} = \omega^{n-3}$. Since the potential is defined to be the label if the position is occupied and 0 otherwise, we get that

$$\phi_i \leq \sum_{n \geq 5} \omega^{n-3} = \frac{\omega^2}{1-\omega} = 1.$$

Since the initial configuration has a finite number of pegs, we get $\phi_i < 1 = \phi(P) \leq \phi_f$ and this represents a contradiction with the fact that ϕ never increases. This proves one cannot send a peg on level five above the base line. □

Starting with the board in Figure 10 below, one can send a peg to the fourth level. This initial board has 21 pegs on it. We do not know if this is the minimum number of pegs on the initial board such that one peg can reach level four.

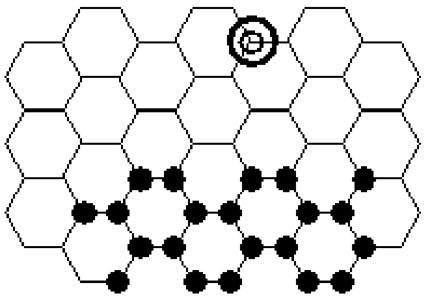

Fig. 10. Initial configuration allowing one peg to reach the fourth level.

4. SOLITAIRE ARMY ON GRAPH BOARDS

Solitaire Army as well as all the games presented above have a common feature: the board can be represented as a graph and the moves are made along the edges of this graph. In this section we will describe this more general setting. This will help understand how potential functions were chosen for the above games.

The more general setting is as follows: the board is an undirected connected graph $G(V, E)$ and the set of positions (holes) is V. The initial set of positions is a given set of vertices, $V_1 \subset V$. All pegs are initially on positions from V_1. There is also given a target vertex t which the player must reach in order to win. The moves: a peg at position x can jump over a peg at position y and land at position z provided that $(x, y) \in E$, $(y, z) \in E$ and z was empty before the move. The peg at position y is removed from the board. We need a potential function ϕ such that the following conditions are met:

a. ϕ is always positive;

b. ϕ cannot increase after one makes a move.

Denote by $d(x)$ the shortest distance between x and t in G and define $\phi(x) = \omega^{d(x)}$ to be the label of a given position on the board. Here ω is the positive root of the equation $\omega^2 + \omega = 1$; ω was chosen this way in order to be able to prove b. This is not an artificial choice, as it will be seen from the next proof. The potential ϕ of the board is the sum of the labels of occupied positions. Obviously, condition a. holds because ω is positive. In order to prove b. note that if x and y are adjacent vertices of G, then $\mid d(x) - d(y) \mid \leq 1$. This happens because of the triangle inequality for distances in a connected graph. Now, let us see what happens after a peg at position x jumps over a peg at position y and lands at position z (where $(x, y) \in E$, $(y, z) \in E$ and z was empty before the move). The increase in potential is exactly $\omega^{d(z)} - \omega^{d(x)} - \omega^{d(y)} \leq \omega^{d(z)} - \omega^{d(z)+2} - \omega^{d(z)+1} = 0$. Therefore, the potential of the board never increases.

Now that we proved that ϕ verifies both a. and b., it is clear how we chose the potentials for the games in the previous section. For example, on a rectangular grid, the distance between two nodes is just the Manhattan distance on that grid and that is why in Solitaire Army we chose the label of a position to be $\omega^{|i|+|j|}$. Another example: for triangular boards, the distance from the top vertex to any other vertex x is just the level (below the top) of x. For hexagonal grids we used a slightly different potential, but that was just for the sake of simplicity.

Lemma 8. *Even if we allow longer jumps, ϕ verifies condition* b.

Proof. First, let us make clear what a longer jump means. This is a jump of a peg at position x over pegs at positions $y_1, \ldots, y_n$ $(n \geq 2)$ provided that it lands at the empty position z. The condition is that $x, y_1, \ldots, y_n, z$ (in this order) form a chain. Obviously, the increase in potential is $\omega^{d(z)} - \omega^{d(x)} - \sum_{i=1}^{n} \omega^{d(y_i)} \leq \omega^{d(z)} - \omega^{d(y_{n-1})} - \omega^{d(y_n)} \leq \omega^{d(z)} - \omega^{d(z)+2} - \omega^{d(z)+1} = 0$. We used property a. and the distance argument used previously. This completes the proof of the fact that ϕ never increases even if we allow longer jumps. □

How do we apply this theory? As one can see, the proofs we did were proofs by contradiction. In order to prove that one cannot send a peg to the target position t, it is enough to prove that $\phi_{initial} < 1$. Because $\phi_f \geq \phi(t) = 1$, we will get a contradiction to the fact that ϕ never increases.

5. OTHER POTENTIAL FUNCTION

Different potential functions were tried, but only the ones yielding the best results have been presented so far. In this section, other potential functions are presented along with their disadvantages.

a) In Solitaire Army, consider the labelling $\phi(i,j) = f(|i|) + f(|j|)$, where f is a positive function. The corresponding potential cannot be bounded because $f(|i|)$ appears infinitely many times on each line i.

b) In the same game, for the labelling $\phi(i,j) = \omega^{|i| \cdot |j|}$, the corresponding potential can increase after one makes a move.

c) In Solitaire Army we can get a bound using a grid labelling of the form $\phi(i,j) = \frac{1}{(|i|+|j|+c)^3}$, where c is a well chosen constant. However, this bound is much worse than 5, which is the best possible bound.

d) In Solitaire Army, for the labelling of the form $\phi(i,j) = \frac{1}{|i|^{|j|}}$, the corresponding potential can increase after one makes a move.

e) For Solitaire Army when diagonal moves are allowed, if we use the same potential as in Solitaire Army, but ω is the positive root of $\omega^4 + \omega^2 = 1$, we obtain a bound which is worse than what we already proved. Wrinkler [14] presents a proof based on this potential function.

f) The hexagonal grid can be viewed as a subgrid of a rectangular grid for which we can apply the potential described in Solitaire Army, where ω is taken to be the root of the equation $\omega^3 + \omega^2 = 1$. We obtain a bound two times worse than the best bound.

6. THE REPULSION GAME

In this section, we will present a game called REPULSION. In this one-person game, initially n pegs are placed at n different positions on an infinite rectangular grid. Whether we consider the positions to be the nodes or the cells of the grid is equivalent. A move is allowed when there exist four consecutive positions on a row (or column) such that the two middle ones hold a peg each and the other two are empty. The goal is to be able to make as many moves as possible. An initial position and the legal moves are presented in Figure 11.

Another way to look at this game is to see the pegs as particles that repel each other when they come too close. Then the system tends to inflate with each repulsion and eventually it seems natural to reach an equilibrium where no repulsions are possible. The theorem below describes this result.

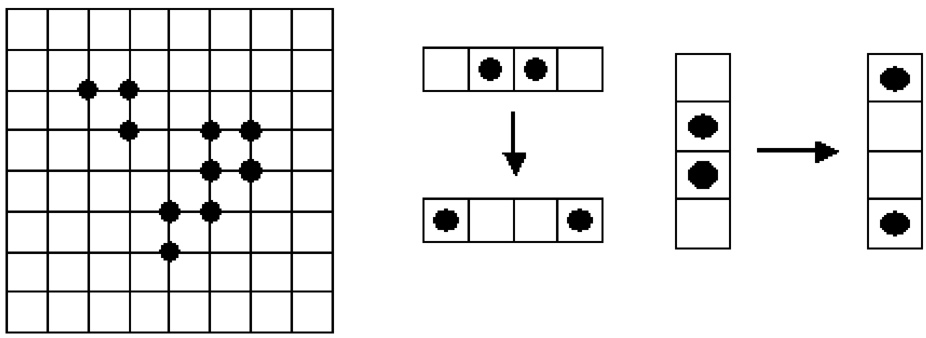

Fig. 11. REPULSION: a possible configuration and the legal moves.

THEOREM 9. *There exists a number a_n such that from every initial configuration with n pegs one cannot move more than a_n times.*

Proof. We will prove the theorem by induction on n. Obviously we can take $a_1 = 0$ and $a_2 = 1$. Assume that the statement of the theorem is true for all $k \leq n-1$. Take $b = \binom{n}{2} \cdot (n + \sum_{k=1}^{n-1} a_k)$. We will prove that no more than $b + 2 \cdot \sum_{k=1}^{n-1} a_k$ moves can be made starting with a configuration of n pegs.

Consider the potential ϕ to be the sum of Manhattan distances between pairs of pegs (each pair is counted once). As it can be very easily observed, every repulsion increases this potential by at least two units. We have two possibilities:

(i) The game ends in at most b moves no matter what the initial configuration (of n pegs) is and no matter what moves one makes. Then we can take $a_n = b$ and we are done.

(ii) There exists an initial configuration with n pegs and a sequence of at least $b+1$ moves that can be made starting from that configuration. Denote by ϕ_b the potential after the bth move. As we noted before, ϕ increases by at least two units after each repulsion and therefore $\phi_b > 2b$ (the initial potential is more than $\binom{n}{2}$). By Dirichlet's Principle, after the bth move there exist two pegs x and y such that $d(x,y) > 2 \cdot (n + \sum_{k=1}^{n-1} a_k)$, where d stands for the Manhattan distance.

Consider the algorithm:

1. Start with $k = 1$, $i = 1$ and $C_0 = \{x\}$.

2. Let C_i be the region that contains C_{i-1} and all positions at distance at most $2 + a_k + a_{n-k}$ from C_{i-1}.

3. If $C_i \setminus C_{i-1}$ does not contain any pegs or C_i contains all pegs, then STOP.

4. Let k be the number of pegs in C_i, let $i = i + 1$ and REPEAT Step 2.

Obviously, this algorithm ends after going through Step 2. at most n times (because then it will exhaust the whole set of pegs). However, if it ends with C_i containing all pegs, that means

$$d(x, y) \leq 2 \cdot \left(n + \sum_{k=1}^{n-1} a_k \right),$$

which is a contradiction. Therefore, the last C_i contains $1 \leq k \leq n - 1$ pegs. Because the algorithm ended, all these k pegs are also in C_{i-1} It follows that the other $n - k$ pegs are outside C_i and at distance at least $a_k + a_{n-k} + 2$ from C_{i-1}. Now, we apply the induction hypothesis and we get that at most $a_k + a_{n-k}$ additional moves can be made in total in these two sets of pegs. Even if we do all these moves, the two sets cannot interact with each other because of the fact that all $n - k$ pegs were at distance at least $a_k + a_{n-k} + 2$ from the other k pegs. In conclusion, the total number of moves can be bounded by

$$b + \max_k \{a_k + a_{n-k}\} \leq b + 2 \cdot \sum_{k=1}^{n-1} a_k.$$

From the analysis of the above two cases, one can infer that the number of moves is bounded by

$$b + 2 \cdot \sum_{k=1}^{n-1} a_k$$

and therefore we can take

$$a_n = b + 2 \cdot \sum_{k=1}^{n-1} a_k.$$

This completes the induction step. The proof of the theorem is now complete. □

7. FUTURE WORK

The results presented in the previous sections suggest some obvious directions for future work. Maybe, the most important improvement that can be done is to tighten the bounds in the proved theorems, perhaps even finding exactly the farthest position that can be reached from an initial configuration. Once this position is found, another question that can be asked is: what is the minimum number of pegs on the initial board that would make that position reachable?

Most puzzles presented in this paper are particular cases of Solitaire Army on boards which are planar graphs. In Section 4 we showed that the defined potential of a graph board does not increase when one moves. However, in order to prove that a position cannot be reached, we still need to show that the initial potential is less than the final potential. It would be interesting to find some sufficient conditions for this inequality to be satisfied.

In Theorem 9 we showed that, starting from any initial configuration with n pegs, one cannot make more than a_n moves. However, this upper bound for the number of possible moves has complexity higher than exponential. It would be interesting to see if one can find the maximum number b_n of moves that can be done in this game, starting from any configuration with n pegs. We conjecture that b_n is a polynomial function. Note that a_n does not depend on the dimensionality of the problem. Thus, one may also be interested in finding better bounds when the dimensionality of the problem is fixed.

8. CONCLUSIONS

We reviewed Solitaire Army game along with some of its already studied variations. In addition, some new and interesting extensions were presented. These variations included Solitaire Army on different shape boards (hexagonal, multidimensional, graph boards) and Solitaire Army with non-classical moves (diagonal moves, longer jumps).

For each such variant of the game, we tried to show how far from the initial position one peg can reach during the game. For some of these problems, the exact maximum distance was found while for others only bounds were presented. For example, in the case of three-dimensional Solitaire Army, the maximum reachable level is either six or seven, but in order to be seven, we showed that one should start from an initial configuration with at least 193 pegs.

At the end of the paper, a new and interesting puzzle, similar to Solitaire Army, was described. For this new one-person game we proved that it will end in a finite number of moves that depends on the initial number of pegs on the board, and does not depend on the dimensionality of the board.

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