ON THE CONVERGENCE OF A SEQUENCE

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We consider the sequence defined by : $a_0 = \sqrt{n}$ where n is a natural number and not a perfect square and $a_{k+1} = \frac{1}{\{a_k\}}$ for all $k \geq 0$ ($\{x\}$ is the fractional part of the real number x). We want to find n so that the sequence is convergent.
We shall prove some properties of the given sequence that will help us achieve our goal. The following Lemma was proved in ([1]).

LEMMA1 *If n is a natural number and not a perfect square, then, for every natural number k, there exist u_k, z_k natural numbers, $z_k \neq 0$, so that $a_k = \frac{u_k+\sqrt{n}}{z_k}$. Moreover, $z_k z_{k+1} + u_{k+1}^2 = n$.*
Proof. In order to prove this lemma we shall use the induction on k.
For $k = 0$ we have $a_0 = \sqrt{n} = \frac{0+\sqrt{n}}{1} > 1$, with $u_0 = 0, z_0 = 1$. Moreover, $z_0 | (n - u_0^2), u_0 \leq [\sqrt{n}]$.
Let b_k be the integer part of a_k(i.e. $b_k = [a_k]$).
Assuming $a_k = \frac{u_k+\sqrt{n}}{z_k} > 1$, where $u_k, z_k \in N, z_k \neq 0$ and moreover
$z_k | (n - u_k^2), u_k \leq [\sqrt{n}]$ for a specified $k \geq 0$, we get: $a_k \neq b_k$(because $\sqrt{n}$ is not an integer) and

$$a_{k+1} = \frac{1}{a_k - b_k} = \frac{1}{\frac{\sqrt{n}+u_k-b_k z_k}{z_k}} = \frac{\sqrt{n} + b_k z_k - u_k}{\frac{n-(u_k-b_k z_k)^2}{z_k}}.$$

From the definition of b_k we get $0 < a_k - b_k < 1$ which is equivalent to $a_{k+1} > 1$.
On the other hand we have $b_k = [a_k] = \left[\frac{u_k+[\sqrt{n}]}{z_k}\right]$. Then $u_k + [\sqrt{n}] = b_k \cdot z_k + t$, where $0 \leq t < z_k, t \in N$. From the induction hypothesis $u_k \leq [\sqrt{n}]$ and then we get $2[\sqrt{n}] \geq b_k z_k + t > t(b_k + 1)$. From $a_k > 1$ we have $b_k = [a_k] \geq 1$. Then we obtain

$$2[\sqrt{n}] > 2t \quad \textit{or} \quad [\sqrt{n}] - t > 0. \tag{1}$$

Using the above results, we have

$$\begin{aligned} u_{k+1} = {} & b_k \cdot z_k - u_k = [\sqrt{n}] - t \\ z_{k+1} = {} & \frac{n-(u_k-b_k z_k)^2}{z_k} = \frac{n - (t - [\sqrt{n}])^2}{z_k} = \frac{n - u_{k+1}^2}{z_k}. \end{aligned} \tag{2}$$

From (1) and (2) we have $u_{k+1} > 0$ and $u_{k+1} \in N$. Moreover, $z_{k+1} = \frac{n-u_k^2}{z_k} + 2u_kb_k - b_k^2z_k$. From the induction hypothesis $z_k|(n-u_k^2)$, and then we have that $z_{k+1} \in Z$. On the other hand $z_{k+1} = \frac{n-u_{k+1}^2}{z_k} > 0$ $(0 \leq [\sqrt{n}] - t \leq [\sqrt{n}])$, so $z_{k+1} \in N$. From $z_{k+1} \in N$ and $z_{k+1} = \frac{n-u_{k+1}^2}{z_k}$ we get that $z_{k+1}|(n-u_{k+1}^2)$ and $z_kz_{k+1} + u_{k+1}^2 = n$. The proof by induction is now complete. □

LEMMA2 *The sequence $(a_k)_{k\geq 0})$ is periodical and the length of the period is less than $2n$.*

Proof. Let u_k, z_k be the numbers defined in Lemma1. From Lemma1 we have $a_k = \frac{u_k+\sqrt{n}}{z_k}$ with u_k, z_k natural numbers, $z_k \neq 0$ and $z_{k+1}z_k + u_{k+1}^2 = n$. It is easy to see that $u_{k+1} \leq \sqrt{n}$ for all $k \geq 0$. From the definition of $(a_k)_{k\geq 0}$ we have $a_k > 1$ for all $k \geq 0$ which is equivalent to $z_k < u_k + \sqrt{n}$. That means $z_k < 2\sqrt{n}$ for all $k > 0$. For $k = 0$ we have $u_0 = 0 < \sqrt{n}$ and $z_0 = 1 < 2\sqrt{n}$. Because both u_k and z_k are natural numbers, the number of possible pairs $(u_k, z_k)_{k\geq 0}$ is less than $2n$. From Dirichlet's principle, there exist s, l natural numbers, $l > s > 0$ with $l - s < 2n$ and $(u_s, z_s) = (u_l, z_l)$. This is equivalent to $a_s = a_l$. Thus, the sequence $(a_k)_{k\geq 0}$ is periodical and the length of the period is $l - s < 2n$. □

Next we prove the main result of this paper.

THEOREM *The sequence $(a_k)_{k\geq 0}$ is convergent if and only if $n-1$ is a perfect square and $n > 1$.*

Prof. First, let us suppose that $n-1$ is a perfect square and $n > 1$. Then there exists x natural number so that $n = x^2+1$ and $x > 0$. We have $a_0 = \sqrt{x^2+1}$, $[a_0] = x$, $a_1 = \frac{1}{\sqrt{x^2+1}-x} = \sqrt{x^2+1}+x$, $[a_1] = 2x$, $a_2 = \frac{1}{\sqrt{x^2+1}+x-2x} = \sqrt{x^2+1}+x = a_1$. That means the sequence $(a_k)_{k\geq 0}$ is constant, begining with a_1. So, the sequence is convergent.

Now, let us suppose that the sequence is convergent. We have $n > 1$ because n is not a perfect square. From lemma2 we have that our sequence is periodical. So, $(a_k)_{k\geq 0}$ is convergent if and only if there exists k so that the sequence is constant begining with a_k. That means $a_{k+1} = a_k$ which is equivalent to $\{a_k\} = \frac{1}{a_k}$ which is equivalent to $a_k - \frac{1}{a_k} = [a_k] = b_k$. From lemma1 $a_k = \frac{u_k+\sqrt{n}}{z_k}$. This implies $b_k = \left(\frac{u_k}{z_k} + \frac{u_kz_k}{n-u_k^2}\right) + \sqrt{n}\left(\frac{1}{z_k} - \frac{z_k}{n-u_k^2}\right)$. Because n is not a perfect square we have $\frac{1}{z_k} = \frac{z_k}{n-u_k^2}$ and $\frac{u_k}{z_k} + \frac{u_kz_k}{n-u_k^2} = b_k$. It follows that $n = u_k^2 + z_k^2$ and $\frac{u_k}{z_k} + \frac{u_kz_k}{z_k^2} = b_k$.So, $2u_k = z_kb_k$.

We prove that $z_k = 1$. We shall prove by induction on i that $z_k|z_i$ and $z_k|2u_i$

for all $i \in \{0, 1, \cdots, k, k+1\}$. We proved this was true for $i = k$ and $i = k+1$(because $a_k = a_{k+1}$). Let us suppose that the statement above is true for $i, i+1, \cdots, k, k+1$. We have from lemma1:
$z_{i-1} = \frac{n-u_i^2}{z_i} = \frac{n-(b_i z_i - u_{i+1})^2}{z_i} = -b_i^2 z_i + 2b_i u_{i+1} + \frac{n-u_{i+1}^2}{z_i} =$
$= -b_i^2 z_i + 2b_i u_{i+1} + z_{i+1}$.
This implies that $z_k | z_{i-1}$ because $z_k | z_i$,$z_k | z_{i+1}$ and $z_k | 2u_{i+1}$ (from the induction step.) From the proof of lemma1 we also have $u_{i-1} = z_{i-1}b_{i-1} - u_i$ which is equivalent to $2u_{i-1} = 2z_{i-1}b_{i-1} - 2u_i$. Thus $z_k | 2u_{i-1}$ because $z_k | z_{i-1}$ and $z_k | 2u_i$ (from the induction step.)
The proof of the statement is now complete.So, we have $z_k | z_i$ and $z_k | 2u_i$ for all $i \in \{0, 1, \cdots, k, k+1\}$. As a particular case, for $i = 0$ we have $z_k | z_0$ where $z_0 = 1$ because $\sqrt{n} = \frac{0+\sqrt{n}}{1}$. So, $z_k = 1$ and $n = u_k^2 + 1$. Thus, $n - 1$ is a perfect square. The theorem is now completely proved. $\square$

References

[1] **Niculescu FL., R.** *On an algorithm of representing irational numbers using continuous fractions*