# Inverse Kinematics

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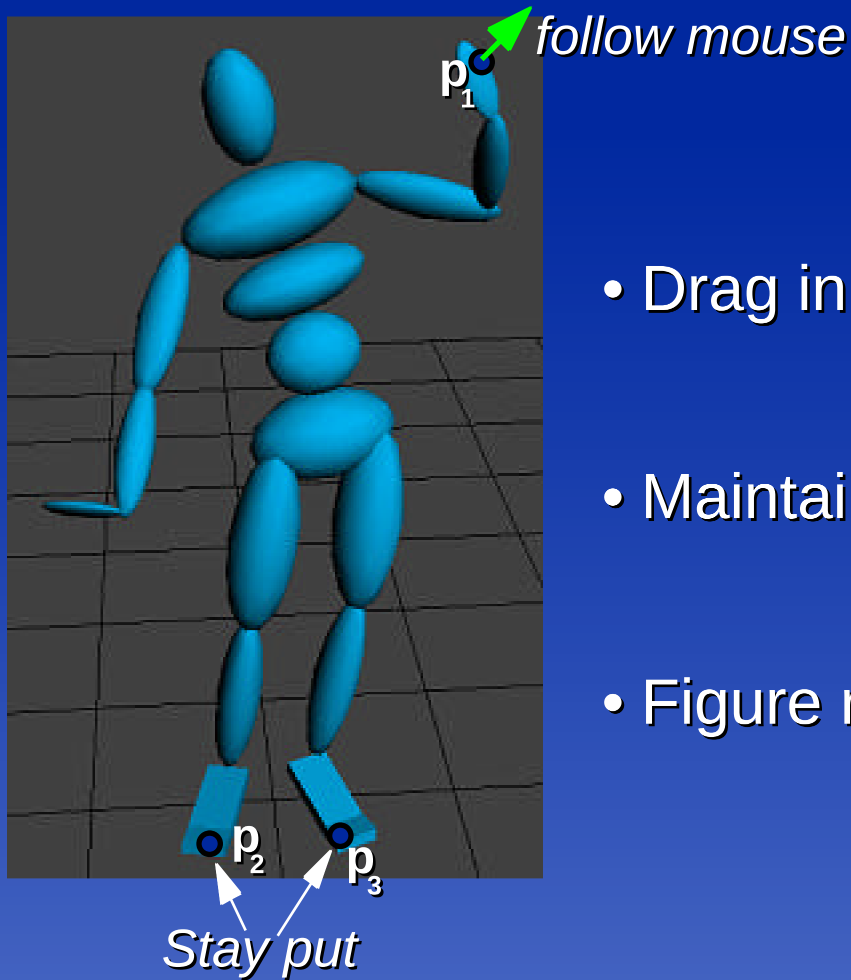
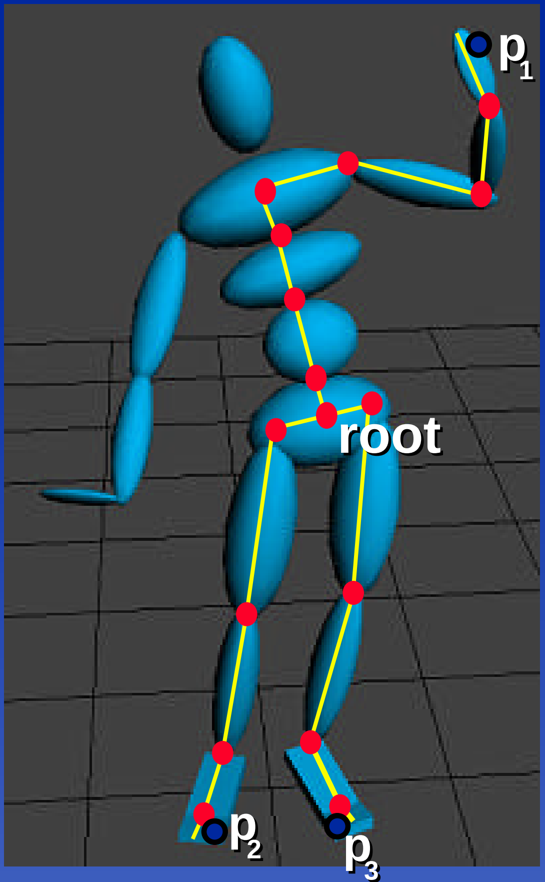
# Define the Problem




## Goals

- Drag in realtime with mouse
- Maintain multiple constraints
- Figure responds in “reasonable” ways

# ... Then Name the System!




Use standard transformation hierarchy

$$p_1 = T(x, y, z)R(\theta, \phi, \varphi)TR(\gamma)...TR(\sigma)\, p_b$$

$$q(t) = [\ \ |\ \ |\ \ |\ \ |\ \ |\ \ |\ \ |\ \ |\ \ ]\ ...\ [\ \ |\ \ |\ \ ]\ ...$$

$$p_1 = C(q(t))$$

# Idea #1

**Let**

$p_m(t) =$ 3D mouse positioner at time *t*

$C(q(t)) =$ position of hand at time *t*

**... then solve for $q$ such that**

$$C(q(t)) = p_m(t)$$

# Unfortunately...

$C(q(t)) = p_m(t)$ contains rotations, therefore nonlinear, and very difficult to solve

**Can be solved for:**

- articulated chains

**Has real problems with:**

- branching hierarchies
- multiple constraints
- convincing motion

# Idea #2

Let's differentiate!

$$\frac{\partial}{\partial t}\left(p_m(t) = C(q(t))\right)$$

$$\Downarrow$$

$$\dot{p}_m = \frac{\partial C}{\partial q}\,\dot{q}$$

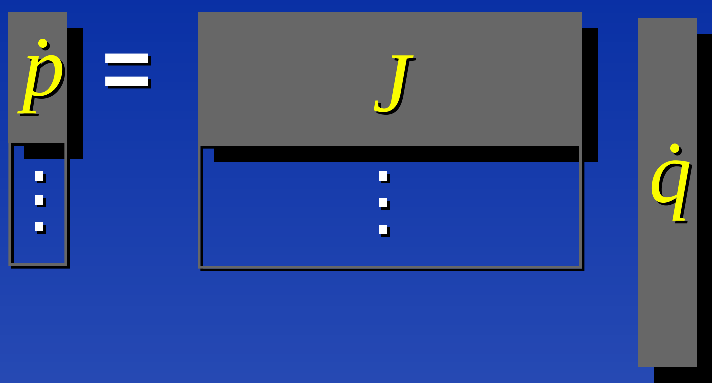
desired – actual

Jacobian $J$

state space velocity

# A Diffy Q by any other name...

$$\dot{p} = J\dot{q}$$




Underdetermined linear system
Can solve for $\dot{q}$

Iterate during interaction using:

$$q(t+\Delta t) = q(t) + \Delta t\,\dot{q}(t)$$

# Are we there yet?

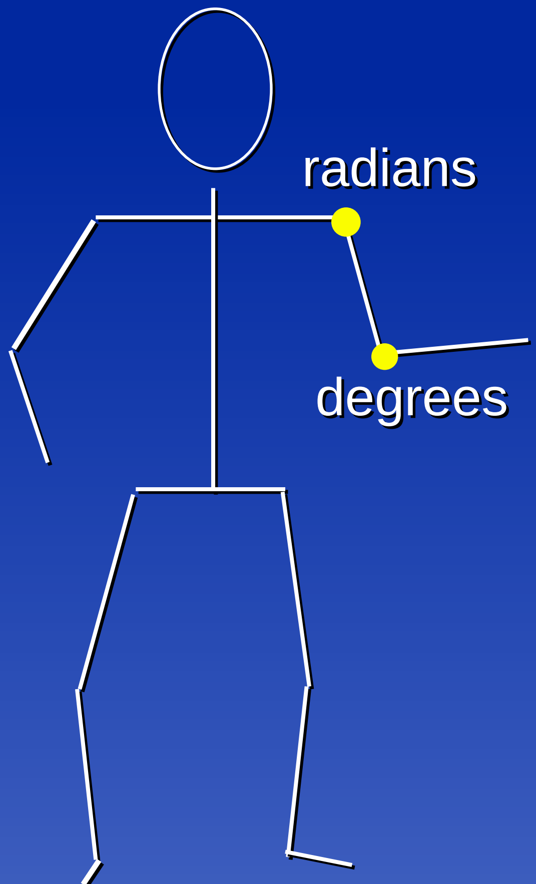
Alas, no.

We got *a* solution for $\dot{q}$

but it may not be a *reasonable* one.

**Why not?**

# Scaling




When there's a choice,
least squares solver always
moves shoulder before elbow

Measuring both angles in radians
still leaves figure stiff-armed

What can we do about this?

# Let’s get physical!

## Aristotlean physics:

$$f = Mv$$

*Why not* $f = Ma$ ?

$$f_G + f_C = M\dot{q}$$

Generalized Force

Constraint Force

Mass Matrix

# Blend on High for 2 minutes...

***Recall***

Constraints

$$p = C(q)$$

Solve

$$\dot{p} = J\dot{q}$$

Where

$$J = \frac{\partial C}{\partial q}$$

***Define:*** $W = M^{-1}$

$$f_G + f_C = M\dot{q}$$

*Lagrange Multipliers*

$$f_G + J^T\lambda = M\dot{q}$$

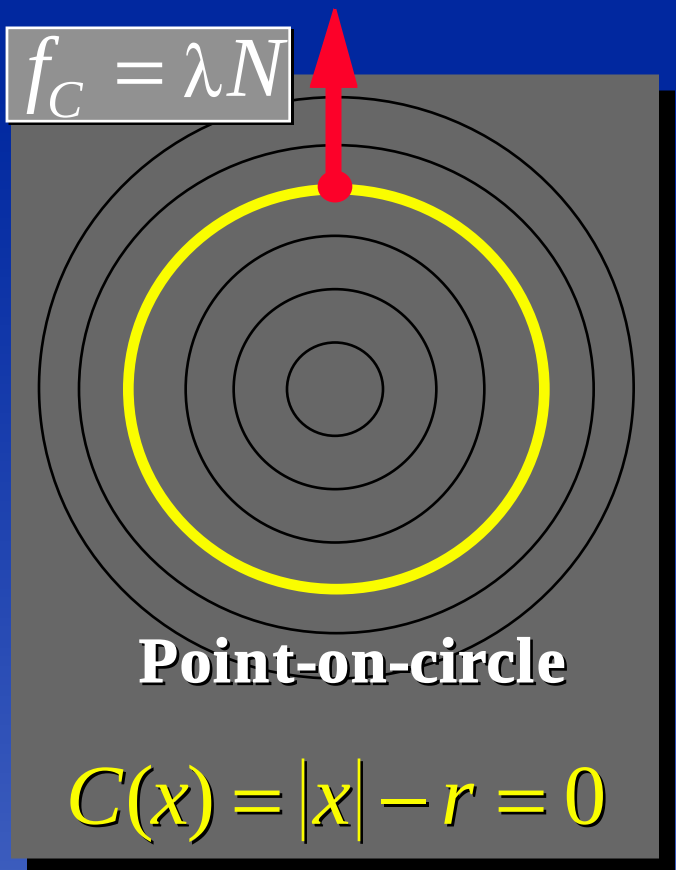
substitute

$$\dot{q} = W\left(f_G + J^T\lambda\right)$$

get

$$\dot{p} = JWf_G + JWJ^T\lambda$$

# Recalling Lagrange Multipliers




**Then...**

Assumption of *passive* constraints stipulated that $f_C$ point along constraint gradient $\frac{\partial C}{\partial x} = N$

**Now...**

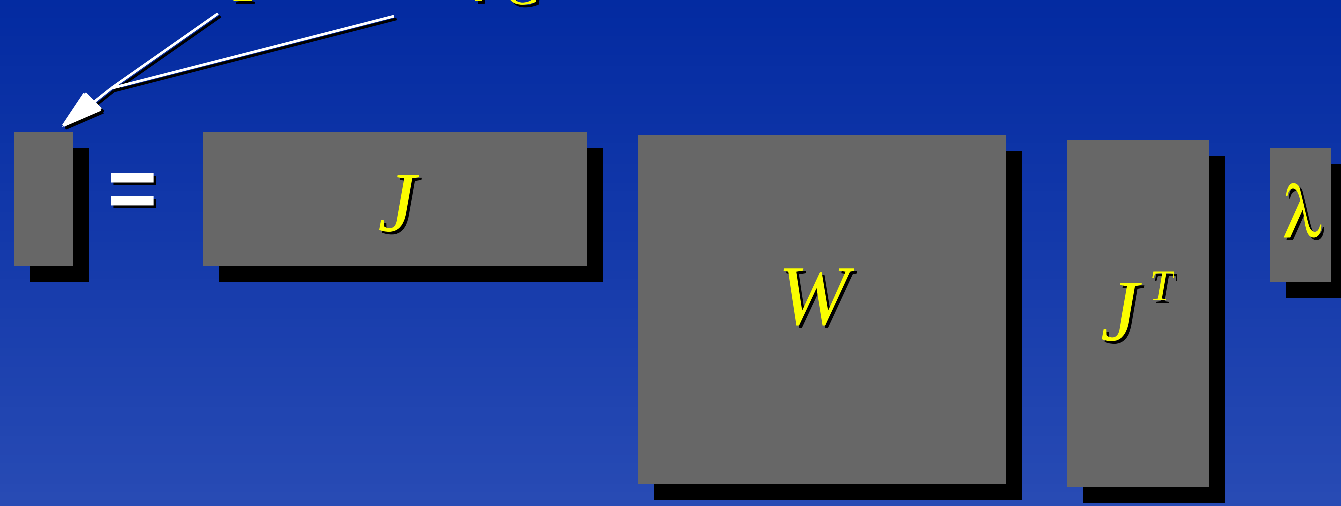
Constraint gradients: $\frac{\partial C}{\partial q} = J$

$f_C$ is a *linear combination* of constraint gradients:

$$f_C = J^T \lambda$$

# and wind up with:

$$\dot{p} - JWf_G = JWJ^T\lambda$$




Can solve for $\lambda$, plug into formula for $\dot{q}$

# Have we solved the scaling problem?

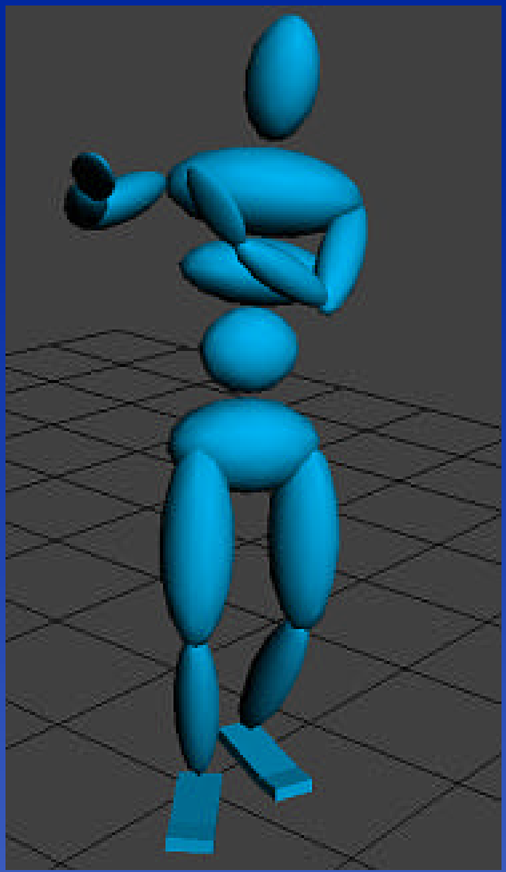
We get a velocity $\dot{q}$ that obeys

the "physical law" $f = Mv$ ...

$M$ determines how the system responds to applied forces.

So it all depends on what $M$ is.

# How well can we do?

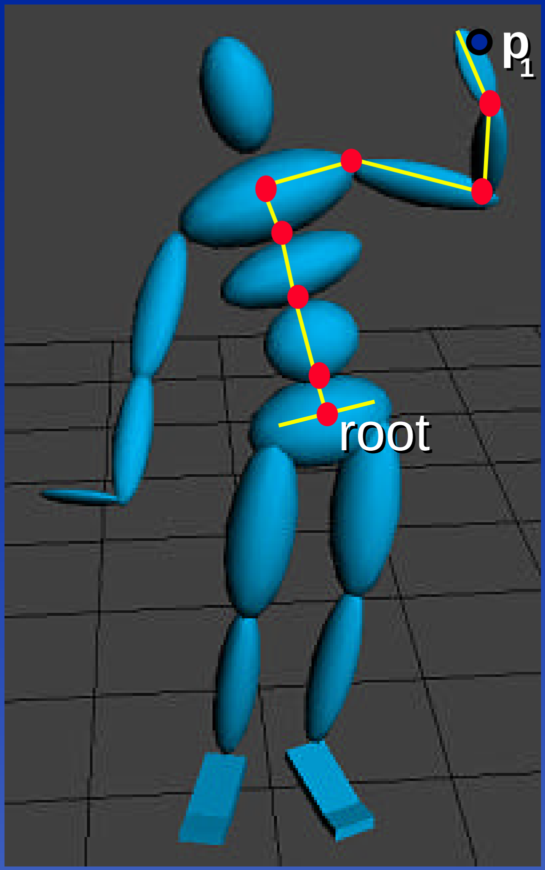


If approximate figure by rigid, uniform density, simple shapes...

Can derive and compute an $M(q)$ from minimizing kinetic energy of articulated figure

Minimum energy consumption is good!

# A word about J




$p_1$ only depends on ancestors in hierarchy

Therefore, $J$ is sparse. Exact pattern depends on ordering of $q$.

Can compute all derivatives efficiently in recursive tree traversal

# What's in the online notes

- More detailed derivations
- Formula for computing Mass Matrix
- Pseudocode for tree traversal
- Bibliography

# That's Just the Beginning...

- One sided constraints $\rightarrow$ joint limits
- Hierarchical constraints
- Spring "muscles"
- Build complex constraint functions for higher level behavior