Algorithm for Combinatorial Coalition Formation and Payoff Division in an Electronic Marketplace *

Cuihong Li
Graduate School of Industrial Administration
Carnegie Mellon University
5000 Forbes Avenue
Pittsburgh, PA 15213
cuihong@andrew.cmu.edu

Katia Sycara
The Robotics Institute
Carnegie Mellon University
5000 Forbes Avenue
Pittsburgh, PA 15213
katia@cs.cmu.edu

ABSTRACT

In an electronic marketplace, coalition formation allows buyers to enjoy a price discount for each item, and combinatorial auction enables buyers to place bids for a bundle of items that are complementary. Coalition formation and combinatorial auctions both help to improve the efficiency of a market, and they have received much attention from economists and computer scientists. But there has not been work studying the situations where both coalition formation and combinatorial auctions exist. In this paper we consider an e-market where each buyer places a bid on a combination of items with a reservation cost, and sellers offer price discounts for each item based on volumes. By artificially dividing the reservation cost of each buyer among the items, we can construct optimal coalitions with respect to each item. These coalitions satisfy the complementarity of the items by reservation cost transfers, and thus induce the optimal solution. We focus on the systems with linear price functions and present a polynomial-time algorithm to find a semi-optimal solution and a payoff division scheme that is in the core of the coalition. Simulation results show that the algorithm obtains a solution close to the optimal value.

1. INTRODUCTION

Coalition formation and combinatorial auctions have received much attention from both economists and computer scientists. Suppliers drive customers to buy in wholesale lots by offering price discounts. By forming coalitions customers can take advantage of the price discount without purchasing more than their real demand([14]). Auction is an efficient market mechanism to determine resource allocation. We say goods g and h are complementary to buyer b if $u_b(\{g,h\}) > u_b(\{g\}) + u_b(\{h\})$, where $u_b(G)$ is the utility of the set of goods G to b([3]). In combinatorial auctions, bidders can explicitly express the complementarity of items by placing bids on combinations of items([8]).

It is common in real markets that price discounts and complementarity among items exist simultaneously. In this condition, coalition formation and combinatorial auctions are both needed to improve the efficiency of the markets. For example, there are two buyers b_1 and b_2. They both need a cellular phone g_1 and a battery g_2. A customer must pay \$500 to buy one unit of g_1 or \$405 for each to buy two units. Also one needs to pay \$50 to buy one unit of g_2 or \$40 for each to buy two units. The utility of both of g_1 and g_2 for each buyer is \$450, while either g_1 or g_2 alone gives utility zero for both b_1 and b_2. Suppose only the mechanism of coalition formation is considered. b_1 and b_2 need to split the value of $\{g_1, g_2\}$ and bid for g_1 and g_2 separately. Assume $p_{b_1}(\{g_1\}) = \$405$, $p_{b_2}(\{g_1\}) = \$400$, $p_{b_1}(\{g_2\}) = \$45$, $p_{b_2}(\{g_2\}) = \$50$, where $p_b(G)$ is the bidding price of buyer b for a set of goods G. Then b_1 and b_2 only get g_2 since $p_{b_1}(\{g_1\}) + p_{b_2}(\{g_1\}) < \$810$, which is the cost for two units of g_1 by coalition. However, getting g_2 alone has no use for the customers, although they enjoy the price discount. On the other hand, suppose only the mechanism of combinatorial auctions is considered. Let $p_{b_1}(\{g_1, g_2\}) = p_{b_2}(\{g_1, g_2\}) = \$450$, which are the highest bidding prices they can afford. In this case neither b_1 nor b_2 can win their bids without coalition since each one needs to pay $\$550 > \$450$ for his request. Finally suppose both of the mechanisms are applied, then both b_1 and b_2 can win their bids and furthermore have \$10 profit together.

Although economists have provided much insight into the stability analysis of coalitions([7]) and the mechanism design of combinatorial auctions([2, 8]), both the determination of optimal coalition structure and stable payoff division in coalition formation problems, and the winner determination in combinatorial auction problems are computationally intractable. There are some discussions about the computational problems by computer scientists in each of these two fields, for instance, [12, 5, 13, 11] in coalition formation, and [3, 10, 15, 1, 9, 4] in winner determination in combinatorial auctions. But no work has been published to date that deals with both behaviors simultaneously.

In this paper we consider an electronic market in which both coalition formation and combinatorial auctions exist.

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Each buyer may want a bundle of items that complement each other and have a reservation cost, the highest cost he can pay for his request. Suppose a one-round sealed auction mechanism(the buyers submit sealed bids and the allocation and payment are determined at one round of bidding) is applied to the market and each buyer unstrategically places a bid that is equal to his reservation cost for the bundle of the items he desires. Assume a part of the bundle has value zero to the buyer[1]. Each seller has a price schedule for each item and offers price discounts in each transaction based on the quantity of the item that is sold. The larger the quantity, the lower the price. By forming coalitions buyers can enlarge the quantity in each transaction and take advantage of price discounts. Such a coalition formation problem is called a Combinatorial Coalition Formation(CCF) problem. Since buyers are self-interested a stable payoff division mechanism is needed so that no coalition members have incentive to leave the coalitions. In this paper we consider the problem of CCF and payoff division in such an electronic market. The objective is to find a subset of the buyers who have the maximum coalition value among all the possible coalitions and distribute the payoff among the coalition members such that the division is in the core[7].

In [16] an efficient algorithm of coalition formation and payoff division was given for the case where each buyer wants an XOR listing of multiple items within a category(for instance, a buyer wants to buy either a camera A for $300 or lower, or a camera B for $400 or lower). Based on the approach with respect to one item in [16], our solution to the CCF problem is as follows. By artificially dividing the reservation cost of each buyer among the items, we can construct optimal coalitions with respect to each item. We then try to make these coalitions satisfy the complementarity of the items by reservation cost transfers, and thus induce the optimal solution. We focus on the systems with linear price functions and present a polynomial-time algorithm to find a semi-optimal solution to CCF and a payoff division scheme that is in the core of the coalition.

The paper is organized as follows. In Section 2 we formulate the CCF problem mathematically. In Section 3 the approach is introduced. The algorithms for optimal subcoalition formation and payoff division, basic transfer scheme, and approximation algorithm of combinatorial coalition formation are presented correspondingly in Section 4, 5 and 6. Section 7 gives the experimental results and Section 8 concludes.

2. PROBLEM FORMULATION

Let $G = \{g_1, g_2, \ldots, g_K\}$ indexed by k and $B = \{b_1, b_2, \ldots, b_N\}$ indexed by n denote the collection of items and buyers respectively. Each buyer b_n places a bid, $bid_n = \{Q_n, r_n\}$, where $Q_n = \{q_n^1, \ldots, q_n^K\}$ is the quantity of each item that b_n requests, and r_n is the reservation cost, the highest cost that b_n can pay for his request Q_n. Denote by $G_n = \{g_k \in G | q_n^k > 0\}$ the set of items that b_n requests, and by $B_k = \{b_n \in B | q_n^k > 0\}$ the set of buyers that request g_k. Denote by $q_C^k = \sum_{b_n \in C} q_n^k$, $C \subset B$, the sum of the bid quantities for g_k by members of C. For each item provided, each seller has a unit price schedule which is a decreasing step function of the volumes sold together. If we assume the sellers have no capacity constraint, we can obtain an integrated price function $p_k(m) : Z^+ \rightarrow R^+$ for each item, which is the minimum price of g_k when m units are sold together among all those price schedules offered by the sellers. The price function $p_k(m)$ is also a decreasing step function.

A **coalition** C is a subset of the buyers with a **coalition value** which is the difference between the sum of the reservation costs of the coalition members and the minimum cost needed to satisfy the requests of all the members:

$$v(C) = \sum_{b_n \in C} (r_n - \sum_{k=1}^{K} q_n^k \times p_k(q_C^k)).$$

A **payoff division** X_C of a coalition C is a vector $(x_C(b) : b \in C)$ with the sum of the elements equal to the value of C:

$$\sum_{b \in C} x_C(b) = v(C).$$

The **core**[7] of a coalition C is the collection of all payoff divisions of C such that each element X_C satisfies, for any $C' \subset C$,

$$v_{C'} \leq x_C(C')$$

where $x_C(C') = \sum_{b \in C'} x_C(b)$. With a payoff division in the core any subset of the coalition members can get at least as much by joining the coalition as the value of the coalition formed by themselves. If the payoff division X_C is in the core of the coalition C, we say C is stable in the core.

The objective of the problem is to find a set of exclusive coalitions such that the sum of the coalition values is maximized, and to distribute the payoff for each coalition such that they are stable in the core. Under the assumption that the price functions are decreasing, we have the conclusion that the values of disjoint coalitions are superadditive:

Claim 1. [2] *If $C^1, C^2 \subset B$, $C^1 \cap C^2 = \emptyset$, then*

$$v(C^1 \cup C^2) \geq v(C^1) + v(C^2).$$

From Claim 1 the optimal coalition configuration always consists of no more than one coalition which is the optimal coalition with the largest non-negative value among all possible coalitions. The problem is to find an optimal coalition C^* and the payoff division in the core of C^*.

Example 1 gives a simple example of the problem:

Example 1. *The set of buyers is $B = \{b_1, b_2, b_3\}$, the set of items is $G = \{g_1, g_2\}$. Buyer b_1 asks for one unit of g_2 with the reservation cost 1, b2 and b3 ask for one unit of both g_1 and g_2 with the reservation cost 5 and 6 respectively.*

b_n	q_n^1	q_n^2	r_n
b_1	0	1	1
b_2	1	1	5
b_3	1	1	6

The price functions for the two items are decreasing step functions:

$$p_1(m) = \begin{cases} 3 & if \quad 0 < m < 3 \\ 2 & if \quad m \geq 3 \end{cases}$$

$$p_2(m) = \begin{cases} 3 & if \quad 0 < m < 2 \\ 2 & if \quad m \geq 2 \end{cases}$$

[1]Otherwise a buyer needs to place multiple bids, one for each set of items that has positive value for him. This case is discussed at the end of the paper.

[2]The proof of Claim 1 and all subsequent claims is in the Appendix.

where m is the quantity of each item sold together.

We can see that an optimal coalition C^* is $\{b2, b3\}$. The value of C^* is $v(C^*) = 1$. The payoff division $X_{C^*} = \{0, 1\}$ is in the core of C^*.

3. APPROACH

Considering that coalition formation is motivated by price discounts of each item, and multiple items requested by a buyer complement each other, we can artificially divide the reservation cost r_n of each buyer to r_n^k, $k = 1, \dots, K$, for each item such that $r_n = \sum_{g_k \in G} r_n^k$, find the optimal coalitions for each item with the *reservation cost division*, and then balance the coalitions to satisfy the complementarity of the items required by the buyers. We call a coalition with respect to an item g_k a *subcoalition* denoted by C_k.

Definition 1. (Reservation Cost Division) A reservation cost division $RD \in R^{K^N}$ is a set of K-dimensional real numbers $\{\{r_n^k\}_{k=1}^K\}_{n=1}^N$ satisfying

$$\sum_{k=1}^{K} r_n^k = r_n$$

and $r_n^k = 0$ if $q_n^k = 0$. Call r_n^k the virtual reservation cost and $p_n^k = r_n^k / q_n^k$ the virtual reservation price of the buyer b_n for the item g_k.

Definition 2. (Subcoalition) A subcoalition $C_k \subset B_k$ with respect to the item g_k is a subset of buyers requesting g_k with the coalition value $v_k(C_k)$ equal to

$$v_k(C_k) = \sum_{b_n \in C_k} (r_n^k - q_n^k \cdot p_k(q_{C_k}^k)).$$

Denote by $C_k^*(RD)$ or C_k^* the optimal subcoalition of the item g_k with or without specifying the reservation cost division RD.

With a reservation cost division, a set of optimal subcoalitions can be constructed one for each item. If a buyer b_n is involved in all or none of the subcoalitions of the items he requests, we say that the subcoalitions are *compatible* with respect to the buyer b_n because b_n will not cause invalidity of the coalition induced by the set of subcoalitions.

Definition 3. (Compatible) A set of subcoalitions $C_1, \dots, C_K$ are compatible with respect to the buyer b_n if $b_n \in \bigcap_{g_k \in G_n} C_k$ or $b_n \notin \bigcup_{g_k \in G_n} C_k$. A set of subcoalitions are compatible if they are compatible with respect to all the buyers. If the set of subcoalitions $C_1, \dots, C_K$ are compatible, we can induce a coalition $C = \bigcup_{k=1}^K C_k$ which is composed of all the members in the subcoalitions.

The optimal subcoalitions have properties stated in Claim 2: The coalition induced by compatible optimal subcoalitions is optimal. Furthermore, if the payoff division for each subcoalition is in the core of the subcoalition, then if we let the payoff of a member in the induced coalition be the sum of his payoffs in the subcoalitions, we get a payoff division which is in the core of the optimal coalition.

CLAIM 2. *If the optimal subcoalitions of all the items are compatible, then $C^* = \bigcup_{k=1}^K C_k^*$ is an optimal coalition and $X_{C^*} = \{x_{C^*}(b_n) | b_n \in C^*, x_{C^*}(b_n) = \sum_{k: b_n \in C_k^*} x_{C_k^*}(b_n)\}$ is in the core of C^*, where $X_{C_k^*}$ is a payoff vector in the core of C_k^*.*

Balancing the subcoalitions to make them compatible can be realized by transferring virtual reservation costs among the items. For one buyer who is involved in the optimal subcoalitions of some of the items he desires, but not in the optimal subcoalitions of the other items, we can make the optimal subcoalitions compatible with respect to him by transferring some virtual reservation cost of the buyer from the former items to the latter ones. If the transfers end up with a reservation cost division such that all the optimal subcoalitions are compatible, then the coalition induced by the subcoalitions is optimal.

EXAMPLE 2. *Suppose in Example 1 the initial reservation cost division RD^0 is:*

b_n	r_n	r_n^1	r_n^2
b_1	1	0	1
b_2	5	2	3
b_3	6	3	3

The optimal subcoalitions with RD^0 of each item are: $C_1^*(RD^0) = \{b_3\}$, $C_2^*(RD^0) = \{b_2, b_3\}$. The two subcoalitions are compatible with respect to b_1 and b_3 but not b_2. If b_2 has 1 virtual reservation cost transferred from item g_2 to g_1, then with the new reservation cost division RD^1

b_n	r_n	r_n^1	r_n^2
b_1	1	0	1
b_2	5	3	2
b_3	6	3	3

we have $C_1^*(RD^1) = \{b_2, b_3\}$, $C_2^*(RD^1) = \{b_2, b_3\}$. The payoff divisions $X_1(C_1^*(RD^1)) = \{0, 0\}$, $X_2(C_2^*(RD^1)) = \{0, 1\}$ are in the core of $C_1^*(RD^1)$ and $C_2^*(RD^1)$ respectively. Then $C^* = \{b_2, b_3\}$ induced from $C_1^*(RD^1)$ and $C_2^*(RD^1)$ is an optimal coalition, and $X(C^*) = \{0, 1\}$ constructed by summing up $X_1(C_1^*(RD^1))$ and $X_2(C_2^*(RD^1))$ is in the core of C^*.

Although we can find an optimal coalition by inducing it from compatible optimal subcoalitions, compatible optimal subcoalitions do not exist for all problem instances. But the existence is always true for the problems with *linear price functions*(the price decreases at a constant step when the quantity increases by one unit).

Definition 4. (Linear price function) A linear price function $p_k(m) : Z^+ \to R^+$ for the item g_k is given by

$$p_k(m) = -d_k \cdot m + a_k$$

where $d_k, a_k \in R^+$ and $m \leq a_k/(2d_k)$. (By bounding m from above we ensure that the purchasing cost $p_k(m) \cdot m$ is an increasing function of the quantity m.)

CLAIM 3. *Suppose the price functions are linear price functions, then there exists a reservation cost division such that the optimal subcoalitions are compatible.*

The focus of the rest of the paper is restricted to the systems with linear price functions. We need to answer the following questions:

- How to efficiently form an optimal subcoalition and distribute the payoff in the core
- How to transfer the virtual reservation cost among items to make the optimal subcoalitions compatible

- How to reduce the computational complexity and construct an approximation algorithm in polynomial time

These problems are solved in Section 4, 5 and 6 respectively. The notations with a "′" are used to denote the terms with the reservation cost division after a transfer.

4. OPTIMAL SUBCOALITION FORMATION AND PAYOFF DIVISION

In [16] an efficient and accurate algorithm for subcoalition formation and stable payoff division is given for the situation where each buyer asks for one unit of each item. The algorithm can be extended to the situation with multiple units.

Claim 4. *Suppose b_i and b_j ask for the same quantity of g_k with the virtual reservation cost r_i^k and r_j^k respectively, $r_i^k > r_j^k$. If $b_j \in C_k^*$, then $b_i \in C_k^*$.*

From Claim 4 to form an optimal subcoalition for the item g_k, we can first sort the buyers with the same bid quantity for g_k by their virtual reservation cost in a descending order. The candidates for the optimal subcoalition can be iteratively constructed by consecutively extracting n_i buyers from each queue i from the head, where n_i goes from zero to the length of the queue i. For example, if we have two queues $\{b_1, b_2\}$ and $\{b_3\}$, then the candidates are $\emptyset$, $\{b_1\}$, $\{b_1, b_2\}$, $\{b_3\}$, $\{b_1, b_3\}$, $\{b_1, b_2, b_3\}$. The subcoalition with the largest value among these candidate subcoalitions is the optimal subcoalition. Let M be the number of different bid quantities for the items[3]. Since the length of each queue is no greater than N, the number of coalitions to be considered is no greater than $(N+1)^M$ and the complexity of the optimal subcoalition formation scheme is $O(N^M)$.

Claim 5 gives a strategy to divide the payoff of the optimal subcoalition such that the payoff division is in the core of the subcoalition.

Claim 5. *Let $cost_k(C_k)$ denote the purchasing cost to satisfy the requests for g_k of the members in C_k, i.e. $cost_k(C_k) = q_{C_k}^k \cdot p_k(q_{C_k}^k)$. The payoff division X_{C_k} is in the core of C_k with*

$$x_{C_k}(b_n) = \begin{cases} (p_n^k - h_{C_k}) \cdot q_n^k & (b_n \in \overline{C}_k) \\ 0 & (b_n \notin \overline{C}_k) \end{cases}$$

where h_{C_k} and $\overline{C}_k$ satisfy

$$cost_k(C_k) = q_{\overline{C}_k}^k \cdot h_{C_k} + \sum_{b_n \in C_k \backslash \overline{C}_k} p_n^k \cdot q_n^k$$

$$\overline{C}_k = \{b_n \in C_k | h_{C_k} \leq p_n^k\}$$

The rule of payoff division for a subcoalition C_k is shown in Figure 1. Each bin by solid lines represents the bid quantity of a buyer for the item g_k, and is divided by dotted lines into units. The area of the bins below h_{C_k} is equal to the purchasing cost of the subcoalition C_k for the item g_k. The shaded area in each bin is equal to the payoff assigned to that buyer.

[3]Based on the goods traded in electronic markets to personal buyers, for instance, books, electrics, parts and accessories, we can reasonably assume M be a small number.

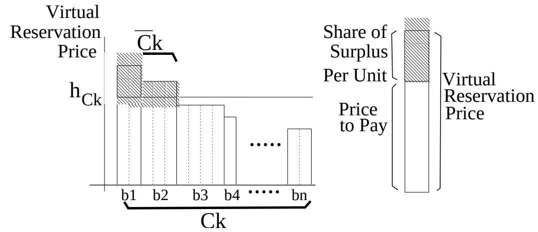

Figure 1: The payoff division rule for a subcoalition

For one subcoalition C_k the payoff division can be implemented in three steps: First, sort the buyers by their virtual reservation prices for the item g_k in a descending order $\{b^1, b^2, ..., b^J\}$. Second, check the buyers one by one from the end and choose the buyer b^{j^*} such that $p_{b^{j^*}}^k \geq h_{C_k} > p_{b^{j^*+1}}^k$. Third, let $\overline{C}_k = \{b^j | j \leq j^*\}$ and $h_{C_k} = (cost_k(C_k) - \sum_{j > j^*} r_{bj}^k)/q_{\overline{C}_k}^k$. The complexities of the three steps are $O(N\log N)$, $O(N)$ and $O(1)$ respectively. Therefore the complexity of payoff division for one subcoalition is $O(N\log N)$. When there are K items, the complexity of payoff division is $O(K \cdot N\log N)$.

5. RESERVATION COST TRANSFER SCHEME

The general reservation cost transfer scheme starts with an initial reservation cost division and checks the buyers one by one. If the optimal subcoalitions are not compatible with respect to the buyer b_n, b_n makes a virtual reservation cost transfer such that the new optimal subcoalitions formed after the transfer are compatible with respect to b_n. This scheme stops when the optimal subcoalitions are compatible with respect to all the buyers.

The amount of reservation cost to be transfered among items in each round for one buyer is decided by his *offers* and *requests*.

Definition 5. (Offer & Request) Assume the reservation cost division of other buyers remains the same.

Offer Off_n^k of the buyer $b_n \in C_k^*$ with respect to the item g_k is the maximum amount of virtual reservation cost of b_n for g_k that can be reduced from r_n^k while keeping $b_n \in C_k^{*'}$, i.e., $Off_n^k = v_k(C_k^*) - v_k(C_k^{\bar{n}})$ where $C_k^{\bar{n}}$ is the optimal subcoalition of g_k that does not include b_n.

Request Req_l^k of the buyer $b_l \notin C_k^*$ with respect to the item g_k is the minimum amount of virtual reservation cost of b_l for g_k that needs to be added to r_l^k, such that $b_l \in C_k^{*'}$, i.e., $Req_l^k = v_k(C_k^*) - v_k(C_k^l)$ where C_k^l is the optimal subcoalition of g_k that includes b_l.

Denote by $dif_n = \sum_{g_k \in G_n : b_n \in C_k^*} Off_n^k - \sum_{g_k \in G_n : b_n \notin C_k^*} Req_n^k$ the difference between the sum of offers and the sum of requests of b_n.

Decision of the transfer amount in each round to have the optimal subcoalitions compatible with respect to one buyer is stated in Claim 6.

Claim 6. *Suppose the optimal subcoalitions are not compatible with respect to b_n. Let $K_n^1 = \{g_k \in G_n | b_n \in C_k^*\}$ and $K_n^2 = \{g_k \in G_n | b_n \notin C_k^*\}$.*

(i) If $dif_n \neq 0$, let $r_n^{k'} = r_n^k - Off_n^k + \gamma_k$ for $g_k \in K_n^1$ and $r_n^{k'} = r_n^k + Req_n^k + \gamma_k$ for $g_k \in K_n^2$, where $\gamma_k \leq 0$ if $dif_n <$

0 and $\gamma_k \geq 0$ if $dif_n > 0$ for $g_k \in G_n$, and $\sum_{k=1}^{K} \gamma_k = dif_n$, then b_n is excluded from all the optimal subcoalitions if $dif_n < 0$, or involved in all the optimal subcoalitions he desires if $dif_n > 0$.

(ii) If $dif_n = 0$, let $r_n^{k'} = r_n^k - Off_n^k$ for $g_k \in K_n^1$ and $r_n^{k'} = r_n^k + Req_n^k$ for $g_k \in K_n^2$, then b_n can either be involved in or excluded from all the optimal subcoalitions he desires.

Claim 7 states that the offer of a member in an optimal subcoalition is an increasing function and the request of a nonmember is a decreasing function of the virtual reservation cost of other buyers for the item.

CLAIM 7. *Make a transfer for $b_n \in B_k$ such that $r_n^{k'} > r_n^k$. Suppose b_i is a member while b_j is a nonmember of the optimal subcoalition of the item g_k both before and after the transfer. Then (1)$Off_i^{k'} \geq Off_i^k$ and (2)$Req_j^{k'} \leq Req_i^k$.*

With respect to the evolution of the optimal subcoalitions in the transfer procedure, we have the following conclusions which are summarized in Claim 8. They guarantee the convergence of the transfer scheme to a set of compatible optimal subcoalitions(Please refer to [6] for a formal statement and proof), and also guarantee the polynomial complexity of the approximation algorithm stated in Section 6.

CLAIM 8. *Make a transfer for $b_n \in B$.*
If $b_n \in C_k^*$ and $b_n \in C_k^{*'}$, then $C_k^{*'} = C_k^*$.
If $b_n \in C_k^*$ and $b_n \notin C_k^{*'}$, then $C_k^{*'} \subset C_k^*$.
If $b_n \notin C_k^*$ and $b_n \in C_k^{*'}$, then $C_k^{*'} \supset C_k^*$.

6. APPROXIMATION ALGORITHM OF COMBINATORIAL COALITION FORMATION

Although the reservation cost transfer mechanism stated in Section 5 leads to an optimal coalition and payoff division in the core of the coalition, polynomial computational complexity is not guaranteed, because the number of iterations may be very large. The heuristic of the approximation algorithm is that once a buyer is excluded from all the optimal subcoalitions, the possibility that he will be involved in the final optimal coalition is very small. We maintain a subset of the buyers $\hat{B}$ which shrinks while excluding buyers. The coalition formation and payoff division is considered within $\hat{B}$ instead of B assuming the buyers out of $\hat{B}$ are not contained in the optimal coalition. In each iteration, a reservation cost transfer is made for one buyer in $\hat{B}$ such that the new optimal subcoalitions are compatible with him after the transfer. If the buyer is discarded by all the optimal subcoalitions he is excluded from $\hat{B}$, else the next buyer in $\hat{B}$ is visited.

Actually even if a buyer is rejected by all the optimal subcoalitions at some time of the transfer procedure it is still possible that he is involved in the optimal coalition. The reason is that there exist dependency relations between the buyers: The joining of a buyer to an optimal subcoalition may have some other buyers also join it, and the leaving of a buyer from an optimal subcoalition may have some other buyers also leave it(Claim 8). Even if a buyer b_i is rejected by an optimal subcoalition C_k^* temporarily, it is possible that he can get into it upon the joining of some other buyer b_j when b_j increases r_j^k. This exception is considered in Option 1, which can be integrated into the main algorithm to improve the solution quality at the expense of increasing the computational complexity. Since the offer of a member in a subcoalition is non-decreasing and the request of a nonmember is non-increasing with the increasing of the virtual reservation cost of other buyers for the item(Claim 7), Option 1 uses a greedy transfer procedure as follows to increase the chance of a buyer to stay. For a buyer b_n whose sum of offers cannot cover the sum of requests, the items desired are visited one by one. When an item g_k is visited, all the buyers desiring g_k transfer all their extra virtual reservation cost(the offers) to g_k from other items, and the offers/requests of b_n are recalculated. The visit to the next item is stopped when the sum of offers covers the sum of requests, in which case a reservation cost division of b_n is constructed by some transfer to have b_n included in all the desired optimal subcoalitions. If all the items desired by b_n have been visited and the stop condition is not reached, b_n is excluded.

The approximation algorithm of combinatorial coalition formation is described in Algorithm 1, and the complexity is analyzed in Claim 9.

ALGORITHM 1.
STEP 0: Initialization
Make an initial reservation cost division. Let $\hat{B} = B$.
STEP 1: Optimal subcoalition formation
Construct the optimal subcoalitions for each item and go to Step 3.
STEP 2: Transfer
For a buyer $b_n \in \hat{B}$ with whom the optimal subcoalitions are not compatible, if $dif_n \geq 0$, change the division of r_n and update the optimal subcoalitions such that b_n is involved in all the optimal subcoalitions he desires; else $\hat{B} = \hat{B} \setminus \{b_n\}$ or go to Option 1.
STEP 3: Termination judgement
If the optimal subcoalitions are compatible, stop; else go to Step 2.

OPTION 1. *Let the item index be $k = 1$, the index in $\hat{B}$ be $i = 1$.*
STEP 4: Let the index of the buyer $\hat{B}[i]$ be j. If $b_j \notin B_k$ or $b_j = b_n$, go to Step 5; else for the buyer b_j make a reservation cost transfer such that $r_j^k = r_j^k + \sum_{l \neq k, b_j \in C_l^*} Off_j^l$, $r_j^l = r_j^l - Off_j^l$ for $l \neq k$ and $C_l^* \ni b_j$, go to Step 5.
STEP 5: $i = i + 1$. If $i \leq |\hat{B}|$, go to Step 4; else update the optimal subcoalition C_k^*.
If $dif_n \geq 0$, change the division of r_n and update the optimal subcoalitions such that b_n is involved in all the optimal subcoalitions he desires, go to Step 3; else go to Step 6.
STEP 6: $k = k + 1$. If $k \leq K$, let $i = 1$ and go to Step 4; else b_n is excluded from $\hat{B}$ and go to Step 3.

CLAIM 9. *The complexity of Algorithm 1 is: $O(K \cdot N^{2+M})$ without Option 1 and $O(K \cdot N^{M+3})$ with Option 1, where M is the number of different bid quantities for the items.*

7. EXPERIMENT

This section reports on an evalution of the performance of our approximation algorithm of combinatorial coalition formation by simulation. The value of the coalition generated by Algorithm 1 with or without Option 1 is compared to the value of the optimal coalition generated by exhausted search. The comparison is conducted in two dimensions:

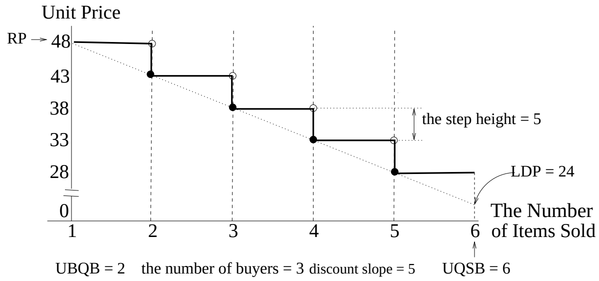

Figure 2: A sample price function

system scale(number of buyers and items) and system characteristics(characterizing the price schedules and bids).

7.1 Instance generation

The framework of instance generation is partially inspired by [16]. An instance is characterized by three parameters: DS(Discount Slope) determines the magnitude of profit to form coalitions, RBMI(the Ratio of Buyers preferring Multiple Items) models the extent of complementarity among the items, RBBR(the Ratio of Buyers Bidding at the Retail Prices) reflects the extent of dependency among the buyers(to how much extent they need to form coalitions) to win their bids.

Price schedule generation:

Let DS(Discount Slope) denote the ratio of the step height to the step length of the price function. The larger DS, the more beneficial to form coalitions. Denote by RP(the Retail Price) the highest price without discount. Assume the sum of quantities requested by all the buyers for one item is bounded from above by UBSQ(the Upper Bound of the Sum of Quantities requested by all the buyers for one item). By having $\mathrm{RP} = 2 \cdot \mathrm{DS} \cdot \mathrm{UBSQ}$ we ensure that the purchasing cost is an increasing function of the quantity requested for the item. Therefore a linear price function is modeled by two parameters, DS and UBSQ. Define LDP(the Lowest Discount Price) to be equal to $\mathrm{RP}/2$, which is the lowest price that could be reached assuming the price function was a continuous linear function with the slope $-\mathrm{DS}$. Figure 1 shows a sample of the linear price function used in the experiment. Assume all the items have identical price schedules without decreasing the complexity of the problem.

Bid generation:

Suppose before the decision of the reservation cost, each buyer b_n has a virtual reservation price p_n^k for each item g_k he desires. The reservation cost is decided by summing up the reservation prices multiplied by the quantity he requests for each item, i.e., $r_n = \sum_{n=1}^{N} p_n^k \cdot q_n^k$. Let the lower bound of the virtual reservation price be equal to LDP and the upper bound equal to RP. Let the ratio of buyers who have virtual reservation prices at RP equal to RBBR(the Ratio of Buyers Bidding at the Retail Prices). Small RBBR means many of the buyers cannot win their bids if they do not form coalitions. The virtual reservation prices of other buyers are randomly distributed between LDP and RP.

The distribution of preferences for multiple items is modeled by RBMI(the Ratio of Buyers preferring Multiple Items). RBMI is a vector $\{rb_1, \ldots, rb_K\}$ where rb_k is the ratio of buyers who desire k items and $\sum_{k=1}^{K} rb_k = 1$. For example, when the number of items is 3, the number of buyers is 10, RBMI $= \{0.2, 0.3, 0.5\}$ means there are 2 buyers bid for 1 item, 3 buyers for 2 items, 5 buyers for 3 items. Large numbers at the end part of RBMI mean that many items are complementary for many buyers. The number of desired items of each buyer is randomly decided following the distribution consistent with RBMI. The quantities requested by each buyer for each item desired are generated randomly in the range $[1, \mathrm{UBQB}]$, where UBQB(the Upper Bound of the Quantity requested by one Buyer for one item) is defined by $\mathrm{UBSQ} = \mathrm{UBQB} \cdot N$.

Table I: Simulation Parameters

Parameter	Values
the number of buyers N	{5, 10, 15, 20, 25}
the number of items K	{3, 5, 7}
UBQB(the largest quantity to be requested for one item by a buyer)	{3}
DS(the discount slope)	{0.2, 0.4, 0.6, 0.8, 1}
RBMI(the ratio of buyers preferring multiple items)	{1.0, 0, 0};{0.7, 0.2, 0.1};{0.5, 0.3, 0.2};{0.333, 0.333,0.334};{1.0, 0, 0, 0, 0};{0.7, 0.2, 0.05, 0.03, 0.02};{0.4, 0.2, 0.15, 0.15, 0.1};{0.2, 0.2, 0.2, 0.2, 0.2};{1.0, 0, 0, 0, 0, 0, 0};{0.5, 0.2, 0.1, 0.05, 0.05, 0.05, 0.05};{0.3, 0.2, 0.2, 0.1, 0.1, 0.05, 0.05};{0.14, 0.14, 0.14, 0.14, 0.14, 0.15,0.15};
RBBR(the ratio of buyers bidding at the retail price)	{0.1, 0.25, 0.4, 0.55, 0.7}

7.2 Results

The simulation is based on the instances generated with combinations of the parameter values listed in Table I. For each set of parameters three instances are randomly generated. For each instance we construct the optimal coalition by exhaustive search, and the approximate solutions by Algorithm 1 with and without Option 1. The average coalition value of the instances with identical parameters is set as the coalition value for the condition with that parameter set. The comparison is made among the optimal value and the value of the coalitions obtained by our algorithm. The largest number of buyers that is used for the comparison is limited to 25 because of the complexity to compute the optimal value.

Comparison with respect to the number of buyers and the number of items:

We calculate the ratio of the approximate value over the optimal value and take the average of the ratios under all the conditions with the same number of buyers and number of items. Figure 3 shows the distribution of the average ratios with respect to the number of buyers and the number of items. From Figure 3 we can see that the value of the coalition generated by Algorithm 1 is very close to the optimal one. Option 1 improves the performance of the algorithm remarkably.

Comparison with respect to DS, RBBR and RBMI:

Since DS, RBMI and RBBR are three characteristic parameters of an instance, we would like to compare the coalition values on the three dimensions to see their impact on the performance of the algorithm. The experimental results imply that generally the performance of the algorithm hurts from the increase of RBMI but does not depend on DS or RBBR. The average performance of the algorithm with respect to RBMI when the number of items is equal to 5 and

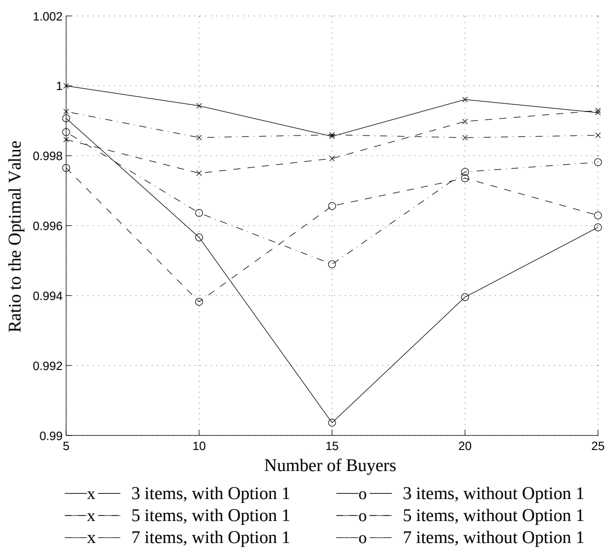

Figure 3: Compare the value of the coalition obtained by Algorithm 1 with the optimal value, with respect to the number of buyers and the number of items

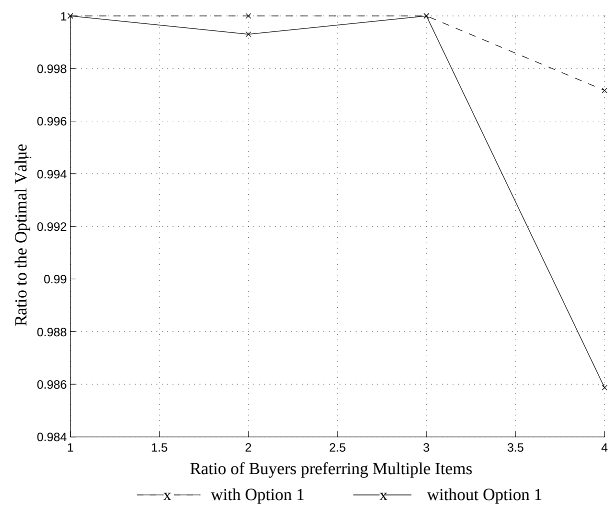

Figure 4: Compare the value of the coalition obtained by Algorithm 1 with the optimal value, with respect to the ratio of buyers preferring multiple items(one integer point on the horizontal axis represents one value of RBMI shown in Table I)

the number of buyers is equal to 20 is shown in Figure 4. The result is understandable since the algorithm aims to generate compatible subcoalitions. When more items are complementary to more buyers, it is more difficult to find compatible subcoalitions.

8. CONCLUSION

In this paper we study the combinatorial coalition formation problem of buyers in a electronic marketplace where sellers offer price discounts based on volumes and buyers have preferences for combinations of items. We focus on linear price functions and present an approximation algorithm in polynomial time for combinatorial coalition formation. The payoff division produced is in the core of the coalition formed. Experimental results show that the algorithm gives a good ratio to the optimal value.

When the price functions are general decreasing step functions, the properties in Claim 8 do not hold any more and the compatible optimal subcoalitions may not exist. In [6] a solution is given for this condition.

In the situation studied in this paper, one buyer can only place one bid. More generally a buyer may have desire for multiple bundles of items and needs to place a bid for each bundle. The relation between the bundles can be OR or XOR. If the bundles of a buyer are exclusively in OR relation, we can generate a dummy buyer for each bundle and this leads to a CCF problem with the dummy buyers. If some bundles are in XOR relation, we can construct all the maximal sets of unconflicting bids for each buyer. The optimal coalition can be obtained by solving a CCF problem for each combination of the sets one from each buyer, and choosing the optimal result.

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APPENDIX

A. PROOF OF CLAIMS

Proof of Claim 1 Since the price functions are non-increasing, $p_k(q^k_{C^1\cup C^2}) \leq min\{p_k(q^k_{C^1}), p_k(q^k_{C^2})\}$, and $p_k(q^k_{C^1\cup C^2})\cdot q^k_{C^1\cup C^2} \leq p_k(q^k_{C^1})\cdot q^k_{C^1} + p_k(q^k_{C^2})\cdot q^k_{C^2}$. But $\sum_{b\in C^1\cup C^2} r_b = \sum_{b\in C^1} r_b + \sum_{b\in C^2} r_b$, it follows that $v(C^1\cup C^2) \geq v(C^1) + v(C^2)$.□

Proof of Claim 2 Let $(C)_k = C \cap B_k$. If a coalition C is induced from a set of compatible subcoalitions $C_1, \ldots, C_K$, $(C)_k = C_k$. Since $\sum_{k=1}^K r^k_n = r_n$, $v(C) = \sum_{k=1}^K \sum_{b_n\in C}[r^k_n - q^k_n \times p_k(q^k_C)]$. But $v_k((C)_k) = \sum_{b_n\in C}[r^k_n - q^k_n \times p_k(q^k_C)]$, it follows that $v(C) = \sum_{k=1}^K v_k((C)_k)$.

Prove C^* is an optimal coalition: Suppose $C \neq C^*$ and $v(C) > v(C^*)$, then $\sum_{k=1}^K v_k((C)_k) > \sum_{k=1}^K v_k((C^*)_k)$. There exists some k such that $v_k((C)_k) > v_k((C^*)_k)$. But $(C^*)_k = C^*_k$.

Prove X_{C^*} is in the core of C^*: For any $C \subset C^*$, $x_{C^*}(C) = \sum_{k=1}^K x_{C^*_k}((C)_k)$. Since C^*_k is stable in the core, $v_k((C)_k) \leq x_{C^*_k}((C)_k)$. Then $v(C) \leq x_{C^*}(C)$. □

Lemma 1. *Let the collection of buyers $B = B_1 \cup B_2$, and the optimal coalitions of B, B_1 be C, C^1 respectively . Then $C \supseteq C^1$ with linear price functions. It follows that the optimal coalition contains all the optimal coalitions of subsets of the buyers.*

Proof: Suppose $C^1 \not\subseteq C$. Let $C^0 = C^1 \cap C$ and $C^{'0} = C^1 \setminus C^0$. Then $C^{'0} \neq \emptyset$. Let $r_C = \sum_{b_n\in C} r_n$. Then $v(C) - v(C^0) = r_{C\setminus C^0} - \sum_{k=1}^K[q^k_{C\setminus C^0}\cdot(-d_k\cdot q^k_{C\setminus C^0} + a_k) - 2q^k_{C^0}\cdot d_k\cdot q^k_{C\setminus C^0}]$ and $v(C\cup C'^0) - v(C^1) = r_{C\setminus C^0} - \sum_{k=1}^K[q^k_{C\setminus C^0}\cdot(-d_k\cdot q^k_{C\setminus C^0} + a_k) - 2q^k_{C^1}\cdot d_k\cdot q^k_{C\setminus C^0}]$. Since $q^k_{C^1} \geq q^k_{C^0}$ and the strict inequality holds for at least one $g_k \in G$, by comparing the above two equations, $v(C \cup C^{'0}) - v(C^1) > v(C) - v(C^0)$. Since $v(C^0) \leq v(C^1)$, $v(C \cup C^{'0}) > v(C)$. But C is optimal.□

Proof of Claim 3 By induction with the number of buyers N: When $N = 1$ it holds straightforwardly. Suppose the statement stands for $N \leq n$.

When $N = n+1$: Let the optimal coalition of $B_0 = \{b_1, b_2, \ldots, b_n\}$ be C^0. Let $B' = (B_0 \setminus C^0) \cup \{b_{n+1}\}$. Let the optimal coalition of $B = B_0 \cup \{b_{n+1}\}$ be C. From Lemma 1 we have $C \supseteq C^0$. Then

$C \setminus C^0$
$= argmax_{T\subseteq(B\setminus C^0)} v(C^0 \cup T)$
$= argmax_{T\subseteq(B\setminus C^0)} \{v(C^0 \cup T) - v(C^0)\}$
$= argmax_{T\subseteq(B\setminus C^0)} \{r_T - \sum_{k=1}^K[q^k_T(-d_k\cdot q^k_T + a_k - d_k\cdot 2q^k_{C^0})]\}$

Therefore $C \setminus C^0$ is an optimal coalition of $B \setminus C^0$ with the price functions $p^{'}_k(m) = -d_k\cdot m + a_k - d_k\cdot 2q^k_{C^0}$. This is a linear price function and from the assumption of induction there exists some reservation cost division for all the buyers $b_i \in B \setminus C^0$ such that $(C \setminus C^0)_k$ is an optimal subcoalition among all the subsets of $(B \setminus C^0) \cap B_k$ with respect to g_k with the linear price function $p^{'}_k(m)$. With this reservation cost division $\{r^k_i\}$, $k = 1, \ldots, K$, $b_i \in B \setminus C^0$,

$(C \setminus C^0)_k$
$= argmax_{T\subseteq(B\setminus C^0)\cap B_k} \{r^k_T - q^k_T(-d_k\cdot q^k_T + a_k - d_k\cdot 2q^k_{C^0})\}$
$= argmax_{T\subseteq(B_k\setminus(C^0)_k)} \{v_k((T \cup C^0)_k) - v_k((C^0)_k)\}$
$= argmax_{T\subseteq(B_k\setminus(C^0)_k)} v_k((T)_k \cup (C^0)_k)$

From Lemma 1 the optimal subcoalition C^k of B_k with respect to g_k includes $(C^0)_k$ for every k. The above equation means that $(C)_k = (C \setminus C^0)_k \cup (C^0)_k$ is an optimal subcoalition with respect to g_k if $(C^0)_k$ is an optimal subcoalition of the buyers in $(C^0)_k$ with respect to g_k. Let the reservation cost division of C^0 to be that with which $(C^0)_k$ is optimal with respect to the item g_k among all the subsets of $(C^0)_k$ for every k.(From the assumption of induction such a reservation cost division exists.) Then $(C)_k$ is an optimal subcoalition of the item g_k with the reservation cost division stated above and $(C)_k$, $k = 1, \ldots, K$ are compatible. □

Proof of Claim 4 If $b_i \notin C^*_k$, construct a subcoalition $C^{'}_k$ by replacing b_j with b_i in C^*_k. This does not change the discount price of the item with the subcoalition since $q^k_i = q^k_j$. But $r^k_i > r^k_j$, hence $v_k(C^*_k) < v_k(C^{'}_k)$ and this contradicts the optimality of C^*_k.□

Proof of Claim 5 The conclusion follows from Proposition 1 in [16] by regarding a buyer b_n asking for q^k_n units of g_k with the reservation cost r^k_n as q^k_n buyers each one asking for 1 unit of g_k with the same reservation price r^k_n/q^k_n.□

Proof of Claim 6 When $dif_n < 0$, more than Off^k_n is transfered out for $g_k \in K^1_n$ and less than Req^k_n is transfered in for $g_k \in K^2_n$. When $dif_n > 0$, less than Off^k_n is transfered out for $g_k \in K^1_n$ and more than Req^k_n is transfered in for $g_k \in K^2_n$. When $dif_n = 0$, after the transfer $v_k(C^{*'}_k) = v_k(C^{\bar{n}'}_k) = v_k(C^{n'}_k)$.

Proof of Claim 7 and 8 Please refer to [6] □

Proof of Claim 9 The complexity of the sub-algorithms are: Construct the optimal subcoalition for one item: $O(N^M)$. Construct the optimal subcoalitions for all items: $O(K\cdot N^M)$. Calculate the offer(request) of one buyer with one item: $O(N^M)$. Calculate the offers(requests) of one buyer with all items: $O(K\cdot N^M)$.

With the number of buyers to be considered currently being $|\hat{B}| = n$ the largest number of iterations needed without Option 1 to exclude a buyer is n. Therefore in the worst case the number of iterations needed without Option 1 is $N + (N-1) + \ldots + 1$ which is of complexity $O(N^2)$. The complexity of the algorithm is $O(K\cdot N^M\cdot N^2) = O(K\cdot N^{2+M})$ without Option 1.

The complexity to construct the optimal subcoalition for one item is $O(N^M)$. There are at most $2KN$ iterations to construct the optimal subcoalition for one item in each round of Option 1. Therefore the complexity of Option 1 is $O(K\cdot N^{M+1})$ and the complexity of Algorithm 1 with Option 1 is $O(K\cdot N^{M+1}\cdot N^2) = O(K\cdot N^{3+M})$. □