

# mean (average)

- mean value of all measurements =  
((each value observed) \*  
(number of times it is observed)) /  
(total number of observations)
- $\langle N \rangle = (\sum_k^{k=0 \text{ to } m} n(k) * k) / (\sum_k^{k=0 \text{ to } m} n(k))$ 
  - $k$  is the count observed in one measurement
  - $n(k)$  is the number of times  $k$  is observed
  - $m$  is the largest count in all the observations

measurements whose outcome is a count (integer)

$$\text{mean: } \langle N \rangle = \frac{\sum_{k=0}^m n(k) \cdot k}{\sum_{k=0}^m n(k)}$$

$k$  is the count observed in one observation  
 $n(k)$  is the number of times  $k$  counts are seen  
 $m$  is the largest value of  $k$  ever seen

Note the measurement and its index are the same.  
More generally

$$\langle X \rangle = \frac{\sum_{k=0}^m n(k) \cdot X_k}{\sum_{k=0}^m n(k)}$$

where  $X_k$   
is the  $k$ -th  
arbitrary-valued  
observation

# width – standard deviation

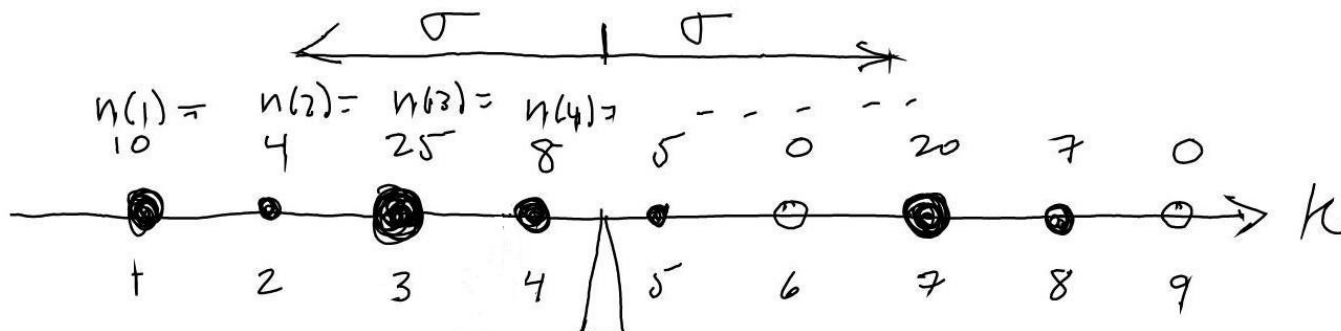
- the most common measure of a distribution's width is its “standard deviation”: the RMS value ( $\sigma$ ) of the deviation of each of the measurements from the mean of all the measurements
- $\sigma^2 = (\sum_k^{k=0 \text{ to } m} n(k) * (k - \langle N \rangle)^2) / (\sum_k^{k=0 \text{ to } n} n(k))$   
(for a counting measurement in which each outcome is an integer; for continuous outcomes you need to replace this with an integral)
- as we discussed, it is observed and provable that the standard deviation ( $\sigma$ ) of a counting experiment whose mean is  $\langle N \rangle$  is  $\langle N \rangle^{1/2}$
- hmm ... should I subtract 1 from the denominator?

measurements whose outcome is an integer (count)  
width (standard deviation,  $\sigma$ ) of the distribution

$$\sigma^2 = \frac{\sum_{k=0}^m n(k) \cdot (k - \langle N \rangle)^2}{\sum_{k=0}^m n(k)}$$

Or generally, when the value and its index differ

$$\sigma^2 = \frac{\sum_{k=0}^m n(k) \cdot (x_k - \langle x \rangle)^2}{\sum_{k=0}^m n(k)}$$



mean =  
4.38

} "center-of-mass"  
} "1st moment"

$$\sigma^2 = \frac{10 \cdot (1 - 4.38)^2 + 4 \cdot (2 - 4.38)^2 + 25 \cdot (3 - 4.38)^2 + \dots}{(10 + 4 + 25 + \dots)}$$

"moment-of-inertia" } "2nd moment"

$$\sigma^2 = 5.27, \quad \sigma = 2.30$$

# shape: the Poisson distribution

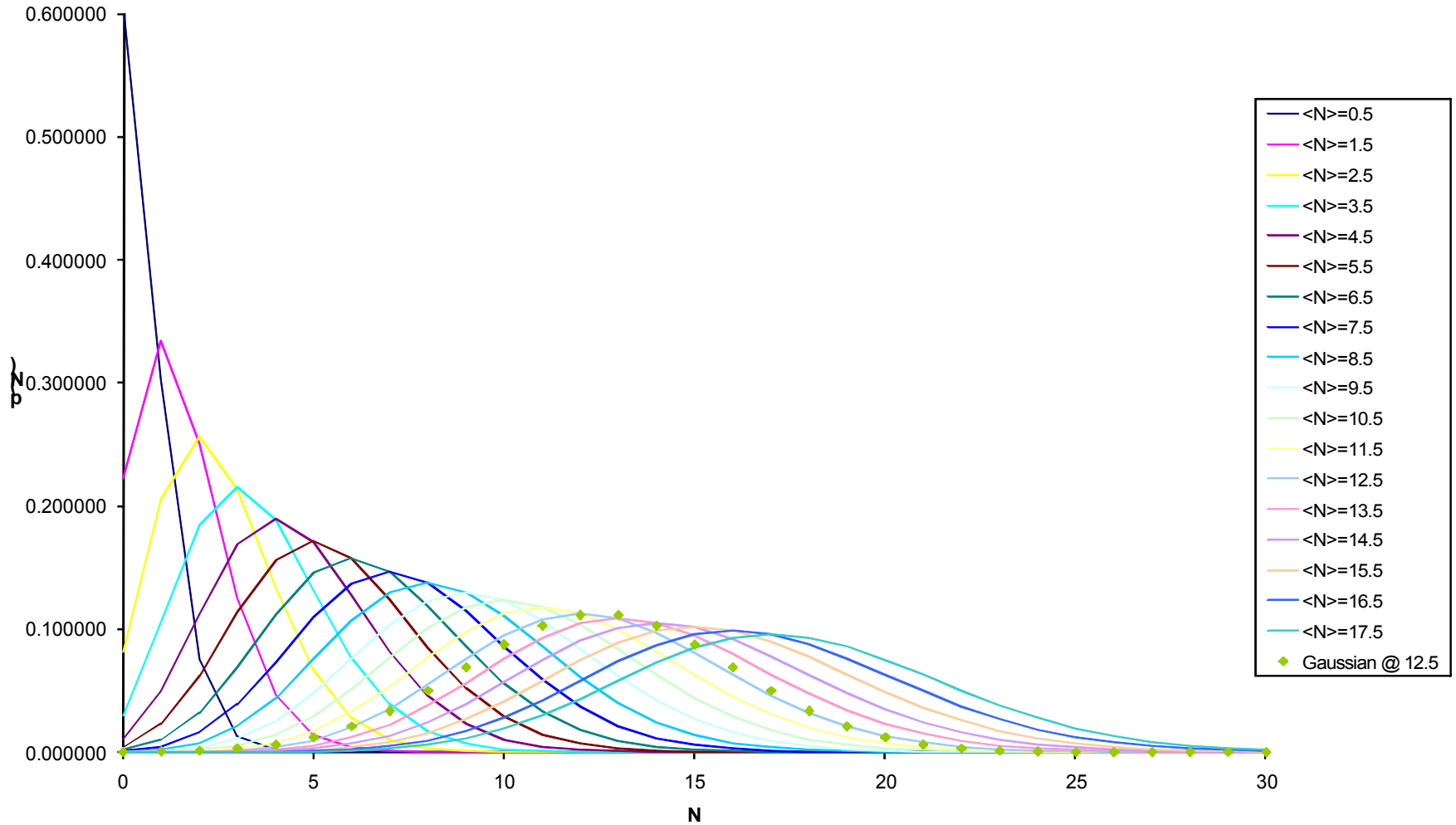
- if the average number of counts in repeated samples is  $\langle N \rangle$  (also written  $\bar{N}$ ) then the probability of a particular experiment giving  $N$  counts is

$$p(N) = \frac{e^{-\bar{N}} \bar{N}^N}{N!}$$

*remember that any sample  $N$  must be an integer*

- but if  $\langle N \rangle$  happens to be an integer it is just a numerical accident ... you should *expect*  $\langle N \rangle$  to be a “floating point” number

Poisson dist:  $p(N)$  for various  $\langle N \rangle$  and Gaussian dist for  $\langle N \rangle = 12.5, \sigma = \text{Sqrt}(12.5)$



# Poisson distribution – origin

$$e^N = 1 + N + \frac{N^2}{2} + \frac{N^3}{6} + \frac{N^4}{24} + \frac{N^5}{120} + O[N]^6$$

$$1 = e^{-N} \left( 1 + N + \frac{N^2}{2} + \frac{N^3}{6} + \frac{N^4}{24} + \frac{N^5}{120} + O[N]^6 \right)$$

$$1 = p(0) + p(1) + p(2) + \dots + p(i) + \dots$$



# the key point to remember

- it is a pervasive rule-of-thumb that in any measurement that can be traced to a counting process, the uncertainty in any particular count is the square root of the average number of counts observed over all the experiments
- for a single counting experiment, your best guess for the uncertainty of the result is the square root of the result

# assignment

(21) A set of counting measurements averages 1.3333... counts per sample; what are the probabilities of the next sample yielding 0, 1, 2, 3, 4, and 5 counts?

(22) Think through and describe *intuitively* the distribution of counting measurement values whose mean is very small, say 0.01; show that your intuition is consistent with what you calculate using the Poisson distribution formula.

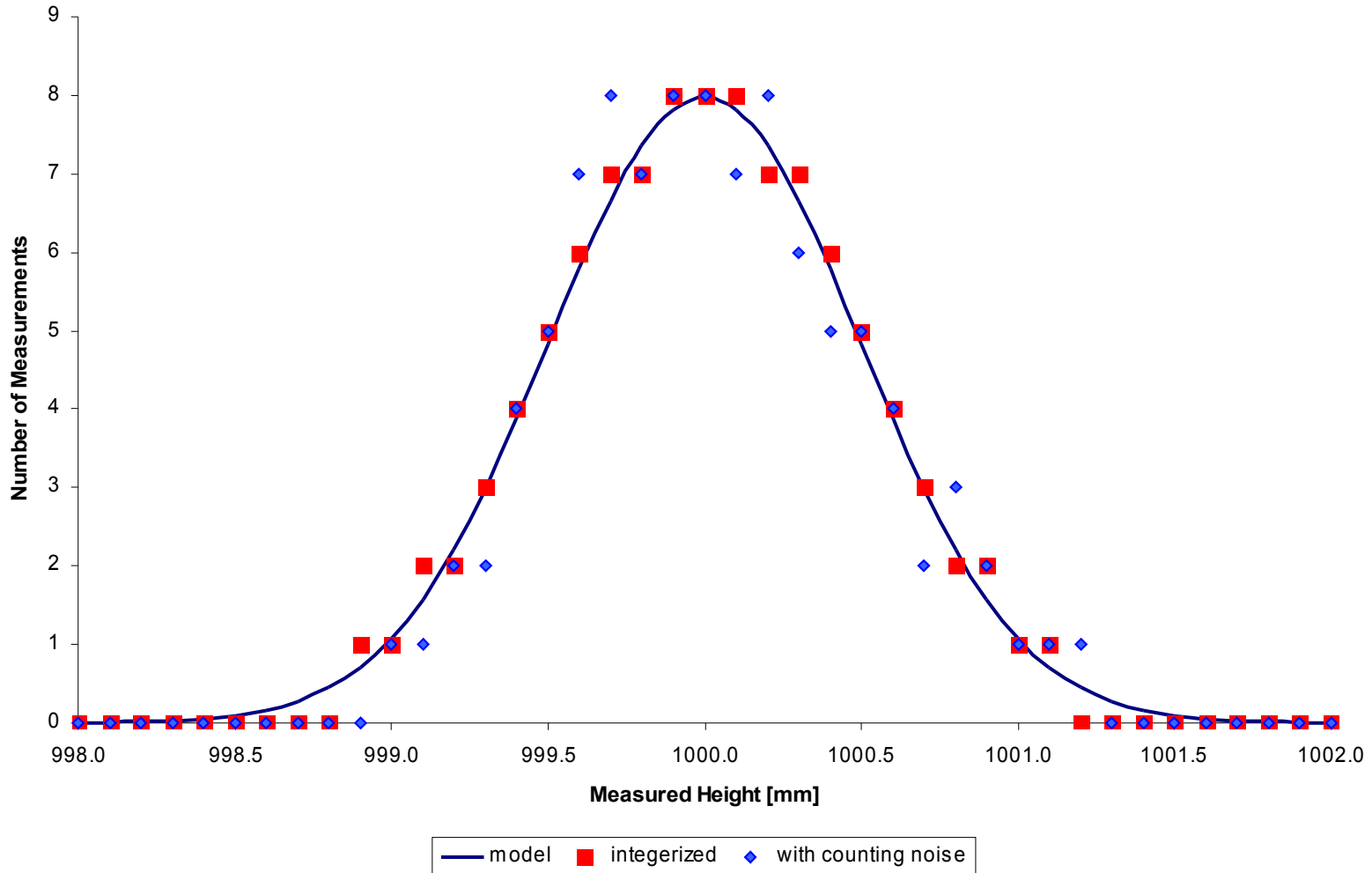
distribution of measurements  
whose outcome can take on  
any value

# there is no such thing ...

- any real measurement is limited by the resolution of the instrument to a finite number of discrete outcomes
  - ◆ how finely are you willing to read the scale?
  - ◆ how many digits are there on the display?
  - ◆ what is the resolution of your computer's ADC?
- but we understand that certain measurements *in principle* take on a continuum of values
- our experiments yield average values of that continuum over the resolution bandwidth

# Gaussian (“normal”) distribution

Distribution of Replicated Measurements of the Height of an Office Desk



- in practice measured values are constrained to certain discrete values that the instrument (or the person reading it) can assign
  - ◆ here the raw measurements are limited to exact steps and multiples of 0.1 mm (the red squares)
  - ◆ in a set of replicated measurements, each of these “quantized” values appears an *integral* number of times
  - ◆ in the actual finite-sized set of replicated measurements, statistical fluctuations are observed (the blue diamonds)

# Gaussian measurement model

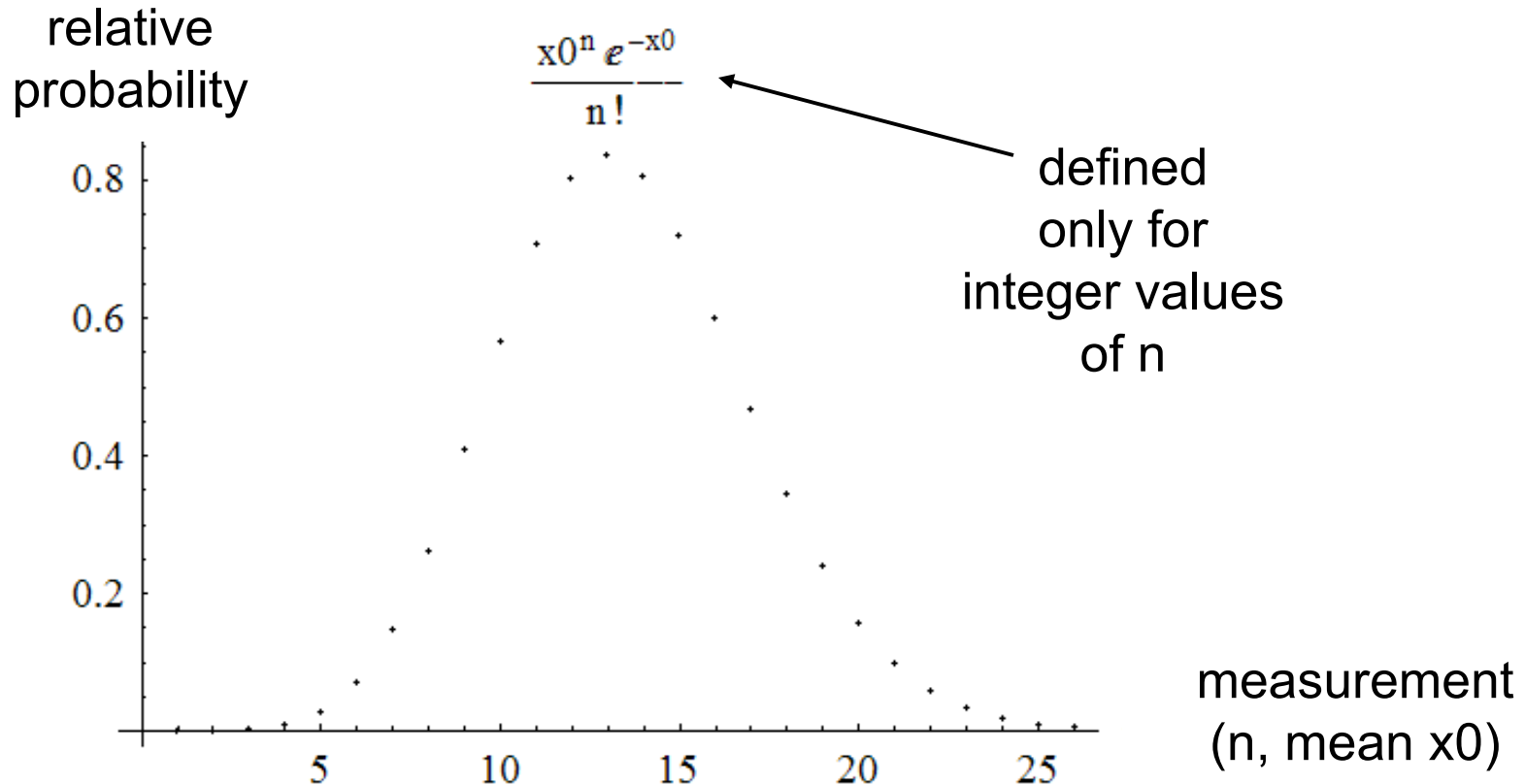
- $$\Delta P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-x_0)^2}{2\sigma^2}} \Delta x$$
  - ◆ in words: the probability of a measurement being between  $x-\Delta x/2$  and  $x+\Delta x/2$  is ...
  - ◆ note:  $\sigma$  has the same units as  $x$ , i.e., millimeters
  - ◆ the distribution has its mean and peak at  $x = x_0$
  - ◆ in this figure  $x_0 = 1000$  mm,  $\sigma = 0.5$  mm
- the Gaussian distribution is *often* a good model, particularly when the measurement scatter is “natural” vs. “technical”

- the actual data (the blue diamonds) plus the model (the Gaussian function of mean  $x_0$  and standard deviation  $\sigma$ ) plus an algorithm for (in some stated sense) minimizing the difference between them together allow you to estimate  $x_0$  (which you state as the outcome of your set of measurements) and  $\sigma$  (which you state as your estimate of the uncertainty of your outcome)
  - ◆ “estimating the parameters of the model”

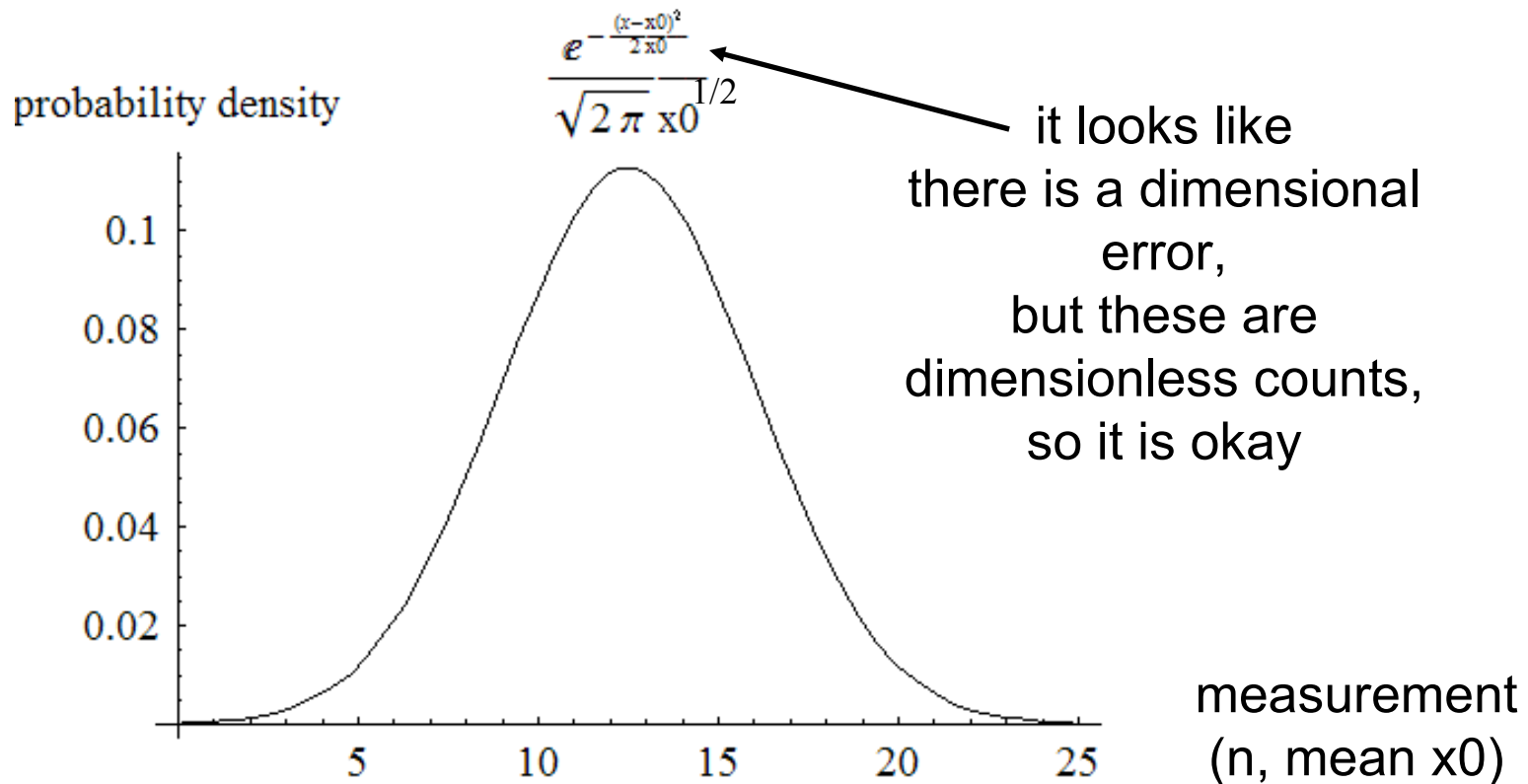


compare and contrast  
Poisson (counting)  
and  
Gaussian (“normal”)  
distributions

measurement whose outcome is an integer count ( $n$ )  
mean value of outcomes = 12.5 (your choice)  
standard deviation of outcomes =  $12.5^{1/2}$  (you have no choice)



measurement  $x$  whose outcome is a continuum value  
mean value of the continuum = 12.5 (your choice)  
standard deviation of outcomes =  $12.5^{1/2}$  (your choice)



# compare & contrast

- ▶ visual examination shows that for  $\langle N \rangle$  greater than about 10 the Poisson distribution is practically indistinguishable from the Gaussian distribution with mean  $\langle N \rangle$  and standard deviation  $\langle N \rangle^{1/2}$
- ▶ but be careful ...
  - ◆ the Gaussian distribution has two independent variables, the mean and the standard deviation (width)
  - ◆ the Poisson distribution has one independent variable, the mean; its standard deviation (width) is completely determined by its mean

measurements with multiple  
sources of uncertainty

vs.

multiple measurements  
of the same quantity made by  
different instruments  
with different uncertainties

# if you take away just one ...

- ... *quantitative* detail from this course, it should be that you know without stopping to think about it ...
  - ◆ how to estimate the error in a measurement composed of measurements of several different quantities for each of which you have an error estimate ...
  - ◆ how to combine separate measurements of the same quantity and estimate the error of their “fusion”

# combining multiple error sources

- in complex measurements there are usually multiple sources of error or uncertainty
  - ◆ measure or estimate each contribution  $\Delta x_i$  to the combined measurement  $X(x_1, x_2, x_3, \dots)$
  - ◆ in the worst possible instance all contributions have the same sign; in the best instance the signs conspire to make the net error zero
  - ◆ in the statistically-typical case they combine with “random phase”, i.e., in quadrature:

$$\Delta X = (\delta x_1^2 + \delta x_2^2 + \delta x_3^2 + \dots)^{1/2}$$

$$\text{where } \delta x_i = (\partial X(x_1, x_2, x_3, \dots) / \partial x_i) \Delta x_i$$

# example: measure the volume of a box

- $V = (L \pm \Delta L) (W \pm \Delta W) (H \pm \Delta H)$   
 $\frac{\partial V}{\partial L} = W H$   
 $\frac{\partial V}{\partial W} = L H$   
 $\frac{\partial V}{\partial H} = L W$
- $\Delta V = ((W H \Delta L)^2 + (L H \Delta W)^2 + (L W \Delta H)^2)^{1/2}$
- $\Delta V/V = (\Delta L/L)^2 + (\Delta W/W)^2 + (\Delta H/H)^2)^{1/2}$
- *note: this ignores the possibility that the box's angles are not precisely 90°; the formula for V should properly include the angles and the consequences of errors in their measurements*



# what if the components have different dimensionality?

- say you are given base area & height:

$$V = (A \pm \Delta A) (H \pm \Delta H)$$

$$\partial V / \partial A = H$$

$$\partial V / \partial H = A$$

- $\Delta V = ((H \Delta A)^2 + (A \Delta H)^2)^{1/2}$

- $\Delta V / V = ((\Delta A / A)^2 + (\Delta H / H)^2)^{1/2}$

- and *if*  $A = L W$

*then*  $\Delta A = ((L \Delta W)^2 + (W \Delta L)^2)^{1/2}$

*so*  $\Delta V = ((L \Delta W)^2 + (W \Delta L)^2 + (A \Delta H)^2)^{1/2}$

# contrasting *error propagation* and *sensor fusion*

- *error propagation*: when a single measurement has multiple error sources, the overall error estimate is  $(\sum \text{component\_error\_sources}^2)^{1/2}$ 
  - ◆ the overall error estimate is (as you should expect) bigger than the biggest single error source
  - ◆ in practice the overall error estimate is often dominated by the *biggest* single error source

- when multiple measurements are made of *the same measurand* using different instruments, their individual contributions are weighted by the *reciprocals* of the squares of their individual error estimates
  - ◆ the overall error estimate is smaller than the smallest individual error source
    - as you should naturally expect!
  - ◆ in practice the overall error estimate is often dominated by the *smallest* single error source
- this is the most basic kind of “sensor fusion”

# example:

## multiple temperature measurements

- three thermometers give measurements  $\pm 1\sigma$   
 $T_1 = 19.0 \pm 2.0 \text{ C}$   
 $T_2 = 18.0 \pm 2.5 \text{ C}$   
 $T_3 = 19.0 \pm 1.0 \text{ C}$
- what is your best estimate of the actual temperature, and what is your estimate of the error in your estimate of the actual temperature?

- ▶ weight each measurement by its “goodness”, i.e., the reciprocal of square of its error estimate:

$$\begin{aligned} T &= (\sum T_i / \sigma_i^2) / (\sum 1 / \sigma_i^2) \\ &= (19./2.^2 + 18./2.5^2 + 19./1.^2) / (1/2.^2 + 1/2.5^2 + 1/1.^2) \\ &= 18.89 \text{ C} \end{aligned}$$

- $\sigma(T) = (\sum \sigma_i^{-2})^{-1/2} = 0.84 \text{ C}$
- as expected, the error estimated for the combined measurement estimate is smaller than any of the individual errors
- as expected, the combined measurement estimate is close to the best individual measurement

# reminder

- make sure this is so clear to you that you never have to think about it twice:
  - ◆ when a single measurement is composed of multiple sub-measurements the overall error estimate is the sum-in-quadrature of the component error contributions
  - ◆ when multiple measurements of the same measurand are available they are fused by averaging with each weighted by the reciprocal of the square of its own error estimate

# assignment

(23) You make five range measurements using an ultrasonic sensor; the sensor's digital display reports ranges  $\{9.9, 10.1, 10.0, 9.9, 10.1\}$  m. The sensor was calibrated at 20 C. The speed-of-sound scales with the square-root of the absolute temperature. Using digital thermometer, you make three measurements:  $\{25.0, 24.9, 25.1\}$  C. Using a mercury-in-glass thermometer you make five measurements:  $\{23.0, 23.5, 26.0, 24.0, 23.5\}$  C. What is your best estimate of the range and its error. (Use one standard deviation for all your error estimates.)

- ▶ some people sometimes use different weights, e.g.,  $\sigma^\alpha$ ,  $1 < \alpha < 2$ , e.g., because they feel it is conservative to give more weight to the less-good data, but  $\alpha = 2$  is usually strictly correct
- ▶ here is a “seat of the pants” derivation, if you are willing to accept some assumptions on faith:
  - ◆ the output measurement is a linear combination of the input measurements. Consider the case of just two:  $m = a m_1 + (1-a) m_2$ ,
    - our aim is to determine optimal weights  $a$ , and  $(1-a)$
  - ◆ the output error depends on *the same* weights applied to the component standard deviations (not their squares):  $\sigma = (a \sigma_1) + ((1-a) \sigma_2)$



- ◆ take the correct value for  $a$  to be the one that minimizes  $\sigma$ :

- $d\sigma/da == 0$

- gives

- $a = \sigma_2 / (\sigma_1 + \sigma_2)$

- $(1-a) = \sigma_1 / (\sigma_1 + \sigma_2)$

- from which:

- $m = (m_1/\sigma_1 + m_2/\sigma_2) / (1/\sigma_1 + 1/\sigma_2)$

- $\sigma = (1/\sigma_1 + 1/\sigma_2)$

- ◆ extend to more than two measurements by applying this result recursively

# next few topics coming up ...

- data acquisition
- light sensors

# updates

- what if you already have an observation of a quantity's mean  $\langle x \rangle$  and standard deviation  $\sigma$
- ... and you make a new measurement  $x'$  with standard deviation  $\sigma'$  ...
- ... and you want to update  $\langle x \rangle$  and  $\sigma$  with your new knowledge of  $x'$  and  $\sigma'$ ?
- it is the same as the problem we just did:  
$$\langle x' \rangle = (\langle x \rangle / \sigma^2 + x' / \sigma'^2) / (1 / \sigma^2 + 1 / \sigma'^2)$$
$$\sigma_{\text{new}} = (1 / \sigma^2 + 1 / \sigma'^2)^{-1/2}$$

# pop quiz

- what do you call an algorithm that implements the above idea, i.e., that the best way to update prior information with new information is to weight each by the square of its standard deviation?
- (just kidding about the quiz :-) )