(A bit more) Finite Element Method

Stelian Coros

From last time…

Goal: find equilibrium configuration i.e., $\mathbf{f}_i = 0 \; \forall i$

Given $\mathbf{x}$ with $\mathbf{f}(\mathbf{x}) \neq 0$, find $\Delta\mathbf{x}$ such that $\mathbf{f}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{0}$

$$\mathbf{f}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \mathbf{K}\Delta\mathbf{x} + O(\Delta\mathbf{x}^2)$$

⇨ Solve $\mathbf{K}\Delta\mathbf{x} = -\mathbf{f}(\mathbf{x})$ for $\Delta\mathbf{x}$

Stiffness matrix

$$\mathbf{K} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

FEM recipe

- Discretize into triangles/tetraderons
- For each element
  - Compute deformation gradient $\mathbf{F} = \boldsymbol{e}\boldsymbol{E}^{-1}$
  - Use material model to define energy density $\Psi(\mathbf{F})$
  - Integrate over element to compute energy: *W*
  - Compute nodal forces as: $\mathrm{f}_{\mathrm{int}} = -\frac{\partial W}{\partial x}$

FEM recipe

St. Venant-Kirchhoff material

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

$$\Psi = \mu\|\mathbf{E}\|_F + \frac{\lambda}{2}\mathrm{tr}^2(\mathbf{E})$$

Neohookean elasticity

$$I_1 = \|\mathbf{F}\|_F^2, \quad J = \det \mathbf{F}$$

$$\Psi = \frac{\mu}{2}(I_1 - 3) - \mu \log(J) + \frac{\lambda}{2}\log^2(J)$$

Area/volume of element

$$\boldsymbol{f} = -\frac{\partial W}{\partial \boldsymbol{x}} = -V\frac{\partial \Psi}{\partial \boldsymbol{F}}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}$$

First Piola-Kirchhoff stress tensor **P**

FEM recipe

St. Venant-Kirchhoff material

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

$$\Psi = \mu\|\mathbf{E}\|_F + \frac{\lambda}{2}\mathrm{tr}^2(\mathbf{E})$$

$$\mathbf{P} = \mathbf{F}\left[2\mu\mathbf{E} + \lambda\mathrm{tr}(\mathbf{E})\mathbf{I}\right]$$

Neohookean elasticity

$$I_1 = \|\mathbf{F}\|_F^2, \quad J = \det \mathbf{F}$$

$$\Psi = \frac{\mu}{2}(I_1 - 3) - \mu\log(J) + \frac{\lambda}{2}\log^2(J)$$

$$\mathbf{P} = \mu(\mathbf{F} - \mathbf{F}^{-T}) + \lambda\log(J)\mathbf{F}^{-T}$$

Area/volume of element

$$\boldsymbol{f} = -\frac{\partial W}{\partial \boldsymbol{x}} = -V\frac{\partial \Psi}{\partial \boldsymbol{F}}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}$$

First Piola-Kirchhoff stress tensor **P**

FEM recipe

St. Venant-Kirchhoff material

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

$$\Psi = \mu\|\mathbf{E}\|_F + \frac{\lambda}{2}\mathrm{tr}^2(\mathbf{E})$$

$$\mathbf{P} = \mathbf{F}\,[2\mu\mathbf{E} + \lambda\mathrm{tr}(\mathbf{E})\mathbf{I}]$$

Neohookean elasticity

$$I_1 = \|\mathbf{F}\|_F^2, \quad J = \det \mathbf{F}$$

$$\Psi = \frac{\mu}{2}(I_1 - 3) - \mu\log(J) + \frac{\lambda}{2}\log^2(J)$$

$$\mathbf{P} = \mu(\mathbf{F} - \mathbf{F}^{-T}) + \lambda\log(J)\mathbf{F}^{-T}$$

$$[\boldsymbol{f}_1 \;\; \boldsymbol{f}_2 \;\; \boldsymbol{f}_3] = -V\boldsymbol{P}\boldsymbol{E}^{-T}; \boldsymbol{f}_4 = -\boldsymbol{f}_1 - \boldsymbol{f}_2 - \boldsymbol{f}_3$$

FEM recipe

- For each element
  - Compute deformation gradient $\mathbf{F} = \boldsymbol{eE}^{-1}$
  - Use material model to define energy density $\Psi(\mathbf{F})$
  - Integrate over element to compute energy: W
  - Compute nodal forces as: $\mathrm{f_{int}} = -\frac{\partial W}{\partial x}$

Assignment 3

- Implement FEM Simulation

Inverse Design Example

- Design an object that can deform like this:

- Need model that predicts the way in which objects deform
  - Design parameters: material parameters, points of application for forces or constraints

“Computational Design of Actuated Deformable Characters”, Skouras et al, 2013

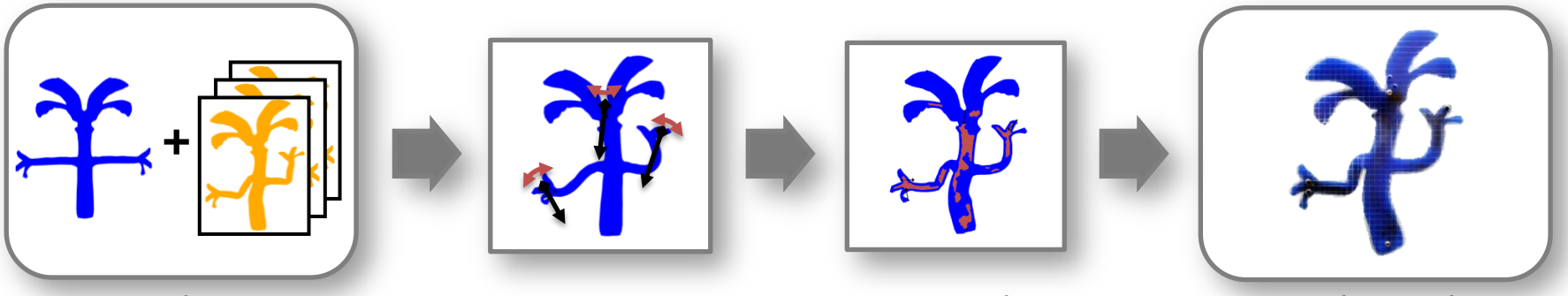
Inverse Design Example

- Design an object that can deform like this:

- FEM Model
  - Design parameters: material parameters, points of application for forces or constraints

“Computational Design of Actuated Deformable Characters”, Skouras et al, 2013

Pipeline



Input and Target Shapes

Actuator Location Optimization

Material Optimization

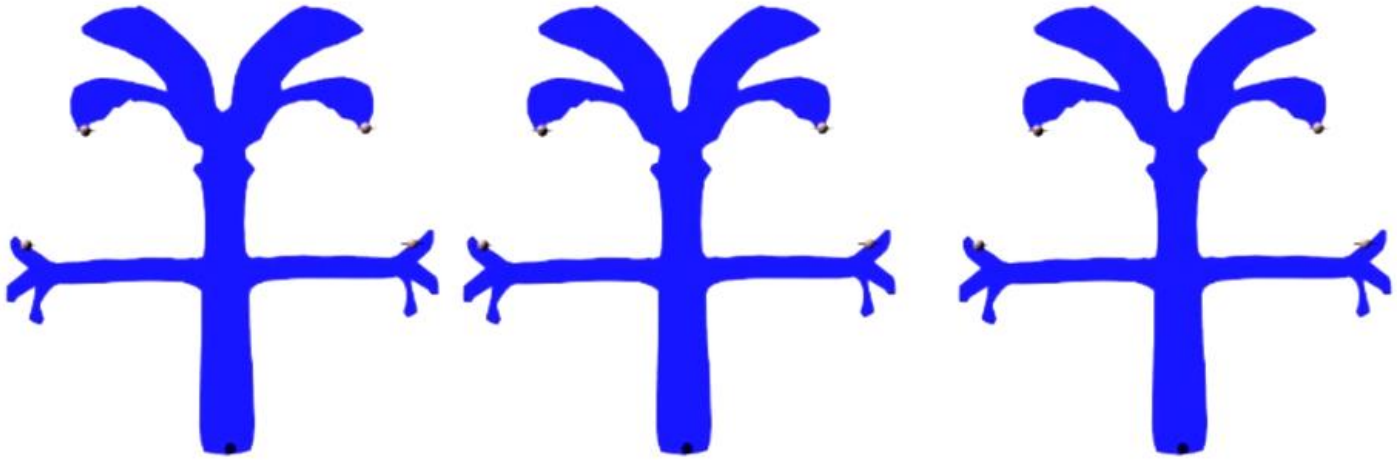
Fabricated Deformable Model

Mathematical Formulation

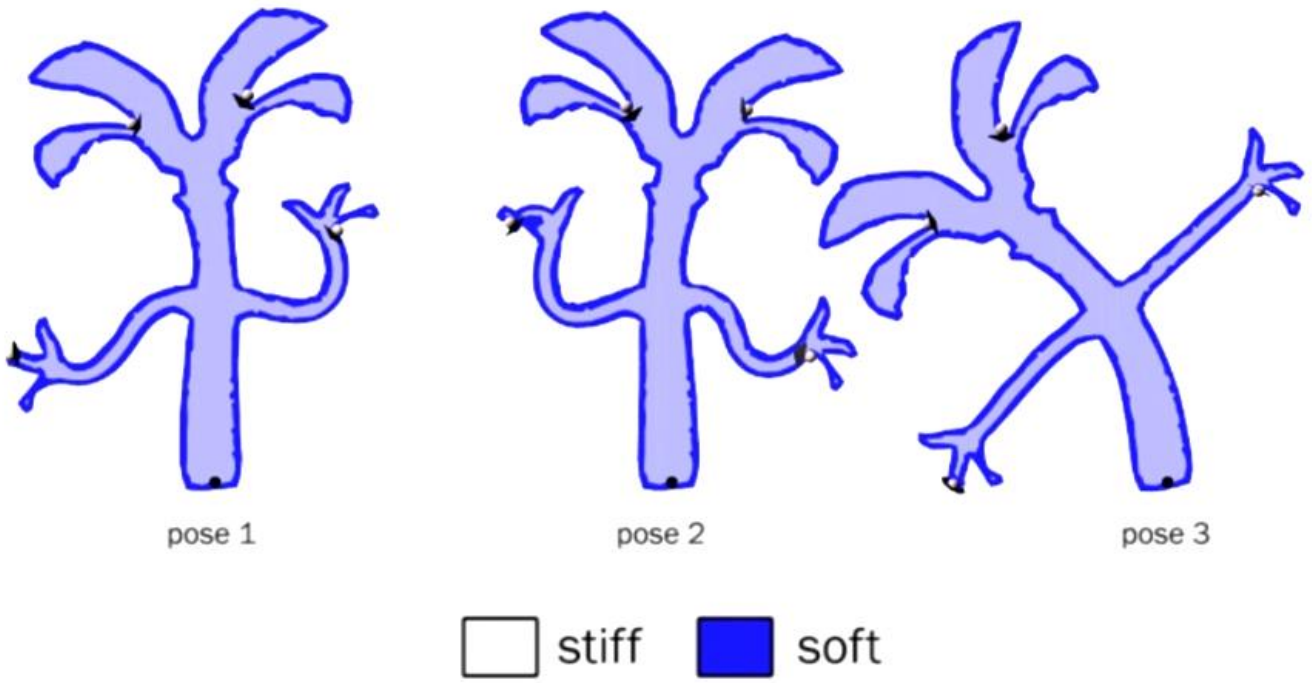
Design Parameters

$$E(\mathbf{x}, \mathbf{p}) = E_{match}(\mathbf{x}, \mathbf{x}_{target}) \quad \text{subject to} \quad \mathbf{f}^{total}(\mathbf{x}, \mathbf{p}) = 0$$

Actuator Location Optimization



Material Distribution Optimization



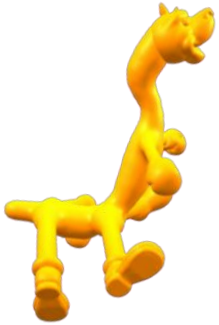
Results

Input Animation

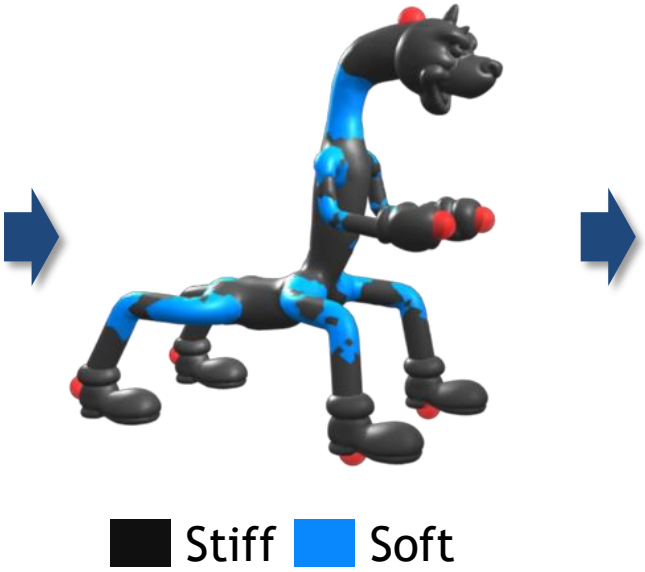
Fabricated Prototype

Results

Rest Pose



Target Pose



Project Proposals and some Project Ideas

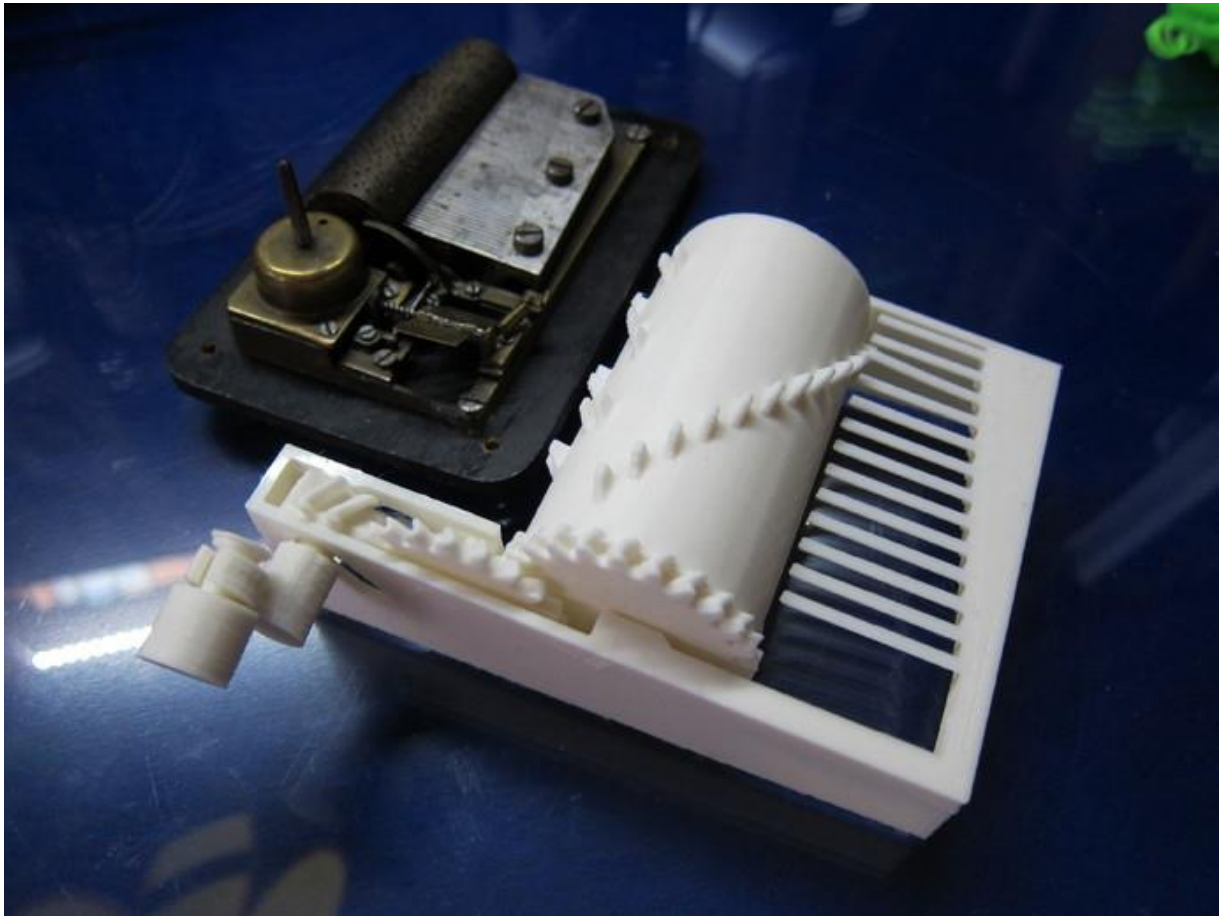
Project Proposals

- 1 page of text/4 paragraphs
  - Problem and motivation
  - Background (include previous work)
  - Proposed technical method
  - Expected results (low and high bar)

Project Proposal Presentations

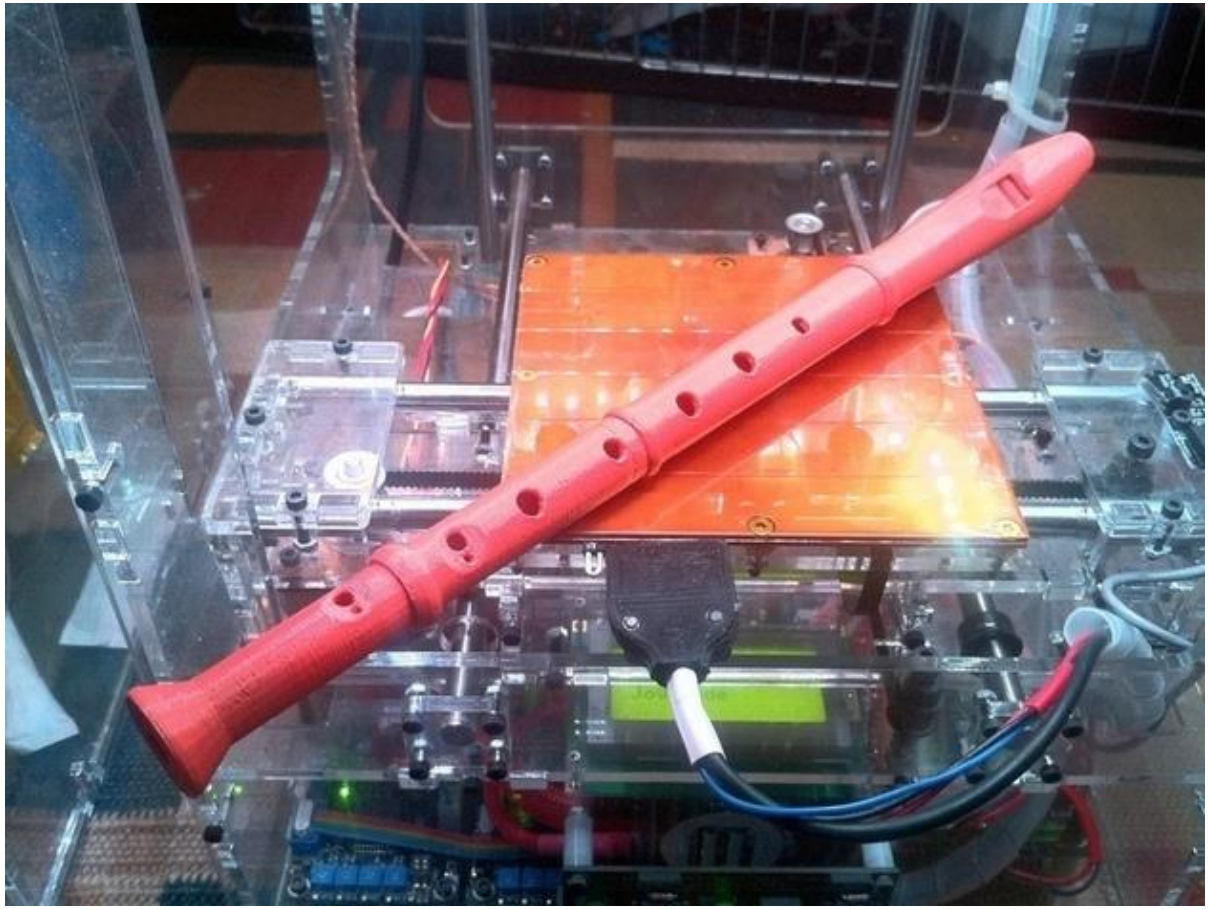
- In-class on March 5th
- 5 minutes/project + discussion/questions
- Slides
  - Problem and motivation (1 slide)
  - Background and Tools (1 slide)
  - Proposed technical method (1-2 slides)
  - Expected results (1 slide)
- Send me slides **at least 1 hour** before the lecture

Parametric music box



http://www.thingiverse.com/thing:53235

Parameterized flute

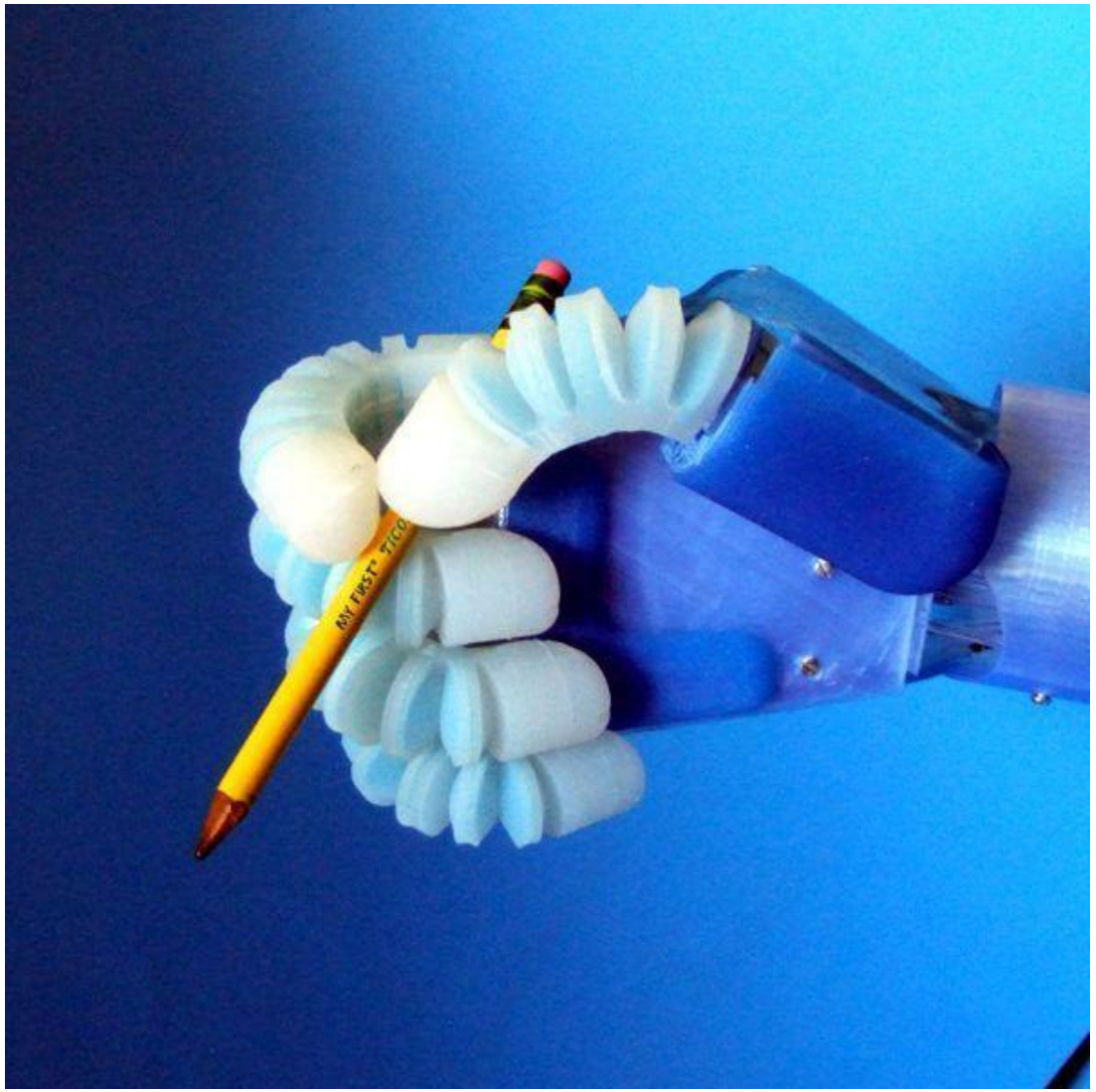


http://www.thingiverse.com/thing:12301

Puzzle Creator

3D Printing animations (zoetrope)

Robot Hands



http://www.3ders.org/articles/20150219-build-an-air-powered-prosthetic-hand-on-a-regular-fdm-3d-printer.html