

WEAK MONOTONICITY CHARACTERIZES
DETERMINISTIC DOMINANT-STRATEGY IMPLEMENTATION

by

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We characterize dominant-strategy incentive compatibility with multi-dimensional types. A deterministic social choice function is dominant-strategy incentive compatible if and only if it is weakly monotone (W-Mon). W-Mon is the following requirement: if changing one agent's type (while keeping the types of other agents fixed) changes the outcome under the social choice function, then the resulting difference in utilities of the new and original outcomes evaluated at the new type of this agent must be no less than this difference in utilities evaluated at the original type of this agent.

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1 Introduction

We characterize dominant-strategy incentive compatibility of deterministic social choice functions in a model with multi-dimensional types, private values, and quasi-linear preferences. We show that incentive compatibility is characterized by a simple monotonicity property of the social choice function. This property, termed weak monotonicity (W-Mon), requires the following: if changing one agent's type (while keeping the types of other agents fixed) changes the outcome under the social choice function, then the resulting difference in utilities of the new and original outcomes evaluated at the new type of this agent must be no less than this difference in utilities evaluated at his original type. In effect W-Mon requires that the social choice function be sensitive to changes in differences in utilities.

It is well known that when agents have multi-dimensional types, characterizations of incentive compatibility are complex. For one-dimensional types, Myerson (1981) showed that a random allocation function in a single-object auction is Bayesian incentive compatible if and only if it is a subgradient of a convex function, which is equivalent to the requirement that each buyer's probability of obtaining the object is non-decreasing in his type. In multi-dimensional environments, while the subgradient condition is still necessary and sufficient for Bayesian incentive compatibility, it is not equivalent to a simple monotonicity requirement.² The subgradient condition is equivalent to the "cyclic monotonicity" condition in Rochet (1987), which is difficult to interpret and use.

Our contribution is to show that when the incentive-compatibility requirement is strengthened to dominant strategy and only deterministic mechanisms are considered, then incentive compatibility in a multi-dimensional types setting is characterized by W-Mon, which is a simple and intuitive condition that generalizes the concept of a non-decreasing function from one to multiple dimensions. As discussed in Section 5,

²See, for example, Rochet (1987), McAfee and McMillan (1988), Williams (1999), Krishna and Perry (1997), Jehiel, Moldovanu, and Stacchetti (1996, 1999), Jehiel and Moldovanu (2001), Krishna and Maennar (2001), and Milgrom and Segal (2002).

the contrast between W-Mon and cyclic monotonicity is the following: the latter is a requirement on every finite selection of type vectors from the domain whereas W-Mon is the same requirement but only for every pair of type vectors. Although cyclic monotonicity is usually stronger and more complicated than W-Mon, in our setting the two turn out to be equivalent. Thus our paper helps delineate the boundaries of multi-dimensional models which permit a characterization that is a simple generalization of Myerson's monotonicity condition.

While other types of monotonicity conditions have been used to characterize dominant-strategy implementability, because we consider smaller domains, these are not sufficient in our model. Maskin monotonicity is a characterization for non-quasilinear settings such as voting models (see Muller and Satterthwaite (1977), and Dasgupta, Hammond, and Maskin (1979)). In quasilinear environments with a complete domain, Roberts (1979) showed that a monotonicity condition called positive association of differences (PAD) is necessary and sufficient for dominant-strategy incentive compatibility. Roberts' complete-domain assumption rules out free disposal and the absence of allocative externalities, and therefore also all environments with private goods such as auctions. In our environment, Roberts' PAD condition imposes no restrictions as all social choice functions satisfy it. W-Mon is the appropriate characterization for the much more restrictive domain of preferences that we consider, one that permits private goods. Chung and Ely (2002) give another characterization for restricted quasilinear environments, which we discuss in Section 5.

Our simplification of the constraint set for incentive compatibility should be helpful in applications such as finding a revenue-maximizing auction in the class of deterministic dominant-strategy auctions. Our characterization also bears upon applications where the mechanism designer is interested in efficiency rather than revenue, such as finding a second-best, dominant-strategy, budget-balanced, double auction. Further, it is well known that because of its computational complexity the Vickrey-Clarke-Groves auction is impractical for selling more than a small number of objects. Several papers investigate computationally feasible (but inefficient) auctions in

private-values settings (see Nisan and Ronen (2000), Lehman et al. (1999), and Holzman and Monderer (2004)). Characterizing the set of incentive-compatible auctions facilitates the selection of an auction that is computationally feasible.

The notion of incentive compatibility in our paper is dominant strategy, which is equivalent to requiring Bayesian incentive compatibility for all possible priors (see Ledyard (1978)). Thus, it is not necessary to assume that agents have priors over the types of all agents (let alone mutual or common knowledge of such priors) for the mechanisms considered here. This weakening of common-knowledge assumptions is in the spirit of the Wilson doctrine (see Wilson (1987)).

In our formulation, we take as primitive a preference order for each agent over the set of outcomes. These orders may be null, partial, or complete, and may differ across agents. We show that W-Mon characterizes dominant-strategy implementability in two environments: (i) when the underlying preference order is partial and a rich-domain assumption holds, and (ii) when the preference order is complete and utilities are bounded. The first environment includes multi-object auctions and the second includes multi-unit auctions with diminishing marginal utilities as special cases. We first prove results for single-agent models, with extensions to many agents being straightforward.

The paper is organized as follows. The characterization of incentive compatibility for a single-agent model is developed in Sections 2 and 3. In Section 4, we extend this characterization many agents. We discuss connections to previous literature in Section 5. The proofs are in an Appendix. A few related examples and results are in the Supplementary Materials to this paper, Bikhchandani et al. (2006).

2 A single-agent model

Let $A = \{a_1, a_2, \dots, a_K\}$ be a finite set of possible outcomes. We assume that the agent has quasilinear preferences over outcomes and (divisible) money. The agent's type,

which is his private information, determines his utility over outcomes. The utility of an agent of type V over outcome a and money m is:

$$\mathcal{U}(a, m, V) = U(a, V) + m, \quad a \in A.$$

It is convenient to assume that the agent's initial endowment of money is normalized to zero and he can supply any (negative) quantity required. We will sometimes write $V(a)$, $V'(a)$ instead of $U(a, V)$, $U(a, V')$ respectively. The domain of V is $D \subseteq \mathfrak{R}_+^K$, with the k th coordinate of type V being $V(a_k)$, this type's utility for outcome a_k .

A *social choice function* f is a function from the agent's report to an outcome in the set A . (The social choice function is deterministic in that the agent's report is not mapped to a probability distribution on A .) As we are interested in truth-telling social choice functions, by the revelation principle we restrict attention to direct mechanisms. Thus, $f : D \rightarrow A$. We assume, without loss of generality, that f is onto A . A *payment function* $p : D \rightarrow \mathfrak{R}$ is a function from the agent's reported type to a money payment by the agent. A *social choice mechanism* (f, p) consists of a social choice function f and a payment function p .

A social choice mechanism is *truth-telling* if truthfully reporting his type is optimal (i.e., is a dominant strategy) for the agent:

$$U(f(V), V) - p(V) \geq U(f(V'), V) - p(V'), \quad \forall V, V' \in D. \quad (1)$$

A social choice function f is *truthful* if there exists a payment function p such that (f, p) is truth-telling; p is said to *implement* f .

Consider the following restriction. A social choice function f is *weakly monotone* (*W-Mon*) if for every V, V' ,

$$U(f(V'), V') - U(f(V), V') \geq U(f(V'), V) - U(f(V), V). \quad (2)$$

If f satisfies W-Mon, then the difference in the agent's utility between $f(V')$ and $f(V)$ at V' is greater than or equal to this difference at V .

W-Mon is a simple and intuitive condition on social choice functions. In effect, it

is a requirement that the social choice function be sensitive to changes in differences in utilities. It is easy to see that W-Mon is a necessary condition for truth-telling:

Lemma 1 *If (f, p) is a truth-telling social choice mechanism then f is W-Mon.*

Proof: Let (f, p) be a truth-telling social choice mechanism. Consider two types V and V' of the agent. By the optimality of truth-telling at V and V' respectively, we have

$$\begin{aligned} U(f(V), V) - p(V) &\geq U(f(V'), V) - p(V') \\ \text{and } U(f(V'), V') - p(V') &\geq U(f(V), V') - p(V) \end{aligned}$$

These two inequalities imply that

$$\begin{aligned} U(f(V'), V') - U(f(V), V') &\geq p(V') - p(V) \\ &\geq U(f(V'), V) - U(f(V), V). \end{aligned}$$

Hence f satisfies W-Mon. *Q.E.D.*

Next, we obtain conditions on D , the domain of the agent's types, under which W-Mon is sufficient for truth-telling.

3 Sufficiency of W-Mon

If the domain of the agent's types, D , is not large enough then W-Mon is not sufficient for truth-telling. This is clear from the following example.

Example 1: There are three outcomes a_1 , a_2 , and a_3 . The agent's type is a vector representing his utilities for these outcomes. The agent has three possible types: $V^1 = (0, 55, 70)$, $V^2 = (0, 60, 85)$, $V^3 = (0, 40, 75)$. That is, $V^1(a_1) = 0$, $V^1(a_2) = 55$, and $V^1(a_3) = 70$ and so on. The domain of types is $D = \{V^1, V^2, V^3\}$.

The social choice function $f(V^1) = a_1$, $f(V^2) = a_2$, and $f(V^3) = a_3$ is W-Mon on the set D because:

$$V^2(a_2) - V^2(a_1) = 60 - 0 \geq 55 - 0 = V^1(a_2) - V^1(a_1)$$

$$\begin{aligned}
V^3(a_3) - V^3(a_2) &= 75 - 40 \geq 85 - 60 = V^2(a_3) - V^2(a_2) \\
V^1(a_1) - V^1(a_3) &= 0 - 70 \geq 0 - 75 = V^3(a_1) - V^3(a_3)
\end{aligned}$$

However, there is no payment function that implements f . Suppose that the agent pays p^1 at report V^1 , p^2 at report V^2 , and p^3 at report V^3 . Without loss of generality, let $p^1 = 0$. For truth-telling we must have $p^2 \geq 55$, else type V^1 would report V^2 . Similarly, $p^3 - p^2 \geq 25$ else type V^2 would report V^3 . Therefore, we must have $p^3 \geq 80$. But then type V^3 would report V^1 .

Even if the domain of types is connected, W-Mon is not sufficient for truthfulness. Let \hat{D} be the sides of the triangle with corners V^1 , V^2 , and V^3 defined above. Let $[V^i, V^j)$ denote the half-open line segment joining V^i to V^j . The allocation rule \hat{f} is as follows: $\hat{f}(V) = a_1, \forall V \in [V^1, V^3)$, $\hat{f}(V) = a_2, \forall V \in [V^2, V^1)$, and $\hat{f}(V) = a_3, \forall V \in [V^3, V^2)$. It may be verified that \hat{f} satisfies W-Mon but there are no payments that induce truth-telling under \hat{f} . \triangle

Requiring W-Mon on a larger domain (than in the example) strengthens this condition. To this end, we define order-based preferences over the possible outcomes.

Order-based domains: We restrict attention to domains $D \subseteq R_+^K$. In certain contexts, regardless of his type, the agent has an order of preference over some of the outcomes in the set A . In a multi-object auction, for instance, where an outcome is the bundle of objects allocated to the agent, if $a_\ell \subset a_k$, then under free disposal it is natural that the agent prefers a_k to a_ℓ and $V(a_\ell) \leq V(a_k)$ for all $V \in D$. Therefore, we take as a primitive the finite set of outcomes A and a (weak) order \succeq on it. This order may be null, partial, or complete.

A type V is *consistent* with respect to (A, \succeq) if $a_k \succeq a_\ell$ implies $V(a_k) \geq V(a_\ell)$. A domain of types D is *consistent* with respect to (A, \succeq) if every type in D is consistent with respect to (A, \succeq) . We will sometimes write domain D on (A, \succeq) to mean D is consistent with respect to (A, \succeq) .

If \succeq is null then D is an *unrestricted* domain in the sense that for any $a_k, a_\ell \in A$, there may exist $V, V' \in D$ such that $V(a_k) > V(a_\ell)$ and $V'(a_k) < V'(a_\ell)$. If, instead,

\succeq is a partial order then D is a *partially ordered* domain: for any $a_k, a_\ell \in A$ if $a_k \succeq a_\ell$, then $V(a_k) \geq V(a_\ell)$ for all $V \in D$. If \succeq is a complete order then D is a *completely ordered* domain: for any $a_k, a_\ell \in A$ either $V(a_k) \geq V(a_\ell)$ for all $V \in D$ or $V(a_k) \leq V(a_\ell)$ for all $V \in D$ depending on whether $a_k \succeq a_\ell$ or $a_k \preceq a_\ell$.

Examples:

(i) As already mentioned, in a multi-object auction the set of outcomes A is a list of possible subsets of objects that the agent might be allocated. The order \succeq is the partial order induced by set inclusion.

(ii) A multi-unit auction is a special case of a multi-object auction in which all objects are identical. Let the outcomes be the number of objects allocated to the agent. Thus, for any $a_k, a_\ell \in A$ either $a_k \leq a_\ell$ or $a_\ell \leq a_k$; accordingly either $a_k \preceq a_\ell$ or $a_k \succeq a_\ell$ and \succeq is a complete order.

(iii) Another special case is when the agent has assignment-model preferences over $K - 1$ heterogeneous objects. Let the outcome a_1 denote no object assigned to the agent, and let a_{k+1} , $k = 1, 2, \dots, K - 1$, denote the assignment of the k th object to the agent. The allocation of more than one object to the agent is not permitted. The underlying order is $a_k \succeq a_1$, for all $k \geq 2$, and $a_k \not\preceq a_\ell$, for all $k, \ell \geq 2$, $k \neq \ell$. \triangle

In an auction, there is an outcome at which the agent does not get any object; the utility of this outcome is 0 for all types of the agent. The proofs in Section 3.2 (but not in Section 3.1) require the existence of such an outcome.

The following definitions will be needed in the sequel. The inverse of a social choice function f is

$$Y(k) \equiv \{V \in D \mid f(V) = a_k\},$$

where the dependence of Y on f is suppressed for notational simplicity. For any $k, \ell \in \{1, 2, \dots, K\}$, define

$$\delta_{k\ell} \equiv \inf\{V(a_k) - V(a_\ell) \mid V \in Y(k)\}. \quad (3)$$

Note that $\delta_{kk} = 0$.

Next, we prove sufficiency of W-Mon for partially ordered domains.

3.1 Partially ordered domains

Recall that the set of outcomes is $A = (a_1, a_2, \dots, a_K)$. Throughout Section 3.1 we make the following assumption on the domain of types.

Rich-domain assumption: *Let D be a domain of types on (A, \succeq) . Then D is rich if every $V \in \mathcal{R}_+^K$ that is consistent with (A, \succeq) belongs to D .*

Thus, if \succeq is null then $D = \mathcal{R}_+^K$. If, instead, \succeq is a partial order then D is the largest subset of \mathcal{R}_+^K satisfying inequalities $V(a_k) \geq V(a_\ell)$ whenever $a_k \succeq a_\ell$ for all $a_k, a_\ell \in A$. It is easily verified that the formulations of the auction examples of the previous section admit rich domains.

Next, we define a payment function that implements a social choice function f satisfying W-Mon on a rich domain. Relabelling the outcomes if necessary, let a_K be an outcome which is maximal under \succeq .³ As D is rich, for each $a_\ell \in A$ there exists a $V \in D$ such that $V(a_K) > V(a_\ell)$. Consider the payment function

$$p_k \equiv -\delta_{Kk}, \quad \forall k = 1, 2, \dots, K. \quad (4)$$

That is, if the agent reports $V \in Y(k)$ then the outcome a_k is selected by f and the agent pays p_k . The next result shows that this payment function enforces incentives between a_K and any other outcome a_ℓ .

Lemma 2 *Let f be a social choice function that is W-Mon. For any $a_\ell \in A$ and $V \in D$,*

- (i) *If $V(a_\ell) - p_\ell < V(a_K) - p_K$ then $f(V) \neq a_\ell$.*
- (ii) *If $V(a_\ell) - p_\ell > V(a_K) - p_K$ then $f(V) \neq a_K$.*

This leads to the main result for partially ordered domains.

³In a multi-object auction, a_K is any maximal subset (with respect to set inclusion) in the range of the mechanism. Thus, if the outcome at which all objects are allocated to the agent is in the range of the mechanism then this outcome is a_K .

Theorem 1 *A social choice function on a rich domain is truthful if and only if it is weakly monotone.*

As already observed, the smaller the domain of types on which the social choice function satisfies W-Mon, the weaker the restriction imposed by W-Mon. Therefore, next we investigate whether W-Mon is sufficient for truth-telling when the domain is not rich, in particular the domain is bounded. To obtain a sufficiency result with smaller-domain assumptions, we make the stronger assumption that the underlying order is complete.

3.2 Completely ordered domains

The order \succeq on the set of outcomes is complete. That is, for any $a_k, a_\ell \in A$, either $a_k \succeq a_\ell$ or $a_\ell \succeq a_k$ but not both. (If all types of the agent are indifferent between two outcomes, then we can combine these two outcomes into one.) Thus, for any domain D consistent with (A, \succeq) either $V(a_k) \geq V(a_\ell)$ for all $V \in D$ or $V(a_\ell) \geq V(a_k)$ for all $V \in D$. We label the outcomes such that $a_k \succeq a_{k-1}$. Define for each type V the marginal (or incremental) utility of the k th outcome over the $(k-1)$ th outcome:

$$v_k \equiv V(a_k) - V(a_{k-1}) \geq 0, \quad k = 1, 2, \dots, K.$$

For notational simplicity, we have $K+1$ outcomes rather than K . Further, we assume that the utility of outcome a_0 is the same for each type in D , and we normalize $V(a_0) \equiv 0, \forall V \in D$.

A multi-unit auction has a completely ordered domain, with the number of units allocated to the buyer being the outcomes. Therefore, we denote the set of outcomes as $A = \{0, 1, 2, \dots, K\}$ (rather than $\{a_0, a_1, \dots, a_K\}$). It is convenient to define the agent's type in terms of marginal utilities $v = (v_1, v_2, \dots, v_K)$ for each successive unit (rather than total utilities $V = (V(1), V(2), \dots, V(K))$). The social choice and payment functions map marginal utilities to an outcome $k = 0, 1, \dots, K$ and to payments respectively. The inverse social choice function $Y(\cdot)$ maps integers $k = 0, 1, \dots, K$ to subsets of types (in marginal utility space).

In this setting, the W-Mon inequality (2) may be restated as follows. A social choice rule f is W-Mon if for every v and v' ,

$$\text{If } f(v') > f(v) \text{ then } \sum_{\ell=f(v)+1}^{f(v')} v'_\ell \geq \sum_{\ell=f(v)+1}^{f(v')} v_\ell. \quad (5)$$

Suppose that f is the allocation rule of a multi-unit auction and that the agent is allocated more units by the mechanism when his (reported) type is v' than when it is v . If f is W-Mon then the agent's valuation at v' for the additional units allocated at v' is at least as large as his valuation at v .

The domain in Example 1 is completely ordered but W-Mon is not sufficient for truthfulness; therefore, we need a larger domain. The following assumption encompasses both bounded utilities and diminishing marginal utilities.⁴

Bounded-domain assumption: *There exist constants $\bar{v}_k \in (0, \infty)$, $k = 1, 2, \dots, K$, such that the domain of agent types, \mathcal{D} , satisfies either (A) or (B) below:*

A. $\mathcal{D} = \prod_{k=1}^K [0, \bar{v}_k]$

B. \mathcal{D} is the convex hull of points $(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k-1}, \bar{v}_k, 0, \dots, 0)$, $k = 0, 1, \dots, K$.

The assumption that $\bar{v}_k < \infty$ for all k is not essential, but does simplify the proofs. Domain assumption A does not restrict the marginal utilities to be decreasing (or increasing). We do not specifically assume that $\bar{v}_k \geq \bar{v}_{k+1}$, but when this inequality holds for all k and domain assumption B is satisfied, then we have diminishing marginal utilities; that is, $v_k \geq v_{k+1}$ for all $v \in \mathcal{D}$. Under domain assumption B, $v = (v_1, v_2, \dots, v_K) \in \mathcal{D}$ if and only if $0 \leq v_\ell \leq \bar{v}_\ell, \forall \ell$ and

$$\frac{v_\ell}{\bar{v}_\ell} \geq \frac{v_{\ell+1}}{\bar{v}_{\ell+1}} \quad \ell = 1, 2, \dots, K - 1. \quad (6)$$

We note that a straightforward modification in the proofs extends our results to the case of increasing marginal utilities, i.e., when \mathcal{D} is the convex hull of points $(0, 0, \dots, 0, \bar{v}_k, \bar{v}_{k+1}, \dots, \bar{v}_K)$, $k = 0, 1, \dots, K$. The assumption of increasing marginal utilities obtains when the objects are complements, such as airwave spectrum rights.

⁴The domain of types is referred to by \mathcal{D} rather than D as types now specify marginal utilities rather than total utilities.

Recalling the definition in (3), note that

$$\delta_{k k-1} = \inf\{v_k \mid v \in Y(k)\} \tag{7}$$

$$\delta_{k-1 k} = -\sup\{v_k \mid v \in Y(k-1)\}.$$

Next, a “tie-breaking at boundaries” assumption, TBB, is invoked to deal with difficulties at the boundary of the domain.

Tie-breaking at boundaries (TBB): *A social choice function f satisfies TBB if:*

(i) $v_k > 0$ for all $v \in Y(k)$, and

(ii) $v_k < \bar{v}_k$ for all $v \in Y(k-1)$.

Consider TBB(i). If $\delta_{k k-1} > 0$ then TBB(i) imposes no restriction. If, instead, $\delta_{k k-1} = 0$ then there exists a sequence $v^n \in Y(k)$ such that $\lim_{n \rightarrow \infty} v_k^n = 0$; the existence of a point $v \in Y(k)$ at which $v_k = \delta_{k k-1} = 0$ is precluded by TBB(i). Similarly, TBB(ii) imposes no restriction if $-\delta_{k-1 k} < \bar{v}_k$, and if, instead, $-\delta_{k-1 k} = \bar{v}_k$ it requires that for any $v \in Y(k-1)$, we have $v_k < \bar{v}_k$.

First, we prove sufficiency of W-Mon and TBB (Lemmas 3 and 4) for truth-telling. We then show (Lemma 5) that (i) for any W-Mon social choice function f there exists a social choice function f' that satisfies W-Mon and TBB and agrees with f almost everywhere, and (ii) the money payments which truthfully implement f' also truthfully implement f .

The next lemma will be used to define payment functions that implement f .

Lemma 3 *Let f be a social choice function on a completely ordered, bounded domain. If f satisfies W-Mon and TBB then $\bar{v}_k \geq \delta_{k k-1} = -\delta_{k-1 k} \geq 0$ for all k .*

It is clear from (7) that $v_\ell \geq \delta_{\ell \ell-1}$ for any $v \in Y(\ell)$ and $v'_\ell \leq -\delta_{\ell-1 \ell}$ for any $v' \in Y(\ell-1)$. This, together with Lemma 3, implies that $v'_\ell \leq -\delta_{\ell-1 \ell} = \delta_{\ell \ell-1} \leq v_\ell$. In other words, the hyperplane $v_\ell = \delta_{\ell \ell-1}$ weakly separates $Y(\ell)$ and $Y(\ell-1)$. Hence, for any payment function that implements f the difference in the payments at points

in $Y(\ell)$ and $Y(\ell - 1)$ must be $\delta_{\ell\ell-1}$. Therefore, consider the following payment function:

$$p_k \equiv \begin{cases} \sum_{\ell=1}^k \delta_{\ell\ell-1}, & \text{if } \ell = 1, 2, \dots, K \\ 0, & \text{if } \ell = 0. \end{cases} \quad (8)$$

The preceding discussion implies that, under this payment function, any type $v \in Y(\ell)$ has no incentive to misreport his type in $Y(\ell - 1)$ or $Y(\ell + 1)$. That the agent has no incentive to misreport his type under this payment function is proved in the next lemma.

Lemma 4 *A social choice function on a completely ordered, bounded domain is truthful if it satisfies W-Mon and TBB.*

The next lemma allows one to dispense with TBB in the sufficient condition for truth-telling.

Lemma 5 *If a social choice mechanism f satisfies W-Mon then there exists an allocation mechanism f' which satisfies W-Mon and TBB such that $f(v) = f'(v)$, for almost all $v \in \mathcal{D}$. Moreover, the payment function p'_k defined as in (8) with respect to f' truthfully implements f .*

Lemma 5 assures us that given any social choice function f that satisfies W-Mon we can construct another social choice function f' which is W-Mon and TBB. By Lemma 4, f' is truthful and by Lemma 5 the payment function which implements f' also implements f . Thus, W-Mon is sufficient for truth-telling. This leads to the main result for completely ordered domains.

Theorem 2 *A social choice function on a completely ordered bounded domain is truthful if and only if it is weakly monotone.*

An alternative characterization for the single-agent completely ordered domain model is through the payment function rather than the social choice function. Consider a multi-unit auction with one buyer. The allocation rule “induced” by any increasing payment function ($p_k \geq p_{k-1} \geq 0$) is implementable. We note that this

characterization becomes considerably more complex when one considers two or more buyers. This is because each buyer's payment function will, in general, depend on others' reported types and for each vector of types, it must be verified that the induced allocation rule does not distribute more units than are available. Our characterization based on W-Mon is easily generalized to multi-agent settings, both for completely ordered and partially ordered domains.

4 Extension to multiple agents

We extend the results of the single-agent model to multiple agents, with each agent having private values over the possible outcomes. For concreteness, we use the set-up of Section 3.1; an identical argument extends the results of Section 3.2.

There are $i = 1, 2, \dots, n$ agents and the finite set of outcomes is $A = \{a_1, a_2, \dots, a_L\}$. Agent i 's type is denoted by $V_i = (V_{i1}, V_{i2}, \dots, V_{i\ell}, \dots, V_{iL})$, where each $V_i \in D_i \subseteq \mathbb{R}_+^L$. The characteristics of all the agents are denoted by $V = (V_1, V_2, \dots, V_i, \dots, V_n)$.⁵ The private-values assumption is that each agent's utility function depends only on his type. Thus, when the types are $V = (V_i, V_{-i})$, agent i 's utility over the outcome a and m units of money is $\mathcal{U}_i(a, m, (V_i, V_{-i})) = U_i(a, V_i) + m$, $a \in A$.

The outcome set A is endowed with (partial) orders, \succeq_i , $i = 1, 2, \dots, n$, one for each agent. The domain of agents' types, $D = D_1 \times D_2 \times \dots \times D_n$, is consistent with $(A, \succeq_1, \succeq_2, \dots, \succeq_n)$ if each D_i , the domain of agent i 's types, is consistent with (A, \succeq_i) . Further, D is rich if each D_i is rich (as defined in Section 3.1).

In an auction, A represents the set of possible assignments of objects to agents (buyers). If buyer i cares only about the objects allocated to him, then the partial order \succeq_i is determined by set inclusion on the respective allocations to buyer i at $a, a' \in A$. Thus, $a \sim_i a'$, (i.e., $U_i(a, V_i) = U_i(a', V_i)$, $\forall V_i \in D_i$) whenever a and a'

⁵In a departure from the notation of Section 3, V now refers to a profile of utilities for n agents rather than for a single agent.

allocate the same bundle of objects to buyer i .

A social choice function f is a mapping from the domain of all agents' (reported) types onto A , $f : D \rightarrow A$. For each agent i there is a payment function $p_i : D \rightarrow \mathfrak{R}$. Let $p = (p_1, p_2, \dots, p_n)$. The pair (f, p) is a social choice mechanism. A social choice mechanism is *dominant-strategy incentive compatible* if truthfully reporting one's type is a dominant strategy for each agent. That is, for every i , V_i, V'_i, V_{-i} ,

$$U_i(f(V_i, V_{-i}), V_i) - p_i((V_i, V_{-i})) \geq U_i(f(V'_i, V_{-i}), V_i) - p_i(V'_i, V_{-i}). \quad (9)$$

A social choice function f is *dominant-strategy implementable* if there exist payment functions p such that (f, p) is dominant-strategy incentive compatible.

The following definition generalizes weak monotonicity to a multiple-agent setting. A social choice function f is *weakly monotone (W-Mon)* if for every i , V_i, V'_i, V_{-i} ,

$$U_i(f(V'_i, V_{-i}), V'_i) - U_i(f(V_i, V_{-i}), V'_i) \geq U_i(f(V'_i, V_{-i}), V_i) - U_i(f(V_i, V_{-i}), V_i). \quad (10)$$

Observe that the requirement of dominant strategy, (9), is the same as requiring truth-telling (i.e. (1)) for each agent i , for each value of V_{-i} . Further, (10) is equivalent to requiring (2) for each agent i , for each value of V_{-i} . Thus, Theorem 1 (and similarly also Theorem 2) generalize:

Theorem 3

(i) *A social choice function on a rich domain is dominant-strategy implementable if and only if it is weakly monotone.*

(ii) *A social choice function on a completely ordered, bounded domain is dominant-strategy implementable if and only if it is weakly monotone.*

5 Relationship to earlier work

In his seminal paper, Myerson (1981) showed that a necessary and sufficient condition for Bayesian incentive compatibility of a single-object auction is that each buyer's

probability of receiving the object is non-decreasing in his reported valuation.⁶ Several authors, including Rochet (1987), McAfee and McMillan (1988), Williams (1999), Krishna and Perry (1997), Jehiel, Moldovanu, and Stacchetti (1996, 1999), Jehiel and Moldovanu (2001), Krishna and Maennar (2001), and Milgrom and Segal (2002), have extended Myerson's analysis to obtain necessary and sufficient conditions for Bayesian incentive-compatible mechanisms in the presence of multi-dimensional types. These results are easily adapted to dominant-strategy mechanisms.

To place our results in the context of this earlier work, let G be a (random) social choice function that maps the domain of agents' types D to a probability distribution over the set of outcomes $A = \{a_1, a_2, \dots, a_L\}$. Thus, for each $V \in D$, $G(V) = (g_1(V), g_2(V), \dots, g_\ell(V), \dots, g_L(V))$ is a probability distribution. Recall that the payment functions are $p = (p_1, p_2, \dots, p_n)$. A social choice mechanism (G, p) induces the following payoff function for agent i :

$$\Pi_i(V_i, V_{-i}) \equiv G(V_i, V_{-i}) \cdot V_i - p_i(V_i, V_{-i}),$$

where $x \cdot y$ denotes the dot product of two vectors x and y . Dominant-strategy incentive compatibility implies that for all i, V_i, V'_i, V_{-i} ,

$$\begin{aligned} \Pi_i(V_i, V_{-i}) &\geq G(V'_i, V_{-i}) \cdot V_i - p_i(V'_i, V_{-i}) \\ &= \Pi_i(V'_i, V_{-i}) + G(V'_i, V_{-i}) \cdot (V_i - V'_i), \quad (11) \\ \implies \Pi_i(V_i, V_{-i}) &= \max_{V'_i} \{G(V'_i, V_{-i}) \cdot V_i - p_i(V'_i, V_{-i})\} \end{aligned}$$

As $\Pi_i(\cdot, V_{-i})$ is the maximum of a family of linear functions, it is a convex function of V_i . Further, for each i and V_{-i} , $G(\cdot, V_{-i})$ is a subgradient of $\Pi_i(\cdot, V_{-i})$. This leads to the following characterization: A social choice function G is dominant-strategy implementable if and only if for each V_{-i} , $G(\cdot, V_{-i})$ is a subgradient of a convex function from D_i to \Re .

⁶Myerson characterized Bayesian incentive compatibility when agents' types are one dimensional; simple modifications to his proofs yield a similar characterization for dominant-strategy incentive compatibility. Myerson's characterization coincides with W-Mon applied to one-dimensional types.

A function $G(\cdot, V_{-i}) : D_i \rightarrow \mathfrak{R}^L$, $D_i \subseteq \mathfrak{R}^L$, is *cyclically monotone* if for every finite selection $V_i^j \in D_i$, $j = 1, 2, \dots, m$, with $V_i^{m+1} = V_i^1$,

$$\sum_{j=1}^m V_i^j \cdot [G(V_i^j, V_{-i}) - G(V_i^{j+1}, V_{-i})] \geq 0. \quad (12)$$

A function is a subgradient of a convex function if and only if it is cyclically monotone (Rockafellar (1970, p. 238)). Thus, cyclic monotonicity of the social choice function also characterizes dominant-strategy implementability. The rationalizability condition of Rochet (1987) generalizes the cyclic-monotonicity characterization of incentive compatibility to settings where the utility function is possibly non-linear.

W-Mon is a weaker condition than cyclic monotonicity. To see this, note that if $m = 2$ then (12) may be restated as $[G(V_i', V_{-i}) - G(V_i, V_{-i})] \cdot (V_i' - V_i) \geq 0$ for all V_i, V_i' . This is the same as (10), with $U_i(G(V_i, V_{-i}), V_i) = G(V_i, V_{-i}) \cdot V_i$, etc. Thus, W-Mon requires the inequality in (12) only for every pair of types whereas Rochet's cyclic-monotonicity condition requires this inequality for all finite selections of types.

For one-dimensional types, cyclic monotonicity is equivalent to W-Mon, which is equivalent to a non-decreasing subgradient function (Rockafellar (1970, p. 240)). Hence, Myerson's characterization of incentive compatibility as a non-decreasing allocation function. W-Mon, which generalizes the concept of a non-decreasing function, does not characterize incentive compatibility in a multi-dimensional setting with random mechanisms; the more complex condition of cyclic monotonicity is needed. Our contribution is to show that when one restricts attention to deterministic social choice functions, dominant-strategy incentive compatibility is characterized by W-Mon.

Although our characterization is significantly simpler, the restriction to deterministic mechanisms may be an important limitation. Manelli and Vincent (2003) and Thanassoulis (2004) show that a multi-product monopolist can strictly increase profits by using a random, rather than deterministic, mechanism. Example S1 in the Supplementary Materials establishes that for random social choice functions W-Mon is not sufficient for dominant-strategy implementability.⁷ Whether there is an intuitive

⁷We are grateful to an anonymous referee for this example.

condition, which in conjunction with W-Mon, is sufficient for incentive compatibility of random social choice functions is an open question.

Chung and Ely (2002) obtain a characterization of dominant-strategy implementability of random social choice functions which they call *pseudo-efficiency*. When restricted to deterministic social choice functions, pseudo-efficiency requires that there exist real-valued functions $w_i(a, V_i)$ such that for each V ,

$$f(V) \in \arg \max_{a \in A} \left(U_i(a, V_i) + w_i(a, V_i) \right), \quad \forall i.$$

For deterministic social choice functions over a finite set of outcomes, W-Mon must be equivalent to pseudo-efficiency. The definition of the latter involves an existential quantifier which makes it hard to verify.

Roberts (1979) characterizes deterministic dominant-strategy mechanisms in quasilinear environments with a “complete” domain. Roberts identifies a condition called *positive association of differences* (PAD) which is satisfied by a social choice function f if for all $V = (V_1, V_2, \dots, V_n)$ and $V' = (V'_1, V'_2, \dots, V'_n)$

$$\begin{aligned} & \text{if } U_i(f(V), V'_i) - U_i(a, V'_i) > U_i(f(V), V_i) - U_i(a, V_i), \quad \forall a \neq f(V), \forall i, \\ & \text{then } f(V') = f(V). \end{aligned} \tag{13}$$

An allocation rule f is an *affine maximizer* if there exist constants $\gamma_i \geq 0$, with at least one $\gamma_i > 0$, and a function $U_0 : A \rightarrow \Re$ such that

$$f(V) \in \arg \max_{a \in A} \left(U_0(a) + \sum_{i=1}^n \gamma_i U_i(a, V_i) \right).$$

Roberts (1979) shows that f is a (deterministic) dominant-strategy mechanism if and only if f satisfies PAD if and only if f is an affine maximizer.

What is the relationship between Roberts’ work and ours? The fundamental difference is that Roberts assumes an unrestricted domain of preferences while we operate in a restricted domain. In particular, Roberts requires that for all $a \in A$, any real number α , and any agent i , there exists a type V_i of agent i such that $U_i(a, V_i) = \alpha$. Thus, taking $(A, \succeq_1, \succeq_2, \dots, \succeq_n)$ and the domain of types as primitives of the two

models, in Roberts' model \succeq_i is a null order and $D_i = \mathfrak{R}^L$ for each agent i , whereas we allow each \succeq_i to be a non-null (even complete) order and the corresponding D_i to be a strict subset of \mathfrak{R}_+^L . We note that $D_i = \mathfrak{R}^L$, for all i , is essential for Roberts' results. Thus, an auction or any mechanism that allocates (private) goods does not satisfy Roberts' domain assumptions as they preclude free disposal and no externalities in consumption. Indeed, in an auction with two or more buyers PAD is vacuous in that all mechanisms satisfy PAD.⁸ W-Mon, however, is not vacuous in this setting and is the appropriate condition for incentive compatibility.⁹ Because a smaller domain (than Roberts') is sufficient for our characterization, one may suspect that W-Mon is stronger than PAD. This is proved in Lemma S1 in the Supplementary Materials. An important difference between these two conditions is that PAD impose restrictions on the social choice function only for changes in types of all players, while W-Mon imposes restrictions for changes in one player's type.¹⁰ Thus, W-Mon and PAD are not equivalent. Further, because of the domain restrictions inherent in our model, our result is not a consequence of Roberts' characterization.

It may be useful conceptually to draw an approximate parallel with the results on dominant-strategy incentive compatibility in various domains. According to the Gibbard-Satterthwaite Theorem, dominant-strategy implementability is equivalent to dictatorship in an unrestricted domain (subject to a range assumption). In the quasilinear model (with otherwise unrestricted domain), Roberts showed that dominant-strategy implementability, PAD, and the existence of affine maximizers are equivalent. In the more restricted economic environments of auctions, where agents care only about their private consumption, the equivalence of these three concepts breaks

⁸Let a differ from $f(V)$ in the allocation to exactly one buyer. Then the hypothesis in (13) is false as the strict inequality holds for at most one and not for all buyers.

⁹In our search for conditions that might be necessary and sufficient on even smaller domains than considered here, we examined two conditions that strengthen W-Mon in a natural way. However, neither of these two conditions is necessary. See Example S3 in Supplementary Materials.

¹⁰As already noted, in multi-agent models PAD does not imply W-Mon. Example S2 in the Supplementary Materials presents a single-agent example in which a social choice function satisfies PAD but not W-Mon; this mechanism is, of course, not truth-telling.

down. The domain restrictions inherent in auctions imply that the class of dominant-strategy incentive-compatible allocation rules is smaller than those satisfying PAD and larger than the set of affine maximizers. If PAD is strengthened to W-Mon, then we recover equivalence between dominant-strategy implementability and W-Mon.¹¹ Although it is stronger than PAD, W-Mon is much weaker than cyclic monotonicity which has been used to characterize incentive compatibility in multi-dimensional settings (Rochet (1987)).

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¹¹W-Mon by itself does not imply affine maximization. Lavi, Mu'alem, and Nisan (2003) identify an additional property which together with W-Mon implies affine maximization.

Appendix

The following lemma is used in the proofs.

Lemma 6 *For any social choice function f and $a_k, a_\ell, a_r \in A$ we have:*

- (i) *If $a_k \succeq a_\ell$ then $\delta_{rk} \leq \delta_{r\ell}$.*
- (ii) *W-Mon implies that $\delta_{k\ell} \geq -\delta_{\ell k}$.*

Proof: (i) As $V(a_k) \geq V(a_\ell)$ for all V , including $V \in Y(r)$, we have $V(a_r) - V(a_k) \leq V(a_r) - V(a_\ell)$, $\forall V \in Y(r)$. Therefore, $\delta_{rk} \leq \delta_{r\ell}$.

(ii) By W-Mon, $V(a_k) - V(a_\ell) \geq V'(a_k) - V'(a_\ell)$, $\forall V \in Y(k), V' \in Y(\ell)$. Thus,

$$\begin{aligned} \delta_{k\ell} = \inf\{V(a_k) - V(a_\ell) \mid V \in Y(k)\} &\geq \sup\{V(a_k) - V(a_\ell) \mid V \in Y(\ell)\} \\ &= -\inf\{V(a_\ell) - V(a_k) \mid V \in Y(\ell)\} = -\delta_{\ell k} \end{aligned}$$

Q.E.D.

Proof of Lemma 2: We first show that p_k is finite. Clearly, $p_K = 0$. If $a_K \succeq a_k$ then $\delta_{Kk} \geq 0$ and $p_k \leq 0$. If $a_K \not\succeq a_k$, $k \neq K$, then select $V' \in Y(k)$. W-Mon implies that $\infty > V(a_K) - V(a_k) \geq V'(a_K) - V'(a_k) > -\infty$ for any $V \in Y(K)$, and therefore $-\delta_{Kk}$ and hence p_k is finite.

(i) By definition, $p_K = 0$ and $p_\ell = -\delta_{K\ell}$. Therefore, $V(a_\ell) - V(a_K) < -\delta_{K\ell} \leq \delta_{\ell K}$, where the second inequality follows from Lemma 6(ii). The definition of $\delta_{\ell K}$ implies that $f(V) \neq a_\ell$.

(ii) In the other direction, $V(a_K) - V(a_\ell) < p_K - p_\ell = \delta_{K\ell}$ implies $f(V) \neq a_K$. *Q.E.D.*

Proof of Theorem 1: In view of Lemma 1, we need only show sufficiency of W-Mon. In particular, we show that the payment function defined in (4) truthfully implements any social choice function f which is W-Mon. Suppose to the contrary that there exists k^*, k and $V \in Y(k^*)$ such that $V(a_{k^*}) - p_{k^*} < V(a_k) - p_k$. Lemma 2(i) and (ii) imply that $k \neq K$ and $k^* \neq K$ respectively (else it would contradict $V \in Y(k^*)$). Further, Lemma 2(i) implies that $V(a_{k^*}) - p_{k^*} \geq V(a_K) - p_K (= V(a_K))$. Choose a

$\gamma > 0$ and a small enough $\epsilon > 0$ such that

$$V(a_{k^*}) + \epsilon - p_{k^*} < V(a_K) + \gamma - p_K < V(a_k) - p_k.$$

Note that $\gamma > \epsilon$. Define $T = \{a_{k^*}\} \cup \{a_\ell \in A \mid a_\ell \succeq a_{k^*} \text{ and } V(a_\ell) = V(a_{k^*})\}$. Let V' be the following type:

$$V'(a_r) \equiv \begin{cases} V(a_r) + \epsilon, & \text{if } a_r \in T \setminus \{a_K\} \\ V(a_r) + \gamma, & \text{if } a_r = a_K \\ V(a_r), & \text{otherwise.} \end{cases}$$

We verify the consistency of V' with \succeq . If $a_{\ell'} \succeq a_\ell$ and $a_\ell \in T$, $a_{\ell'} \notin T$, $a_{\ell'} \neq a_K$ then select $\epsilon > 0$ small enough so that $V'(a_{\ell'}) = V(a_{\ell'}) \geq V(a_\ell) + \epsilon = V'(a_\ell)$. If $a_K \succeq a_\ell$, $a_\ell \in T$, then as $\gamma > \epsilon$, we have $V'(a_K) \geq V'(a_\ell)$ if $V(a_K) \geq V(a_\ell)$. Further, a_K was chosen so that $a_\ell \not\succeq a_K$ for any $\ell \neq K$.

The consistency of V' and the rich-domain assumption implies that $V' \in D$. By Lemma 6(i), $p_{k^*} \leq p_\ell$ for any $a_\ell \in T$. Therefore, as $V'(a_\ell) = V'(a_{k^*})$, for all $a_\ell \in T \setminus \{a_K\}$, we have

$$V'(a_\ell) - p_\ell \leq V'(a_{k^*}) - p_{k^*} < V'(a_K) - p_K, \quad \forall a_\ell \in T \setminus \{a_K\}.$$

Thus, $a_k \notin T$ and Lemma 2(i) implies that $f(V') \neq a_\ell$ for any $\ell \in T \setminus \{a_K\}$. As $V'(a_K) - p_K < V'(a_k) - p_k$, $f(V') \neq a_K$ by Lemma 2(ii). Thus, $f(V') = a_{k'} \notin T \cup \{a_K\}$. But then,

$$0 = V'(a_{k'}) - V(a_{k'}) < V'(a_{k^*}) - V(a_{k^*}) = \epsilon$$

which violates W-Mon. *Q.E.D.*

Proof of Lemma 3: By Lemma 6(ii) and the fact that $\bar{v}_k \geq v_k \geq 0$ for all $v \in \mathcal{D}$, we have $\bar{v}_k \geq \delta_{kk-1} \geq -\delta_{k-1k} \geq 0$.

Let $v^k \equiv (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k, 0, \dots, 0)$ for any $k = 0, 1, 2, \dots, K$ (with $v^0 \equiv (0, 0, \dots, 0)$). Observe that under either bounded-domain assumption A or B, $v^k \in \mathcal{D}$. Thus, $v^k \in Y(q)$ for some $q = 0, 1, \dots, K$. TBB(i) implies that $v^k \notin Y(q)$ for any $q > k$, and TBB(ii) implies that $v^k \notin Y(q)$ for any $q < k$. Therefore, $v^k \in Y(k)$. Next, let $v(t) = (1-t)v^{k-1} + tv^k$, $t \in [0, 1]$, be a point on the straight line joining v^k

and v^{k-1} , $k \geq 1$. Observe that $v(t) = (\bar{v}_1, \dots, \bar{v}_{k-1}, t\bar{v}_k, 0, \dots, 0) \in \mathcal{D}$, $\forall t \in [0, 1]$. Thus, $v(t) \in Y(q)$ for some q . TBB implies that $v(t) \in Y(k-1) \cup Y(k)$. Because $v_k(t) = t\bar{v}_k$ increases in t , there exists a $t^* \in [0, 1]$ such that $v(t) \in Y(k-1)$ for all $t < t^*$ and $v(t) \in Y(k)$ for all $t > t^*$. Thus,

$$t^*\bar{v}_k = \lim_{t \uparrow t^*} v_k(t) \leq -\delta_{k-1k} \leq \delta_{kk-1} \leq \lim_{t \downarrow t^*} v_k(t) = t^*\bar{v}_k$$

Hence, $\delta_{kk-1} = -\delta_{k-1k}$.

Q.E.D.

Proof of Lemma 4: W-Mon implies that

$$\text{If } \sum_{\ell=f(v)+1}^q v'_\ell < \sum_{\ell=f(v)+1}^q v_\ell, \quad \forall q > f(v) \text{ then } f(v') \leq f(v). \quad (14)$$

$$\text{If } \sum_{\ell=q+1}^{f(v)} v'_\ell > \sum_{\ell=q+1}^{f(v)} v_\ell, \quad \forall q < f(v) \text{ then } f(v') \geq f(v). \quad (15)$$

Observe that if v' , v , satisfy the hypotheses in (14) and (15) then $f(v') = f(v)$.

First, we prove that for any $k = 0, 1, 2, \dots, K$,

$$\left\{ v \in \mathcal{D} \mid \sum_{\ell=q}^k v_\ell \geq \sum_{\ell=q}^k \delta_{\ell\ell-1}, \forall q \leq k, \sum_{\ell=k+1}^q v_\ell \leq \sum_{\ell=k+1}^q \delta_{\ell\ell-1}, \forall q > k \right\} \subseteq \text{cl}[Y(k)]. \quad (16)$$

There are two cases to consider. Note that Case B below arises only if domain assumption B holds and (6) is violated by $(\delta_{10}, \delta_{21}, \dots, \delta_{KK-1})$.

Case A: $(\delta_{10}, \delta_{21}, \dots, \delta_{KK-1}) \in \mathcal{D}$.

Consider the point $\hat{v}^k(\epsilon) = (\delta_{10} + \epsilon_1, \dots, \delta_{kk-1} + \epsilon_k, \delta_{k+1k} - \epsilon_{k+1}, \dots, \delta_K - \epsilon_K)$ where $\epsilon_1, \epsilon_2, \dots, \epsilon_K$ satisfy the following conditions:

- (i) If $[q \leq k \text{ and } \delta_{qq-1} = \bar{v}_q]$ or $[q > k \text{ and } \delta_{qq-1} = 0]$ then $\epsilon_q = 0$.
- (ii) If $[q \leq k \text{ and } \delta_{qq-1} < \bar{v}_q]$ or $[q > k \text{ and } \delta_{qq-1} > 0]$ then $\epsilon_q > 0$.

As $(\delta_{10}, \delta_{21}, \dots, \delta_{KK-1}) \in \mathcal{D}$, there exist $\epsilon_1, \epsilon_2, \dots, \epsilon_K$ satisfying (i) and (ii) above such that $\hat{v}^k(\epsilon) \in \mathcal{D}$. (Under domain assumption B, the ϵ_q 's must be chosen to ensure that $\hat{v}^k(\epsilon)$ satisfies (6).) Consider any $q < k$. If $\delta_{q+1q} < \bar{v}_{q+1}$ then as $\hat{v}_{q+1}^k(\epsilon) > \delta_{q+1q}$, we know that $\hat{v}^k(\epsilon) \notin Y(q)$. If, instead, $\delta_{q+1q} = \bar{v}_{q+1}$ then (as $\epsilon_{q+1} = 0$) we have $\hat{v}_{q+1}^k(\epsilon) = \bar{v}_{q+1}$. Thus, TBB(ii) implies that $\hat{v}^k(\epsilon) \notin Y(q)$. Similarly, TBB(i) implies

that $\hat{v}^k(\epsilon) \notin Y(q)$ for $q > k$. Hence $\hat{v}^k(\epsilon) \in Y(k)$. Therefore, (14) and (15) imply that¹²

$$\left\{ v \in \mathcal{D} \mid \sum_{\ell=q}^k v_\ell > \sum_{\ell=q}^k (\delta_{\ell\ell-1} + \epsilon_\ell), \forall q \leq k, \sum_{\ell=k+1}^q v_\ell < \sum_{\ell=k+1}^q (\delta_{\ell\ell-1} - \epsilon_\ell), \forall q > k \right\} \subset Y(k).$$

One can construct a sequence $(\epsilon_1^n, \epsilon_2^n, \dots, \epsilon_K^n) \rightarrow 0$ such that $\hat{v}^k(\epsilon^n) \in \mathcal{D}$. Taking limits as $\epsilon^n \rightarrow 0$, we get

$$\left\{ v \in \mathcal{D} \mid \sum_{\ell=q}^k v_\ell > \sum_{\ell=q}^k \delta_{\ell\ell-1}, \forall q \leq k, \sum_{\ell=k+1}^q v_\ell < \sum_{\ell=k+1}^q \delta_{\ell\ell-1}, \forall q > k \right\} \subset Y(k),$$

which in turn implies (16).

Case B: $(\delta_{10}, \delta_{21}, \dots, \delta_{KK-1}) \notin \mathcal{D}$.

For each $k = 0, 1, 2, \dots, K$ define

$$\mathbf{v}^k(\epsilon) = \left\{ v \mid v_q = \max[v_{q+1} \frac{\bar{v}_q}{\bar{v}_{q+1}}, \delta_{qq-1} + \epsilon_q], \forall q < k, \delta_{kk-1} + \epsilon_k \leq v_k \leq \bar{v}_k, \right. \\ \left. v_q = \min[v_{q-1} \frac{\bar{v}_q}{\bar{v}_{q-1}}, \delta_{qq-1} - \epsilon_q], \forall q > k \right\}.$$

Any $v \in \mathbf{v}^k(\epsilon)$ satisfies (6). Thus, provided $\epsilon_1, \epsilon_2, \dots, \epsilon_K$ satisfy (i) and (ii) defined in Case A, and are small enough, $\mathbf{v}^k(\epsilon) \subset \mathcal{D} [= \cup_{q=0}^K Y(q)]$. For any $v \in \mathbf{v}^k(\epsilon)$, we have $v_q \geq \delta_{qq-1} + \epsilon_q$ for any $q \leq k$; thus $\mathbf{v}^k(\epsilon) \cap Y(q-1) = \emptyset$. Similarly, for any $q > k$, $\mathbf{v}^k(\epsilon) \cap Y(q) = \emptyset$. Thus, $\mathbf{v}^k(\epsilon) \subset Y(k)$ for small enough ϵ_ℓ 's. From (14) and (15) applied to each $v \in \mathbf{v}^k(\epsilon)$, we know that (with the qualification in footnote 12)

$$\left\{ v \in \mathcal{D} \mid v_k > \delta_{kk-1} + \epsilon_k, \sum_{\ell=q}^k v_\ell > \delta_{kk-1} + \epsilon_k + \sum_{\ell=q}^{k-1} \max[v_{\ell+1} \frac{\bar{v}_\ell}{\bar{v}_{\ell+1}}, \delta_{\ell\ell-1} + \epsilon_\ell], \forall q < k, \right. \\ \left. \sum_{\ell=k+1}^q v_\ell < \sum_{\ell=k+1}^q \min[v_{\ell-1} \frac{\bar{v}_\ell}{\bar{v}_{\ell-1}}, \delta_{\ell\ell-1} - \epsilon_\ell], \forall q > k \right\} \subset Y(k).$$

Taking limits as $(\epsilon_1, \epsilon_2, \dots, \epsilon_K) \rightarrow 0$, we see that

$$\left\{ v \in \mathcal{D} \mid v_k > \delta_{kk-1}, \sum_{\ell=q}^k v_\ell > \delta_{kk-1} + \sum_{\ell=q}^{k-1} \max[v_{\ell+1} \frac{\bar{v}_\ell}{\bar{v}_{\ell+1}}, \delta_{\ell\ell-1}], \forall q < k, \right.$$

¹²If for some $q \leq k$, $\delta_{\ell\ell-1} = \bar{v}_\ell$ for all $\ell = q, q+1, \dots, k$ then the corresponding strict inequality in the set on the left hand side is replaced by a weak inequality. A similar change is made if for some $q > k$, $\delta_{\ell\ell-1} = 0$ for all $\ell = q, q+1, \dots, k$. In either case, (i) implies that $\epsilon_\ell = 0$ in the relevant range. This ensures that the set on the left hand side is non-empty; the inclusion of this set in $Y(k)$ is implied by TBB together with (14) and (15).

$$\left\{ \sum_{\ell=k+1}^q v_\ell < \sum_{\ell=k+1}^q \min[v_{\ell-1} \frac{\bar{v}_\ell}{\bar{v}_{\ell-1}}, \delta_{\ell\ell-1}], \forall q > k \right\} \subseteq Y(k)$$

and therefore

$$\left\{ v \in \mathcal{D} \mid v_k \geq \delta_{kk-1}, \sum_{\ell=q}^k v_\ell \geq \delta_{kk-1} + \sum_{\ell=q}^{k-1} \max[v_{\ell+1} \frac{\bar{v}_\ell}{\bar{v}_{\ell+1}}, \delta_{\ell\ell-1}], \forall q < k, \right. \\ \left. \sum_{\ell=k+1}^q v_\ell \leq \sum_{\ell=k+1}^q \min[v_{\ell-1} \frac{\bar{v}_\ell}{\bar{v}_{\ell-1}}, \delta_{\ell\ell-1}], \forall q > k \right\} \subseteq \text{cl}[Y(k)].$$

That this last set inclusion is equivalent to (16) follows from the observation that (6) implies that if $\delta_{kk-1} + \sum_{\ell=q}^{k-1} \max[v_{\ell+1} \frac{\bar{v}_\ell}{\bar{v}_{\ell+1}}, \delta_{\ell\ell-1}] > \sum_{\ell=q}^k v_\ell \geq \sum_{\ell=q}^k \delta_{\ell\ell-1}$ for some $q < k$ or if $\sum_{\ell=q+1}^k \min[v_{\ell-1} \frac{\bar{v}_\ell}{\bar{v}_{\ell-1}}, \delta_{\ell\ell-1}] < \sum_{\ell=q+1}^k v_\ell \leq \sum_{\ell=q+1}^k \delta_{\ell\ell-1}$ for some $q > k$, then $v \notin \mathcal{D}$. This establishes (16) for Case B.

Next, suppose that the set inclusion in (16) is strict. In particular, there exists $k, v \in \text{cl}[Y(k)]$ such that $\sum_{\ell=q'}^k v_\ell < \sum_{\ell=q'}^k \delta_{\ell\ell-1}$, for some $q' < k$. (From the definition of δ_{kk-1} we know that $q' \neq k$.) We may assume WLOG that $\sum_{\ell=q}^k v_\ell \geq \sum_{\ell=q}^k \delta_{\ell\ell-1}$, $\forall q = q' + 1, \dots, k$ and that $v \in Y(k)$. (If $v \in \text{cl}[Y(k)] \setminus Y(k)$, then there exists $v' \in Y(k)$, v' close to v , such that $\sum_{\ell=q'}^k v'_\ell < \sum_{\ell=q'}^k \delta_{\ell\ell-1}$.) Therefore, $v_{q'} < \delta_{q'q'-1} \leq \bar{v}_{q'}$ and $\sum_{\ell=q'}^q v_\ell < \sum_{\ell=q'}^q \delta_{\ell\ell-1}$, $\forall q = q', q' + 1, \dots, k$. Consider the point $\hat{v} \equiv (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{q'-1}, v_{q'} + \hat{\epsilon}, v_{q'+1}, \dots, v_k, 0, \dots, 0)$ where $\hat{\epsilon} > 0$ is small enough that $\hat{v} \in \mathcal{D}$ and $\sum_{\ell=q'}^q \hat{v}_\ell < \sum_{\ell=q'}^q \delta_{\ell\ell-1}$, $\forall q = q', q' + 1, \dots, k$. Thus, (16) implies that $\hat{v} \in \text{cl}[Y(q' - 1)]$. Suppose that $\hat{v} \in Y(q' - 1)$. But this violates (5) as $\sum_{\ell=q'}^k \hat{v}_\ell > \sum_{\ell=q'}^k v_\ell$ and $v \in Y(k)$. If, instead, $\hat{v} \in \text{cl}[Y(q' - 1)] \setminus Y(q' - 1)$ then there exists $v^* \in Y(q' - 1)$ which is arbitrarily close to \hat{v} and (5) is violated. Thus, for any $v \in \text{cl}[Y(k)]$ we have $\sum_{\ell=q}^k v_\ell \geq \sum_{\ell=q}^k \delta_{\ell\ell-1}$, $\forall q \leq k$. A similar proof establishes that if $v \in \text{cl}[Y(k)]$ then $\forall q > k$, $\sum_{\ell=k+1}^q v_\ell \leq \sum_{\ell=k+1}^q \delta_{\ell\ell-1}$. Therefore, the set inclusion in (16) can be replaced by an equality, i.e.,

$$\text{cl}[Y(k)] = \left\{ v \in \mathcal{D} \mid \sum_{\ell=q}^k v_\ell \geq \sum_{\ell=q}^k \delta_{\ell\ell-1}, \forall q \leq k, \& \sum_{\ell=k+1}^q v_\ell \leq \sum_{\ell=k+1}^q \delta_{\ell\ell-1}, \forall q > k \right\}. \quad (17)$$

For any $v \in Y(k)$ and any $q < k$,

$$\sum_{\ell=1}^k v_\ell - \sum_{\ell=1}^k \delta_{\ell\ell-1} \geq \sum_{\ell=1}^q v_\ell - \sum_{\ell=1}^q \delta_{\ell\ell-1} \quad (18)$$

$$\iff \sum_{\ell=q+1}^k v_{\ell} \geq \sum_{\ell=q+1}^k \delta_{\ell \ell-1}.$$

The last inequality follows from (17). Thus, (18) is true; when $v \in Y(k)$ the agent cannot increase his payoffs by reporting a type $v' \in Y(q)$, $q < k$. Similarly, (18) is true for $q > k$. Thus, the payment function p_k defined in (8) implements f . *Q.E.D.*

Proof of Lemma 5: Before describing a procedure which converts f to an f' with the stated properties, we need the following result.¹³

CLAIM: *Let f be an allocation rule that is W-Mon but not TBB. That is there exists $v^k \in Y(k)$ and $v^{k-1} \in Y(k-1)$ such that either $v_k^k = v_k^{k-1} = 0$ or $v_k^k = v_k^{k-1} = \bar{v}_k$.*

Define a new allocation rule which is identical to f except that:

(i) If $v_k^k = v_k^{k-1} = 0$ then allocate $k-1$ (instead of k) units at v^k .

(ii) If $v_k^k = v_k^{k-1} = \bar{v}_k$ then allocate k (instead of $k-1$) units at v^{k-1} .

Then, the new allocation rule is W-Mon.

PROOF: (i) Suppose that $v_k^k = v_k^{k-1} = 0$. At v^k the buyer is allocated $k-1$ units in the new allocation rule. Since f is W-Mon, all we need to check is that v^k satisfies W-Mon inequalities in the new allocation rule. Observe that $0 = \delta_{kk-1} = v_k^k \leq v_k$, $\forall v \in Y(k)$. Thus v^k satisfies the W-Mon inequalities with respect to all $v \in Y(k)$. Therefore, we need to show that for any $v \in Y(q)$, $q \neq k, k-1$,

$$\sum_{\ell=q+1}^{k-1} v_{\ell}^k \geq \sum_{\ell=q+1}^{k-1} v_{\ell}, \quad \text{if } q < k-1 \quad \text{and} \quad \sum_{\ell=k}^q v_{\ell}^k \leq \sum_{\ell=k}^q v_{\ell}, \quad \text{if } q > k. \quad (19)$$

From W-Mon of f we know that for any $v \in Y(q)$, $q \neq k, k-1$,

$$\sum_{\ell=q+1}^k v_{\ell}^k \geq \sum_{\ell=q+1}^k v_{\ell}, \quad \text{if } q < k-1 \quad \text{and} \quad \sum_{\ell=k+1}^q v_{\ell}^k \leq \sum_{\ell=k+1}^q v_{\ell}, \quad \text{if } q > k.$$

This, together with $v_k^k = 0$, implies (19).

(ii) The proof is similar.

Q.E.D.

¹³Throughout this proof, $Y(\cdot)$, $\delta_{k\ell}$ are defined with respect to f and $Y'(\cdot)$, $\delta'_{k\ell}$ are defined with respect to f' .

Consider any f that satisfies W-Mon. From f we obtain an allocation rule f' using the following procedure. First, let $f'(v) \equiv f(v)$, $\forall v$, and then make the following changes to f' :

1. Let $k = K$.
2. If $\delta_{kk-1} = 0$ then for all $v \in Y(k)$ such that $v_k = 0$, let $f'(v) = k - 1$.
3. Decrease k by 1. If $k \geq 1$ then go to step 2; otherwise, go to Step 4.
4. Let $k = 1$.
5. If $-\delta_{k-1k} = \bar{v}_k$ then for all $v \in Y(k-1)$ such that $v_k = \bar{v}_k$, let $f'(v) = k$.
6. Increase k by 1. If $k \leq K$ then go to step 5; otherwise, stop.

By Lemma 6, $\delta_{kk-1} \geq -\delta_{k-1k}$. Thus, if at Step 2 of the procedure, we transfer some v from $Y(k)$ to $Y'(k-1)$, then in Step 5 we will not transfer any v 's from $Y(k-1)$ to $Y'(k)$, and vice versa. The Claim assures us that each time we make changes to f' in Steps 2 or 5, f' continues to satisfy W-Mon; thus the f' obtained at the end of this procedure is W-Mon. By construction, the final f' satisfies TBB. Further, $f(v) = f'(v)$ for almost all $v \in \mathcal{D}$.

Let $p'_k = \sum_{\ell=1}^k \delta'_{\ell\ell-1}$ be the payment function defined in (8) with respect to f' . By Lemma 4, p'_k implements f' . We show that for any $v \in \mathcal{D}$, assuming truthful reporting under either mechanism, the buyer's payoffs are the same under f or f' implemented with p'_ℓ . Therefore, it must also be optimal to tell the truth when f is implemented with prices p'_ℓ . Let $f(v) = k$ and $f'(v) = k'$. We establish that

$$\sum_{\ell=1}^k (v_\ell - \delta'_{\ell\ell-1}) = \sum_{\ell=1}^{k'} (v_\ell - \delta'_{\ell\ell-1}). \quad (20)$$

If $k = k'$, then clearly (20) is true. If, instead, $k' < k$ then, from the above construction, $v_\ell = \delta'_{\ell\ell-1} = 0$, $\ell = k' + 1, k' + 2, \dots, k$. Similarly, if $k' > k$ then $v_\ell = \delta'_{\ell\ell-1} = \bar{v}_\ell$, $\ell = k + 1, k + 2, \dots, k'$. Thus, (20) holds. Therefore, since the prices $\delta'_{\ell\ell-1}$ truthfully implement f' , they also truthfully implement f . *Q.E.D.*

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