Money for Nothing: Exploiting Negative Externalities

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ABSTRACT

We show that existence of negative externalities among market participants competing for a scarce resource, a setting typical for electronic commerce and internet advertising, allows for emergence of the no-allocation equilibrium with positive revenues for the seller. A monopolist selling K indivisible items to a large number of unit-demand buyers who face negative externalities whenever their rivals get the items, can exploit these negative externalities. If the number of buyers is large enough, the no-allocation equilibrium emerges: no items get allocated, yet buyers still pay the seller to avoid a potential exposure to negative externalities. We provide conditions on the magnitude of externalities and on the level of buyer competition that yield optimality of the no-allocation equilibrium.

In the context of internet advertising, the no-allocation equilibrium allows the monopolist seller of a limited number of ad slots to simultaneously (1) optimize revenues by collecting a small payment from each of the potential advertisers who are concerned with negative externality effects, and (2) ensure ad-free experience to its users. Therefore, our results describe settings in which ad-free user experience can be supported not just by charging users, but could be subsidized by potential advertisers whose ads will *not* be shown.

Categories and Subject Descriptors

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1. INTRODUCTION

When market participants compete for a scarce resource, they face a possibility of losses due to resource being allocated to competitors. This is the case in many electronic commerce situations. A typical situation is that of internet advertising. The number of impressions per webpage is limited and could be considered a scarce resource when demand is high. Many advertisers rely on their internet presence and ad placement as the main channel for attracting customers. If an advertiser cannot secure a webpage impression to reach a potential customer who will be shown that webpage, the advertiser might prefer that no competitor's ads are shown to the customer being exposed to competitors' ads.[1]

One should expect that design of optimal allocation and pricing mechanisms could be affected by the fact that market participants not only have private valuations for a scarce resource, but also face negative externalities when a competitor succeeds in obtaining the resource. One theoretical obstacle to finding an optimal mechanism is that information structure of market participants becomes two-dimensional. In addition, a potential usefulness of such optimal mechanism depends on whether such an optimal mechanism (a) allows for a manageable practical implementation, and (b) yields fundamentally different allocation and pricing decisions from those reached without exploiting negative externalities (e.g., if such externalities turn out to be negligible). In this paper we show that a monopolistic seller (or a social planner) could exploit existence of even minuscule negative externalities, provided sufficiently large demand for a scarce resource.

Intuitively, even if negative externalities are small, not allocating any resources might be optimal when the number of buyers is large; a small number of winners would trigger a large total value of negative externalities for all losers. A recent school naming-rights example illustrates this point: thirteen donors gave combined $85 million, with minimum single donor gift of $5 million, to the Wisconsin School of Business at the University of Wisconsin-Madison to *"preserve the Wisconsin name for at least 20 years. During that*

[1]This effect is not limited to e-commerce: a telecom company that does not obtain a frequency license might lose potential future customers who will be captured by a competitor who might have secured that license; a drug developer who does not obtain a regulatory approval is better off if all competitors also fail to obtain an approval, than if any of competitors succeed and are first to the market.

time, the school will not be named for a single donor or entity" [17]. Not only did the Wisconsin School of Business raise $85 million for NOT naming the school (compared, e.g., with $55 million naming gift for the Tepper School of Business at Carnegie Mellon University [16]), but they can also try to sell the name again in 2027.

This example indicates a potential for implementing such no-allocation equilibrium in the context of electronic commerce, and internet advertising specifically. A monopolist seller of limited ad space could attempt to collect a small amount of money from each of the potential advertisers who are concerned with negative externality effects, and by doing so optimize revenues and ensure ad-free experience to its users. Thus, our results show that ad-free user experience can be supported not just by a user-subscription fee model (used by several popular content providers), but could be subsidized by potential advertisers whose ads will *not* be shown.

In sponsored search, intense competition among internet advertisers for user attention and actions (measured by click-through rates and conversion rates), induces negative externalities whenever competing ads are displayed together. (This point has been argued in [7].) User attention and actions that an ad can attract in sponsored search depends on the total number of the ads shown simultaneously [12, 15], as well as on the relative position of the ad [5, 9]. The externalities due to user attention and click through rates, also labeled quantity effects [4], are analyzed in cascade models (e.g., [1, 11, 8]).

Since click through rates can be learned by a search engine as the data are observable, the quantity externalities are different from the value externalities, i.e., the value conditioned on receiving a click. These value externalities are private to advertisers [4]. The private information about value externalities may not be one dimensional, due to the dependence on the conversion probability [7]. Furthermore, only the ad owner has information on conversions, so the value of negative externality imposed onto a competitor might actually be private information of the ad owner. Such information structure has been studied in economics literature [10] in the limited context of a single unit resource. We will build on this model and extend it to an arbitrary but finite number of resources and study relationships involving externality valuations and intensity of competition.

A reasonable conjecture would be that incorporating externality valuations in the ad auction design ought to improve expressiveness, and could therefore improve both efficiency and the seller's revenue (for general discussion of expressiveness, see, e.g., [14]). Some recent papers explore mechanisms that use externality valuation information in different formats. In [7], extensions are designed and equilibria are analyzed for a Generalized Second Price auction in which an advertiser's private value depends on whether the ad is allocated exclusively or not. In their model advertiser places two bids: one for exclusive placement and one for being placed with multiple other ads. (The difference in the valuations correspond to negative externality of competitors' ads being placed.) The model in [6] assumes the value of an ad to an advertiser depends on the relative quality of the ad compared to the other ads that are shown simultaneously. In [4], a framework of the unit-bidder constraints for value externalities is considered: a bidder is allowed to specify a set of single competitor constraints, where each such constraint prohibits the bidder's ad being shown together with the ad of the competitor identified by the constraint. Clearly, the focus of such modeling approach is to deal with externalities by limiting possible allocations. Given somewhat complicated valuation structure that accounts for all externalities, even computing optimal allocation in the complete information setting could be unmanageable. In [3], a general representation of settings with externalities is provided and then the efficient computation of optimal outcomes are studied. (There is also a large body of literature on advertising, see [2].)

In this paper, we provide a theoretical analysis supporting the intuition behind optimality of the non-allocation. Our setting is that of a monopolistic seller of K identical indivisible items with n potential unit-demand buyers. The buyers have independent private values for the item, as well as independent private values for (negative) externality they would impose onto every rival if they get the item. This is the informational setting of the seminal work of Jehiel et al. [10]. They manage to provide the optimal mechanism for the single item case ($K = 1$), despite two-dimensional valuation structure. The optimal mechanism with $K > 1$ items (Theorem 1) is a straightforward generalization of the methodology in [10].

Our focus is on exploring properties of the optimal mechanism with multiple items and a large number of buyers. If there are sufficiently many buyers willing to pay a small amount in order to avoid negative externalities being imposed on them, the optimal mechanism will not allocate any items. We formalize this observation by specifying the joint effect of the number of buyers and the number of items to the emergence of an equilibrium with no allocation. Moreover, under the no-allocation equilibrium, the seller's expected profit increases in the number of buyers and the number of items.

Since negative externalities are the driving force behind possibility of the no-allocation equilibrium, our analysis includes a variety of externality valuation structures. Specifically, we show conditions for emergence of the no-allocation equilibrium under the assumption of externalities being independent of item valuations (independent externalities), as well as under the assumption of externalities depending on item valuations (dependent externalities). We also allow for scaling the magnitude of externality valuations relative to the magnitude of item valuations. This case is important for settings in which negative externality imposed onto rivals is much smaller than the value of the items (e.g., the value of the ad shown is much higher than the negative externality if rival's ad is shown.)

The multiple items setting of this paper also provides a way for understanding the relationship between exclusivity pricing and pricing negative externalities. In our model, the exclusivity premium is set to the payoff difference between an exclusive allocation and a sharing allocation, and is equal to the magnitude of aggregate externalities.

Our results establish emergence of the no-allocation equilibrium when negative externalities are present (possibly small) and when competition is intense. Therefore, auction mechanisms that exploit negative externalities could, in contrast to standard mechanisms that ignore negative externality information, yield optimal pricing and (no-)allocation, which in turn makes them relevant and desirable for use in practice.

The paper is organized as follows. In the next section, we describe our model. In section 3, we show the optimal mechanism. Section 4 illustrates our findings regarding conditions for the no-allocation equilibrium, for uniformly distributed valuations, with both independent and dependent externalities. The generalization of our results for general distributions is demonstrated in section 5. Brief concluding remarks close the paper.

2. MODEL

A risk-neutral seller is selling K identical indivisible items to n $(n > K)$ potential risk-neutral buyers i, $i = 1, ..., n$. Each buyer has unit demand. We adopt the information structure of [10], and assume that buyer i's private information is given by a two-dimensional type: $t_i = (t_i^1, t_i^2) = (\pi_i, \alpha_i)$, where π_i is i's valuation of the item, and α_i is the externality imposed on buyer j $(j \neq i)$ if buyer i obtains the item.[2] Note that $-\alpha_i$ can be viewed as a negative externality on buyer j when buyer i gets the item and buyer j does not, i.e., buyer i creates the negative (positive) externality on buyer j when $\alpha_i > 0$ $(\alpha_i < 0)$. The seller's type is set to $t_0 = (0, 0)$. Buyer types are independent across buyers, so the type space is $T = T_1 \times T_2 \times ... \times T_n$ where types are drawn from $T_i = [\underline{\pi}, \overline{\pi}] \times [\underline{\alpha}, \overline{\alpha}]$, according to the joint probability density function (hereafter, PDF) f_i and joint cumulative density function (hereafter, CDF) F_i.

If valuations and externalities are independent, $f_i(t_i^1, t_i^2) = f_i^1(t_i^1) f_i^2(t_i^2)$.

We also make the following standard assumption:

Assumption 1. $\frac{1-F_i^1(\cdot)}{f_i^1(\cdot)}$ is a decreasing function for all i.

We next define the set of buyers with K highest valuations and the set of buyer i's rivals with K highest externalities imposed on i.

Definition 1. $K^1\left(\left\{t_j^1\right\}_{j=1}^n\right) \triangleq \{j(n),\, j(n-1),\, ...,\, j(n-K+1)\}$, where $t^1_{j(n)} \geq t^1_{j(n-1)} \geq ... \geq t^1_{j(n-K+1)} \geq ... \geq t^1_{j(1)}$.

Definition 2. $K^2(\left\{t_j^2\right\}_{j=1}^n, i) \triangleq \{j(n),\, j(n-1),\, ...,\, j(n-K+1)\}$, where $t^2_{j(n)} \geq t^2_{j(n-1)} \geq ... \geq t^2_{j(n-K+1)} \geq ... \geq t^2_{j(2)}$ and $t^2_{j(1)} \triangleq t^2_i$.

[2]Our analysis and all results readily extend to the environment with different externalities on different buyers, i.e., where α_i is a length $n-1$ vector $(\alpha_{i1}, ..., \alpha_{i(i-1)}, \alpha_{i(i+1)}, ..., \alpha_{in})$. For simplicity of exposition, in this paper we present the model with identical externalities imposed on different buyers, i.e., $\alpha_{ij} = \alpha_i$ for all $j \neq i$.

A direct revelation mechanism is defined as (x, p, ρ). x is the payment vector from the buyer to the seller with $x_i : T \to R$. p is the allocation (probability) vector with $p : T \to \{z \in R^n_+ \mid \sum z_i \leq K \text{ and } z_i \leq 1\}$. The probability $p_i(t_1, t_2, ..., t_n)$ is defined as the probability that buyer i obtains one of the K items, regardless of which one she occupies. ρ is the trigger strategy vector when buyer i refuses to participate with $\rho_i : T_{-i} \to \{z \in R^n_+ \mid \sum z_i \leq K$ and $z_i \leq 1\}$. We also define the interim payment as

$$y_i(t_i) = \int_{T_{-i}} x_i(t_1,\, ..., t_n)\phi_{-i}(t_{-i})dt_{-i}, \tag{1}$$

and interim allocation rule as

$$q_i(t_i) = \int_{T_{-i}} p_i(t_1,\, ..., t_n)\phi_{-i}(t_{-i})dt_{-i}, \tag{2}$$

where $\phi = f_1 \times f_2 \times ... \times f_n$.

Note that when $K > 1$, externalities may be imposed on buyer i even when she obtains the item since buyer j could obtain the item as well. (Case $K = 1$ corresponds to the model in [10].)

In addition, we assume that each buyer's utility is additively separable. The interim utility of buyer i, when she reports s_i with true type t_i and her rivals truthfully report, is thus

$$\begin{aligned} &U_i(s_i, t_i) \\ &= q_i(s_i)t_i^1 - \textstyle\sum_{j\neq i} \int_{T_{-i}} p_j(s_i, t_{-i}) t_j^2 \phi_{-i}(t_{-i}) dt_{-i} \\ &\quad - y_i(s_i). \end{aligned}$$

Obviously, the seller's optimal trigger strategy for buyer i is to sell the K items to buyer i's opponents who have the K largest externalities. This strategy imposes the severest punishment on the buyer if she rejects to participate, i.e.,

$$\begin{aligned} \rho^i_{v(i,t_{-i})}(t_{-i}) &= 1, \\ \rho^i_j(t_{-i}) &= 0, \text{ for } j \neq v(i, t_{-i}) \\ &\quad \text{and} \\ v(i, t_{-i}) &\in K^2(\left\{t_j^2\right\}_{j=1}^n, i). \end{aligned} \tag{3}$$

Under this optimal trigger strategy, the seller's problem is

$$\max_{\{x,p\}} \sum_{i=1}^n \int_{T_i} y_i(t_i) f_i(t_i) dt_i \tag{4}$$

subject to

$U_i(t_i, t_i) \geq U_i(s_i, t_i)$
for all i and all s_i, $t_i \in T_i$ (ICC)

$U_i(t_i, t_i) \geq A_i$
for all i and all $t_i \in T_i$ (IRC)

where

$$A_i \triangleq -\int_{T_{-i}} \Big(\sum_{j \in K^2(\left\{t_h^2\right\}_{h=1}^n, i)} t_j^2 \Big) \phi_{-i}(t_{-i}) dt_{-i}. \tag{5}$$

Inequality (ICC) is the Incentive Compatibility Constraint, which ensures that truthfully reporting is a Nash equilibrium. (IRC) is the Individual Rationality Constraint, under which there is no incentive for the buyers to reject participation.

In order to make analysis tractable, we will assume that distributions from which buyer types, $t_i = (\pi_i, \alpha_i)$, are drawn are i.i.d. across buyers. However, π_i and α_i may or may not be correlated.

3. OPTIMAL MECHANISM

We follow approach of [10] and their Proposition 2 that uses standard Myerson technique [13] to obtain expression for the seller's *ex ante* profit in the case $K = 1$, and thus we obtain

$$\begin{aligned} &EP \\ &= -\sum_{i=1}^n A_i \\ &\quad + \int (\sum_{i=1}^n [\pi_i - \tfrac{1-F_i^1(\pi_i)}{f_i^1(\pi_i)} - (n-1)E_i] p_i(\pi_1, \ ..., \pi_n)) \cdot \\ &\qquad f_i^1(\pi_1) \cdots f_i^1(\pi_n) d\pi_1 \cdots d\pi_n \end{aligned}$$

where

$$E_i = \int_{\underline{\alpha}}^{\overline{\alpha}} \tau f_i^2(\tau) d\tau. \tag{6}$$

when externalities are independent of item valuations, and

$$\begin{aligned} &EP \\ &= -\sum_{i=1}^n A_i \\ &\quad + \int (\sum_{i=1}^n [\pi_i - \tfrac{1-F_i^1(\pi_i)}{f_i^1(\pi_i)} - (n-1)g_i(\pi_i)] p_i(\pi_1, \ ..., \pi_n)) \cdot \\ &\qquad f_i^1(\pi_1) \cdots f_i^1(\pi_n) d\pi_1 \cdots d\pi_n \end{aligned}$$

for externalities perfectly correlated with item valuations, i.e., for $\alpha_i = g_i(\pi_i)$.

Therefore, we have the following theorem illustrating the optimal allocation rules and the optimal interim payment rules.

THEOREM 1. *1) If item valuations and externalities are independent, the optimal allocation rule is*

$$p_i^*(\pi_1, \ ..., \ \pi_n) = \begin{cases} 1 & \text{if } i \in K^1(\{\pi_j - \tfrac{1-F_j^1(\pi_j)}{f_j^1(\pi_j)} - (n-1)E_j\}_{j=1}^n) \\ & \text{and} \\ & \pi_i - \tfrac{1-F_i^1(\pi_i)}{f_i^1(\pi_i)} - (n-1)E_i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

2) If item valuations and externalities are perfectly correlated, i.e. $\alpha_i = g_i(\pi_i)$*, the optimal allocation rule is*

$$p_i^*(\pi_1, \ ..., \ \pi_n) = \begin{cases} 1 & \text{if } i \in K^1(\{\pi_j - \tfrac{1-F_j^1(\pi_j)}{f_j^1(\pi_j)} - (n-1)g_j(\pi_j)\}_{j=1}^n) \\ & \text{and} \\ & \pi_i - \tfrac{1-F_i^1(\pi_i)}{f_i^1(\pi_i)} - (n-1)g_i(\pi_i) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

3) The optimal interim payment is given by

$$\begin{aligned} y_i^*(t_i) &= -A_i + t_i^1 q_i^*(t_i) - \textstyle\int_{\underline{\pi}_i}^{t_i^1} q_i^*(v, t_i^2) dv \\ &\quad - \textstyle\sum_{j \neq i} \int_{T_{-i}} p_j^*(t_i, t_{-i}) t_j^2 \phi_{-i}(t_{-i}) dt_{-i}, \end{aligned} \tag{7}$$

where $q_i^*(t_i)$ *is obtained by substituting* $p_i^*(\pi_1, \ ..., \ \pi_n)$ *into the definition of the interim allocation rule (2).*

4. UNIFORMLY DISTRIBUTED TYPES

Throughout this section we will assume that types are uniformly distributed. This will allow us to illustrate the conditions on the number of buyers n and the number of items K that yield the no-allocation equilibrium. We will also assume negative externalities, i.e. $\alpha > 0$.

We first consider the externality independence case with K items, i.e., $f_i(t_i^1, t_i^2) = f_i^1(t_i^1) f_i^2(t_i^2)$. The distribution of externalities can be scaled up or scaled down relative to valuations by some positive number c. We end this illustration by analyzing the externality dependence case.

4.1 Independent Externalities

We assume the valuation π_i is drawn from $[0, 1]$ uniform distribution, while the externalities α_i are drawn from a scaled down (up) $[0, c]$ uniform distribution, where $c < 1$ ($c > 1$). Then $F^1(\pi_i) = \pi_i$, $f^1(\pi_i) = 1$, $F^2(\alpha_i) = \alpha_i / c$, and $f^2(\alpha_i) = 1/c$. By the definition of E (6), the externality is

$$E = \int_0^c \tau \frac{1}{c} d\tau = \frac{c}{2}.$$

By Theorem 1, the items will be sold to those buyers whose valuations π_i are among the first K largest, i.e. $i \in K^1$, and, satisfy

$$\pi_i - \frac{1 - F^1(\pi_i)}{f^1(\pi_i)} - (n-1)E \geq 0.$$

With uniform distribution, the condition is

$$\pi_i \geq \frac{1}{2}[1 + \frac{c}{2}(n-1)]$$

Therefore, the items will not be allocated if

$$n > 1 + \frac{2}{c}.$$

Since, for $j = 0, ..., K-1$, the density of the $(n-1-j)^{\text{th}}$ largest out of $n-1$ i.i.d. random variables with CDF $F(x)$ and PDF $f(x)$, is

$$f_{x^{(n-1-j)}} = \frac{(n-1)!}{(n-j-2)! j!} F(x)^{n-j-2} (1 - F(x))^j f(x),$$

the threat A can be rewritten as

$$A = -\frac{n-1}{n}c\left[\left(\sum_{j=1}^{K-1}\frac{(n-2)!}{(n-j-2)!j!}\prod_{i=0}^{j-1}\frac{j-i}{n-(j-i)}\right)+1\right] \quad (8)$$

The calculation also involves two integrals,

$$\int_0^c (n-1)\left(\frac{x}{c}\right)^{n-1}dx = \frac{n-1}{n}c$$

and

$$\int_0^c \left(\frac{x}{c}\right)^{n-j-1}\left(1-\frac{x}{c}\right)^j dx = \frac{j}{n-j}\int_0^c \left(\frac{x}{c}\right)^{n-j}\left(1-\frac{x}{c}\right)^{j-1}dx \quad \text{for } j \geq 1.$$

The seller's expected profit when the seller does not allocate any items is

$$\begin{aligned}
EP &= -nA \\
&= (n-1)c\left[\left(\sum_{j=1}^{K-1}\frac{(n-2)!}{(n-j-2)!j!}\prod_{i=0}^{j-1}\frac{j-i}{n-(j-i)}\right)+1\right] \\
&= (n-1)c\cdot \left[\left(\sum_{j=1}^{K-1}\frac{(n-2)!}{(n-j-2)!j!}\frac{j\cdot(j-1)\cdot\ldots\cdot 1}{(n-1)\cdot(n-2)\cdot\ldots(n-j)}\right)+1\right] \\
&= (n-1)c\left[\left(\sum_{j=1}^{K-1}\frac{(n-2)!}{(n-j-2)!j!}\frac{j!(n-j-1)!}{(n-1)!}\right)+1\right] \quad (9)\\
&= c\left[\sum_{j=1}^{K-1}(n-j-1)+(n-1)\right] \\
&= nc\sum_{j=0}^{K-1}\frac{n-j-1}{n} \\
&= c\left[(n-1)K - \frac{K(K-1)}{2}\right]
\end{aligned}$$

where $\sum_{j=0}^{K-1}\frac{n-j-1}{n}$ is the sum of expected values of the K largest order statistics for the uniform distribution.

In the next proposition we summarize the no-allocation result and describe how the number of buyers and the number of items affect the optimal expected profit.

PROPOSITION 1. *Suppose there are K items and n buyers with types (π_i, α_i) independently drawn from $U[0,1]\times U[0,c]$. If $n > 1+2/c$, the optimal mechanism will not allocate any of the items. Moreover, the seller's expected profit is increasing in the number of buyers n, and in the number of items K.*

Note that $n > 1+2/c$ is a sufficient condition to make the seller generate revenues without allocating any items. The result is intuitive: a small c means low externalities, and thus it requires a large number of buyers to make the no-allocation equilibrium feasible. On the other hand, c can be considered as a function of n. It follows from Proposition 1 that the no-allocation equilibrium emerges when $c(n) > 2/(n-1)$. From (9), we can observe that, when n is large enough, the seller can implement the no-allocation equilibrium and collect a small payment (close to c) from each buyer. This is achieved by threatening the buyers with allocating the items to other buyers, in particular, threatening to allocate to K buyers with the largest negative externality values.

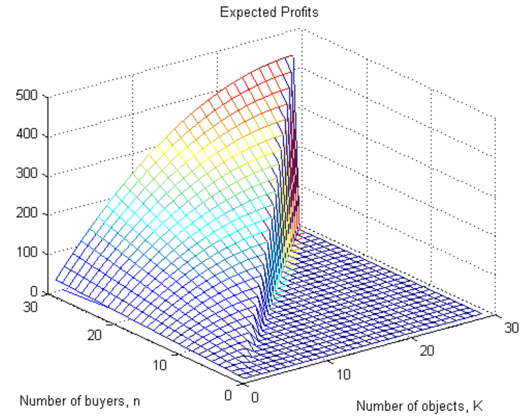

Figure 1: Figure 1 Expected Profit with n Buyers and K items. ($c = 1$)

Since the optimal expected profit is concave in K, the seller could achieve the largest expected profit when

$$K^* = n - 1/2.$$

Therefore, the expected profit increases in the number of items, K, because $K \leq n-1$.

Figure 1 shows how expected profits change with the number of buyers and the number of items, with $c = 1$, $1+\frac{2}{c} \leq n \leq 30$, and $1 \leq K \leq n-1$. The seller's expected profit increases linearly with the number of buyers n.

We further simplify the results above for a special case of the single item, $K = 1$, and for $c = 1$, which corresponds to the setting in [10].

By the definition of A_i (5), the expected utility of a non-participating buyer is given by

$$A = -(n-1)\int_0^1 \tau^{n-1}d\tau = -\frac{n-1}{n}.$$

By the definition of E (6), the externality is still

$$E = \int_0^1 \tau d\tau = \frac{1}{2}.$$

According to Theorem 1, the single item will be assigned to the buyer $i^* = \arg\max_i\{\pi_i - \frac{1-F^1(\pi_i)}{f^1(\pi_i)} - (n-1)E, 0\} = \arg\max_i\{2\pi_i - 1 - \frac{1}{2}(n-1), 0\}$. This requires $\pi_i \geq (n+1)/4$. Thus, the allocation rule is specified as

$$p_{i^*}(t_1, ..., t_n) = 1 \text{ and } p_{j\neq i^*}(t_1, ..., t_n) = 0.$$

There are two cases.

Case 1: $(n+1)/4 > 1$. Since $\pi_i \in [0,1]$, $p_i(t_1, ..., t_n) = 0$ for all i with the trigger strategy (3). Therefore, the interim payment for buyer i is $y_i^*(\pi_i) = -A+0-0-0 = (n-1)/n$. The expected profit of the seller at this case is thus $EP = n-1$.

Case 2: $(n+1)/4 \le 1$. If $\pi_i < (n+1)/4$, buyer i will not get the single item, and by (7) the interim payment is

$$y_i^*(\pi_i) = -A - 0 - 0$$
$$-(n-1)\int_{\frac{n+1}{4}}^{1} \pi_j^{n-2} dt_j^1 \int_0^1 t_j^2 dt_j^2$$
$$= \tfrac{n-1}{n} - \tfrac{1}{2}[1 - (\tfrac{n+1}{4})^{n-1}].$$

If $\pi_i \ge (n+1)/4$, buyer i may get the item. The optimal interim allocation rule is $q_i^*(t_i) = \int_{T_{-i}} p_i^*(t_i, t_{-i}) dt_{-i} = \pi_i^{n-1}$, and the optimal interim payment (7) is specified as

$$y_i^*(\pi_i)$$
$$= -A + t_i^1 q_i^*(t_i) - \int_0^{\pi_i} q_i^*(v, t_i^2) dv$$
$$-(n-1)\int_{\pi_i}^{1} \pi_j^{n-2} dt_j^1 \int_0^1 t_j^2 dt_j^2$$
$$= \tfrac{n-1}{n} + \pi_i^n - \tfrac{1}{n}\left[\pi_i^n - \left(\tfrac{n+1}{4}\right)^n\right]$$
$$-\tfrac{1}{2}(1 - \pi_i^{n-1})$$
$$= \tfrac{n-1}{n} + \tfrac{n-1}{n}\pi_i^n + \tfrac{1}{n}\left(\tfrac{n+1}{4}\right)^n$$
$$-\tfrac{1}{2} + \tfrac{1}{2}\pi_i^{n-1}.$$

Therefore, the seller's expected profit in this case is

$$EP$$
$$= \sum_{i=1}^n \int_0^1 y_i^*(\pi_i) d\pi_i$$
$$= n\int_0^1 y_i^*(\pi_i) d\pi_i$$
$$= \tfrac{n(n+1)}{4}\left[\tfrac{n-1}{n} - \tfrac{1}{2}\left[1 - \left(\tfrac{n+1}{4}\right)^{n-1}\right]\right]$$
$$+n\left(1 - \tfrac{n+1}{4}\right)\left[\tfrac{n-1}{n} - \tfrac{1}{2} + \tfrac{1}{n}\left(\tfrac{n+1}{4}\right)^n\right]$$
$$+n\int_{\frac{n+1}{4}}^{1}\left(\tfrac{n-1}{n}\pi_i^n + \tfrac{1}{2}\pi_i^{n-1}\right) d\pi_i.$$

Numerically, when $n = 1$, $EP = 0.25$; when $n = 2$, $EP = 1.115$ (consistent with the results in [10]); when $n = 3$, $EP = 2$; and when $n = 4$, $EP = 3$. For $n \le 3$, we are in Case 2. For $n \ge 4$, we are in Case 1. Therefore, for sufficiently large n, starting from $n = 4$ in this example, the single item will not be allocated to any buyer. However, the seller could raise revenues from the buyers by the trigger strategy. This is because each buyer would like to pay a small amount of money in order to avoid the impact of negative externalities. Moreover, the expected profit increases in the number of buyers since the expected profit increases in n when $n = 1$, 2, 3 and 4 and the expected profit is $n - 1$ when $n \ge 4$.

4.2 Dependent Externalities

Here we assume valuation π_i is drawn from $[0, 1]$ uniform distribution, and consider a perfectly linear correlation case, $\alpha_i = c_i\pi_i$, where c_i is publicly known, and $c_i > 0$. $c_i < 1$ means the externality is just a fraction of the valuation. Then, the allocation condition becomes

$$i \in K^1\left(\pi_i - \tfrac{1-F^1(\pi_i)}{f^1(\pi_i)} - (n-1)c_i\pi_i\right)$$
$$\text{and } \pi_i - \tfrac{1-F^1(\pi_i)}{f^1(\pi_i)} - (n-1)c_i\pi_i \ge 0.$$

With uniform distribution, it is

$$[2 - (n-1)c_i]\pi_i \ge 1.$$

Therefore, there will be no allocation if $2 - (n-1)c_i \le 0$ or $0 < 2 - (n-1)c_i < 1$.

Since π_i is drawn from $[0, 1]$ uniform distribution, α_i is uniform on $[0, c_i]$. By the definition of A_i (5), we get the same expression as in the previous subsection,i.e. (8)

$$A_i = -\tfrac{n-1}{n}c_i\cdot$$
$$\left[\left(\sum_{j=1}^{K-1}\tfrac{(n-2)!}{(n-j-2)!j!}\Pi_{i=0}^{j-1}\tfrac{j-i}{n-(j-i)}\right) + 1\right].$$

In particular, if $K = 1$,

$$\begin{aligned} A_i &= -(n-1)\int_0^1 c_i\tau^{n-1} d\tau \\ &= -\frac{n-1}{n}c_i. \end{aligned}$$

Consequently, the expected profit under the no-allocation equilibrium is the same as in the externality independence setting.

The following proposition summarizes these results.

Proposition 2. *Suppose there are K items and n buyers with types (π_i, α_i) where π_i is drawn from $U[0, 1]$ and $\alpha_i = c_i\pi_i$. If $n > \max_i\{1 + 1/c_i\}$, the optimal mechanism will not allocate any of the items. Moreover, the seller's expected profit is increasing in the number of buyers n, and in the number of items K.*

The condition from the proposition is only sufficient for the existence of the no-allocation equilibrium. In fact, when externality valuations are perfectly correlated with item valuations, buyers with highest item valuations are simultaneously the buyers who would impose highest negative externalities on rivals. Thus, with the same scales, a smaller number of buyers than in the independent externalities case will make negative externalities dominate item valuations and yield the no-allocation equilibrium. On the other hand, buyers with larger downward scales, i.e., a smaller c_i, impose smaller negative externalities on rivals. Thus, a larger number of buyers is required to yield the no-allocation equilibrium.

Corollary 1. *Under conditions of either Proposition 1 or Proposition 2, the expected profits under the no-allocation equilibrium are*

$$\begin{aligned} EP &= -\sum_{i=1}^n A_i \\ &= \sum_{i=1}^n \frac{c_i}{n}\left[(n-1)K - \frac{K(K-1)}{2}\right]. \end{aligned}$$

5. NO-ALLOCATION EQUILIBRIUM WITH INDEPENDENT EXTERNALITIES

We now show that the no-allocation equilibrium emerges for a large enough number of buyers, regardless of the distributional assumptions. We assume externality α_i to be independent from the item valuations π_i.

THEOREM 2. *Suppose there are K items for sale to n buyers with independent valuations and externalities. There exist N such that for $n > N$, it is optimal for the seller not to allocate any item. Moreover, the seller's expected profit is positive under the no-allocation equilibrium, and is increasing in both K and n.*

PROOF. See Appendix A. □

The independence assumption is not a critical one and the same proof strategy would go through even when externalities depend on the item valuations. The exact conditions for the no-allocation result would be determined by the dependence structure of externalities and valuations.

6. CONCLUDING REMARKS

In this paper we discuss an optimal mechanism that a monopolist seller could use to exploit possible (negative) externalities among large number of buyers interested in the limited number of items that are being sold. Specifically, we demonstrate that no allocation is an equilibrium even with modest negative externalities, provided sufficiently large demand. This provides an opportunity for the seller to generate revenues without having to sell any of the items. For example, a monopolistic seller of the ad space could generate revenues by charging potential advertisers for not showing any ads (thereby providing ads-free experience to the end users that are initial targets by the interested advertisers). For this strategy to be viable, (i) advertisers would have to be (ever so slightly) negatively impacted if rival's ad is shown, and (ii) the number of interested advertisers has to be large relative to the number of available ad slots.

Our formal analysis of the properties of the optimal mechanism is first illustrated in the case of uniformly distributed valuations. We show that it is optimal for the seller not to allocate any item to any buyer when there are many buyers. Moreover, under the no-allocation equilibrium, the seller's optimal expected profit is increasing in the number of buyers and the number of items. These results hold both in the case of independent externalities and in the case of dependent externalities. We then proceed to show the robustness of the results in a general distribution case.

A critical piece of our model is the two-dimensional valuation structure that we adopt from [10]. In addition to the private valuation of the item, externalities imposed to rivals are also private information of the buyer who creates these externalities by getting the item. For example, only the ad owner knows the conversion rates and can have an estimate of the negative externalities imposed on competitors whose ads are not shown. We discuss a slight modification of the negative externalities in the Appendix B and show that our main result holds regardless if negative externality is imposed on all competitors or on auction losers only.

It would be interesting to consider the case in which the externalities imposed on a buyer who does not get the item is privately known to that buyer, and not to a rival who got the item. However, the problem of analytically finding an optimal mechanism in such setting remains intractable, except in special cases. (For example, certain information structures could be transformed into setting of this paper.) This raises a question whether the emergence of the no-allocation equilibrium is a mere consequence of the valuation structure in our model or whether it holds for a variety of valuation structures. We conjecture that the intuitive reasoning (that we confirmed analytically within our model) ought to hold for any reasonable valuation structure: there should be no allocation when the sum of negative externalities imposed on a large number of buyers who do net get an item is larger than the total surplus of the small number of buyers who would get the items. In fact, special cases of our model such as commonly known value of negative externalities have direct special case analogues in other reasonable valuation structure models. Also, we can numerically confirm the results for a limited number of buyers with privately held values for the allocation in which negative externalities are imposed on them, but have to limit calculations to a small finite number of buyer types.

Our findings provide theoretical foundation for plausibility of existence of the no-allocation equilibrium. Thus, such option should not be ignored whenever negative externalities are present and competition is intense. Hence, design of computationally manageable and implementable direct mechanisms that exploit privately held information about negative externalities could have implications in practice.

We leave for future research an investigation of optimal mechanisms for valuation structures in which each buyer has private information on its valuation for any allocation. Furthermore, investigating optimal mechanisms in such valuation model would likely allow for important insights on the relationship between exclusivity and negative externalities that go beyond basic observations in this paper.

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APPENDIX

A. PROOF OF THEOREM 2

PROOF. By the Assumption 1, the condition

$$\pi_i - \frac{1-F^1(\pi_i)}{f^1(\pi_i)} - (n-1)E \geq 0$$

can be reduced to the condition

$$\pi_i \geq \widetilde{\pi}(n),$$

where $\widetilde{\pi}(n)$ increases with n. (Note that this argument may not hold in the externality dependence case for an arbitrary $f^1(\cdot)$ function and an arbitrary number c.)

As a result, we may have $\widetilde{\pi}(N_1) > \overline{\pi}$ for some large enough N_1. This is analogous to the no-allocation result from the uniform distribution case.

In order to study the expected profit under the no-allocation equilibrium, we consider $K = 1$ first. The interim payment is

$$\begin{aligned} & y_i(\pi_i) \\ &= (n-1)\int_{\underline{\alpha}}^{\overline{\alpha}} \tau(F^2(\tau))^{n-2} f^2(\tau)d\tau \\ &= \overline{\alpha} - \int_{\underline{\alpha}}^{\overline{\alpha}} (F^2(\tau))^{n-1} d\tau. \end{aligned}$$

Obviously,

$$\begin{aligned} & \tfrac{\partial y_i(\pi_i)}{\partial n} \\ &= -\int_{\underline{\alpha}}^{\overline{\alpha}} (F^2(\tau))^{n-1}\ln(F^2(\tau))d\tau \geq 0 \end{aligned}$$

since $F^2(\tau) \in [0,1]$.

Since we are considering negative externalities, i.e. $\overline{\alpha} \geq \underline{\alpha} \geq 0$, we have $\tau(F^2(\tau))^{n-2}f^2(\tau) \geq 0$ for $\underline{\alpha} \leq \tau \leq \overline{\alpha}$. This indicates $y_i(\pi_i) > 0$ since $n > 1$.

Hence,

$$EP = ny_i(\pi_i)$$

is increasing in n for $n > N_1$.

Now we consider $K \geq 1$ under the no-allocation equilibrium. For $j = 0, ..., K-1$, the density of the $(n-1-j)^{\text{th}}$ largest out of $n-1$ i.i.d. random variables with CDF $F(x)$ and PDF $f(x)$, is

$$\begin{aligned} & f_{x^{(n-1-j)}} \\ &= \tfrac{(n-1)!}{(n-j-2)!j!} F(x)^{n-j-2}(1-F(x))^j f(x). \end{aligned}$$

Therefore, the seller's expected profit is

$$\begin{aligned} & EP \\ &= n\sum_{j=0}^{K-1}\int_{\underline{\alpha}}^{\overline{\alpha}} \tfrac{(n-1)!}{(n-j-2)!j!} F^2(\tau)^{n-j-2} \cdot \\ & \qquad (1-F^2(\tau))^j f^2(\tau)\tau d\tau. \end{aligned}$$

With negative externalities, for each j,

$$\begin{aligned} & h(n,j) \\ & \triangleq \int_{\underline{\alpha}}^{\overline{\alpha}} \tfrac{(n-1)!}{(n-j-2)!j!} F^2(\tau)^{n-j-2}(1-F^2(\tau))^j f^2(\tau)\tau d\tau \\ & \geq 0. \end{aligned}$$

This indicates EP is increasing in K by adding more non-negative terms.

For each $j > 0$, we do the integration by parts,

$$\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-2}(1-F^2(\tau))^j f^2(\tau)\tau d\tau$$

$$= \int_{\underline{\alpha}}^{\overline{\alpha}} \frac{1}{n-j-1}(1-F^2(\tau))^j \tau dF^2(\tau)^{n-j-1}$$

$$= \frac{1}{n-j-1}(1-F^2(\tau))^j F^2(\tau)^{n-j-1}\tau|_{\underline{\alpha}}^{\overline{\alpha}}$$

$$-\frac{1}{n-j-1}\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}\cdot$$

$$\left((1-F^2(\tau))^j d\tau - j\left(1-F^2(\tau)\right)^{j-1}\tau dF^2(\tau)\right)$$

$$= \frac{1}{n-j-1}\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1} j\left(1-F^2(\tau)\right)^{j-1}\tau dF^2(\tau)$$

$$-\frac{1}{n-j-1}\int_{\underline{\alpha}}^{\overline{\alpha}} \left(F^2(\tau)\right)^{n-j-1}(1-F^2(\tau))^j d\tau$$

$$= \frac{1}{n-j-1}\frac{1}{n-j} j(1-F^2(\tau))^{j-1}F^2(\tau)^{n-j}\tau|_{\underline{\alpha}}^{\overline{\alpha}}$$

$$-\frac{1}{n-j-1}\frac{1}{n-j}\int_{\underline{\alpha}}^{\overline{\alpha}} jF^2(\tau)^{n-j}\cdot$$

$$[(1-F^2(\tau))^{j-1}d\tau - (j-1)\left(1-F^2(\tau)\right)^{j-2}\tau dF^2(\tau)]$$

$$-\frac{1}{n-j-1}\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j d\tau.$$

If $j > 1$, the first term is zero and we can also continue the iteration of integration by parts,

$$\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-2}(1-F^2(\tau))^j f^2(\tau)\tau d\tau$$

$$= 0 + \frac{1}{n-j-1}\frac{1}{n-j} j\cdot$$

$$\int_{\underline{\alpha}}^{\overline{\alpha}} (j-1)\left(1-F^2(\tau)\right)^{j-2} F^2(\tau)^{n-j}\tau dF^2(\tau)$$

$$-\frac{1}{n-j-1}\frac{1}{n-j} j\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j}(1-F^2(\tau))^{j-1}d\tau$$

$$-\frac{1}{n-j-1}\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j d\tau.$$

For $j > 0$, we thus have the general form

$$\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-2}(1-F^2(\tau))^j f^2(\tau)\tau d\tau$$

$$= \left(\frac{1}{n-j-1}\Pi_{l=0}^{j-1}\frac{1}{n-j+l}(j-l)\right)\overline{\alpha}$$

$$-\frac{1}{n-j-1}\sum_{l=0}^{j-1}\left(\Pi_{m=0}^{l}\frac{1}{n-j+m}(j-m)\right)\cdot$$

$$\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j+l}(1-F^2(\tau))^{j-l-1}d\tau$$

$$-\frac{1}{n-j-1}\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j d\tau$$

$$= \frac{(n-j-2)!}{(n-1)!} j!\overline{\alpha}$$

$$-\frac{1}{n-j-1}\sum_{l=0}^{j-1}\frac{j!(n-j-1)!}{(j-l-1)!(n-j+l)!}\cdot$$

$$\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j+l}(1-F^2(\tau))^{j-l-1}d\tau$$

$$-\frac{1}{n-j-1}\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j d\tau.$$

Therefore, for each $j > 0$

$$\begin{aligned} & h(n,j) \\ = & \int_{\underline{\alpha}}^{\overline{\alpha}} \frac{(n-1)!}{(n-j-2)!j!} F^2(\tau)^{n-j-2}(1-F^2(\tau))^j f^2(\tau)\tau d\tau \\ = & \ \overline{\alpha} \\ & -\frac{(n-1)!}{(n-j-1)!j!}\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j d\tau \\ & -\sum_{l=0}^{j-1}\frac{(n-1)!}{(j-l-1)!\,(n-j+l)!}\cdot \qquad (10) \\ & \quad \int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j+l}(1-F^2(\tau))^{j-l-1}d\tau. \qquad (11) \end{aligned}$$

This formulation also yields the case $j = 0$.

Since for $j > 0$

$$(n-1)-(n-j-1) = j,$$

and for $0 \leq l \leq j-1$

$$(n-1)-(n-j+l) = j-l-1.$$

Hence,

$$\frac{(n-1)!}{(n-j-1)!j!}$$

has finite n terms in the numerator, and for each l

$$\frac{(n-1)!}{(j-l-1)!\,(n-j+l)!}$$

also has finite n terms in the numerator. Therefore,

$$-\frac{(n-1)!}{(n-j-1)!j!}\cdot \int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j d\tau$$

and

$$-\sum_{l=0}^{j-1}\frac{(n-1)!}{(j-l-1)!\,(n-j+l)!}\cdot \int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j+l}(1-F^2(\tau))^{j-l-1}d\tau$$

both increase in n for $n > N_2$, where N_2 is a sufficiently large number. To see this, take the first one for example. The derivative is

$$\frac{\partial}{\partial n}\left(-\frac{(n-1)!}{(n-j-1)!j!}\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j d\tau\right)$$

$$= -\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j d\tau \frac{\partial}{\partial n}\left(\frac{(n-1)!}{(n-j-1)!j!}\right)$$

$$-\frac{(n-1)!}{(n-j-1)!j!}\frac{\partial}{\partial n}\left(\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j d\tau\right).$$

By Leibniz's rule, we have

$$-\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j d\tau \frac{\partial}{\partial n}\left(\frac{(n-1)!}{(n-j-1)!j!}\right)$$

$$-\frac{(n-1)!}{(n-j-1)!j!}\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}\cdot$$

$$(1-F^2(\tau))^j \ln\left(F^2(\tau)\right) d\tau$$

$$=\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j\cdot$$

$$\left(-\frac{(n-1)!}{(n-j-1)!j!}\ln\left(F^2(\tau)\right)-\frac{\partial}{\partial n}\left(\frac{(n-1)!}{(n-j-1)!j!}\right)\right)d\tau.$$

Since $F^2(\tau) \in [0,1]$, the derivative is positive for sufficiently large n.

As a result, for arbitrary K,

$$EP = n\sum_{j=0}^{K-1} h(n,j)$$

is increasing in n for $n > N \triangleq \max\{N_1, N_2\}$. □

B. AN ALTERNATIVE MODEL

We discuss the scenario where negative externalities are imposed only if the buyer does not obtain any of the item. Hence, the interim utility function is revised as

$$U(s_i,t_i)$$

$$= q_i(s_i)t_i^1 - y_i(s_i) - (1-q_i(s_i))\cdot \quad .$$

$$\sum_{j\neq i}\int_{T_{-i}} p_j(s_i,t_{-i})t_j^2\phi_{-i}(t_{-i})dt_{-i}$$

As a result, the envelope theorem gives potentials similar to the standard case, i.e.,

$$\frac{dU(t_i,t_i)}{dt_i^1} = q_i(t_i)$$

and

$$\frac{dU(t_i,t_i)}{dt_i^2} = 0.$$

The interim payment is thus

$$y_i(t_i) = -A_i + t_i^1 q_i(t_i) - \int_{\underline{\pi}_i}^{t_i^1} q_i(v,t_i^2)dv$$

$$-(1-q_i(t_i))\sum_{j\neq i}\int_{T_{-i}} p_j(t_i,t_{-i})t_j^2\phi_{-i}(t_{-i})dt_{-i}$$

where A_i has the same definition.

Hence, with independent externalities, the seller's expected profit is

$$EP = -\sum_{i=1}^n A + \int(\sum_{i=1}^n[\pi_i - \frac{1-F^1(\pi_i)}{f^1(\pi_i)} - (n-1)E_i]\cdot$$

$$p_i(\pi_1,\ \ldots,\pi_n))f^1(\pi_1)\ldots f^1(\pi_n)d\pi_1\ldots d\pi_n$$

$$+\sum_{i=1}^n\int q_i(t_i)(\sum_{j\neq i}\int_{T_{-i}} p_j(t_i,t_{-i})t_j^2\phi_{-i}(t_{-i})dt_{-i})\cdot$$

$$f^1(t_i^1)f^2(t_i^2)dt_i$$

where

$$E_i = \int_{\underline{\alpha}}^{\overline{\alpha}} \tau f_i^2(\tau)d\tau.$$

We focus on the last term, which is

$$Ext \triangleq \sum_{i=1}^n\int q_i(t_i)\cdot$$

$$(\sum_{j\neq i}\int_{T_{-i}} p_j(t_i,t_{-i})t_j^2\phi_{-i}(t_{-i})dt_{-i})f^1(t_i^1)f^2(t_i^2)dt_i$$

$$=\sum_{i=1}^n\int p_i(t_i,t_{-i})\cdot$$

$$(\sum_{j\neq i} E\int_{T_{-i}^1} p_j(t_i,t_{-i})\phi_{-i}^1(t_{-i}^1)dt_{-i}^1)f^1(t_i^1)\phi_{-i}^1(t_{-i}^1)dt_i^1$$

$$=\int(\sum_{i=1}^n\sum_{j\neq i} p_i(t_i,t_{-i})E\int_{T_{-i}^1} p_j(t_i,t_{-i})\phi_{-i}^1(t_{-i}^1)dt_{-i}^1)\cdot$$

$$f^1(\pi_1)\cdots f^1(\pi_n)d\pi_1\cdots d\pi_n \geq 0.$$

Since the allocation probabilities should satisfy

$$0 \leq \sum_{i=1}^n p_i(t_i,t_{-i}) \leq K,$$

the Ext must be bounded above by some finite positive number $\overline{Ext}$.

Moreover, with the i.i.d. assumption about the distributions and from (10), we obtain

$$-nA$$

$$= n\sum_{j=0}^{K-1} h(n,j)$$

$$= n\sum_{j=0}^{K-1}(\overline{\alpha} - \frac{(n-1)!}{(n-j-1)!j!}\cdot$$

$$\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j-1}(1-F^2(\tau))^j d\tau$$

$$-\sum_{l=0}^{j-1}\frac{(n-1)!}{(j-l-1)!(n-j+l)!}\cdot$$

$$\int_{\underline{\alpha}}^{\overline{\alpha}} F^2(\tau)^{n-j+l}(1-F^2(\tau))^{j-l-1}d\tau)$$

$$\to \infty \text{ as } n \to \infty$$

because the last two terms converge to zero as $n \to \infty$.

In addition, $-(n-1)E_i$ goes to $-\infty$ as $n \to \infty$. Therefore, when n is sufficiently large, it is still optimal to implement the no-allocation equilibrium and the seller's expected profit is

$$EP^* = -nA,$$

which is positive and increasing in n and K as illustrated in Theorem 2.