CS 15-892 Foundations of Electronic Marketplaces
Homework 2

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Due October 28th in the beginning of class.

You may use any sources that you want. If you do so, you must cite the sources that you use. Teamwork is not allowed.

1. (10pts) Consider a private-values auction of a single good to a single bidder with quasilinear utility function and knowledge of its own valuation $v \sim F$. The auctioneer posts a fixed reserve price r; the bidder then accepts the good at that price or leaves the auction. Derive the revenue-maximizing r for this setting.

2. (30pts) Consider designing a mechanism (where participation is *ex post* individually rational) for the following setting. You have one company (*Profit & Gamble)* to sell. You don't care about keeping it or getting rid of it. There are two bidders with quasilinear utility functions. Bidder 1's valuation is drawn from a uniform distribution on [0,1] (1 = one billion dollars). Bidder 2's valuation is independently drawn from a uniform distribution on [1,4].

   (a) Design a mechanism that uses take-it-or-leave-it offers (at most one offer to each bidder) and attempts to maximize revenue subject to that. What is your expected revenue in your mechanism? What is your worst-case revenue in your mechanism? Is your mechanism Pareto efficient? Justify your answer.

   (b) What is the optimal (i.e., revenue-maximizing) auction for the setting? What is your expected revenue in that auction? What is your worst-case revenue in that auction? Is that auction Pareto efficient? Justify your answer.

3. (40pts) (War of attrition [from Auction Theory, 2nd edition, by Krishna]) Consider a two-bidder war of attrition in which the bidder with the highest bid wins the object but both bidders pay the losing bid. The bidders' values are independently and identically distributed according to F.

   (a) Derive a symmetric equilibrium bidding strategy.

   (b) Directly show the symmetric equilibrium strategy and the seller's revenue when the bidders' values are uniformly distributed on $[0, 1]$.

   Hint: Use the final equation of the following proof of the revenue equivalence theorem [slightly adapted from Auction Theory, 2nd edition, by Krishna].

Consider any auction where the highest bidder wins, and fix a symmetric equilibrium β of it. Let p(v) be the equilibrium expected payment by a bidder with value v. Suppose β is such that $p(0) = 0$.

Consider the expected payoff of bidder 1 with value v when he bids $\beta(z)$ instead of $\beta(v)$. Bidder 1 wins when his bid $\beta(z)$ exceeds the highest competing bid $\beta(Y_1)$, i.e., when $z > Y_1$. His expected payoff is

$$u(z, v) = G(z)v - p(z)$$

where $G(z) = F(z)^{N-1}$ is the distribution of Y_1.

Maximization results in the first-order condition

$$\frac{\partial u(z, v)}{\partial z} = g(z)v - \frac{d}{dz}p(z) = 0$$

At an equilibrium it is optimal to bid according to $z = v$, so we obtain that for all y,

$$\frac{d}{dy}p(y) = g(y)y$$

Thus,

$$\begin{aligned} p(v) &= p(0) + \int_0^v yg(y)dy \\ &= \int_0^v yg(y)dy \\ &= G(x)\ E[Y_1|Y_1 < x] \end{aligned}$$

Since the right hand side does not depend on the particular auction form, revenue equivalence follows. □

4. (20pts) For this problem, assume valuation functions are continuous and bounded. We will show the following.

Theorem 1 2-*bidder, independent and identically-distributed private-valuations, first-price auctions* with a random and unknown reserve price *have no assymmetric equilibria.*

   (a) (5pts) As a warm up, let's assume that Theorem 1 is true. Prove the following corollary.

   Corollary 1 *For $n > 2$, all n-bidder, independent and identically-distributed private-valuations, first-price auctions* with no reserve price *have no assymmetric equilibria.*

   (b) (5pts) Next, prove the following claim for general (continuous, bounded) bid functions b_1 and b_2 and allocation $\mathbf{x}$, where x_i refers to the allocation for a specific agent i.

Claim 1 *Assume bid functions b_1 and b_2 cross exactly twice. For any v, if $b_1(v) > b_2(v)$, then $x_1(v) > x_2(v)$. Also, if $b_1(v) = b_2(v)$, then $x_1(v) = x_2(v)$.*

Hint: Figure 1 (courtesy Jason Hartline) visualizes the two-crossing property. Valuations v are shown on the x-axis, while bid functions b_1 and b_2 are shown on the y-axis. Between distinguished valuations v' and v'', we have $b_1(v) > b_2(v)$ (also: $b_1(v') = b_2(v')$ and $b_1(v'') = b_2(v'')$). What does $x_i(v)$ for $v \in (v', v'')$ look like as a function of the random reserve price and the other agent's valuation draw? Specifically, how does the value of x_1 compare to x_2 for that range of v?

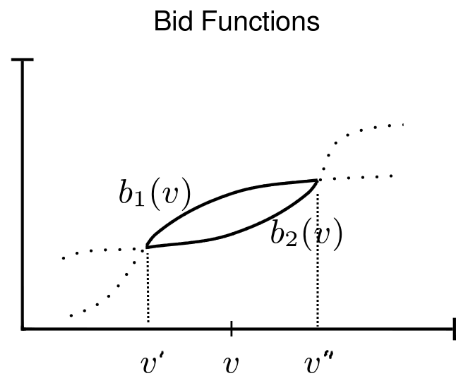

Figure 1: Visual hint for Claim 1.

(c) (10pts) Recall from class that we can write the utility $u_i(v_i)$ of an agent i given its valuation v_i in two ways.

- By the first-price payment rule

$$u_i(v_i) = (v_i - b_i(v_i)) \cdot x_i(v_i), \tag{1}$$

  where b_i is the agent's bid function and x_i is the allocation for i from the allocation $\mathbf{x}$. That is, the agent receives the utility of the good minus what it paid, weighted by the probability that it wins the auction.

- By the revenue equivalence theorem

$$u_i(v_i) = \int_0^{v_i} x_i(z) dz. \tag{2}$$

Prove Theorem 1. You will likely use Equations 1 and 2, as well as Claim 1.