Cake Cutting

Teacher: Ariel Procaccia (on behalf of Sandholm)
• Single heterogeneous good, represented as $[0,1]$

• Set of players $N = \{1, \ldots, n\}$

• Piece of cake $X \subseteq [0,1]$: finite union of disjoint intervals
Each player $i$ has a valuation $V_i$ that is:

- **Additive**: $\alpha + \beta$
- **Normalized**: 1
- **Divisible**: $\lambda \alpha$
Fairness, formalized

• Our goal is to find an allocation $A_1, \ldots, A_n$

• Proportionality:

$$\forall i \in N, V_i(A_i) \geq \frac{1}{n}$$

• Envy-Freeness (EF):

$$\forall i, j \in N, V_i(A_i) \geq V_i(A_j)$$
Fairness, formalized

Poll 1: What is the relation between proportionality and EF?
1. Proportionality $\Rightarrow$ EF
2. EF $\Rightarrow$ proportionality
3. Equivalent
4. Incomparable
**Cut-and-Choose**

- Algorithm for $n = 2$ [Procaccia and Procaccia, circa 1985]
- Player 1 divides into two pieces $X, Y$ s.t.
  \[ V_1(X) = 1/2, V_1(Y) = 1/2 \]
- Player 2 chooses preferred piece
- This is EF and proportional
The Robertson-Webb model

- What is the time complexity of C&C?
- Input size is $n$
- Two types of queries
  - $\text{Eval}_i(x, y)$ returns $V_i([x, y])$
  - $\text{Cut}_i(x, \alpha)$ returns $y$ such that $V_i([x, y]) = \alpha$

Diagram:

- Eval output $\rightarrow \alpha$
- $x \rightarrow y \leftarrow \text{cut output}$
The Robertson-Webb model

- Two types of queries
  - $\text{Eval}_i(x, y) = V_i([x, y])$
  - $\text{Cut}_i(x, \alpha) = y \text{ s.t. } V_i([x, y]) = \alpha$

#queries needed to find an EF allocation when $n = 2$?
Dubins-Spanier

• Referee continuously moves knife
• Repeat: when piece left of knife is worth $1/n$ to player, player shouts “stop” and gets piece
• That player is removed
• Last player gets remaining piece
Poll 2: What is the complexity of DS in the RW model?

1. $\Theta(n)$
2. $\Theta(n \log n)$
3. $\Theta(n^2)$
4. $\Theta(n^2 \log n)$
1/3
1/3  1/3  1/2
Even-Paz

• Given \([x, y]\), assume \(n = 2^k\)
• If \(n = 1\), give \([x, y]\) to the single player
• Otherwise, each player \(i\) makes a mark \(z\) s.t.

\[
V_i([x, z]) = \frac{1}{2} V_i([x, y])
\]

• Let \(z^*\) be the \(n/2\) mark from the left
• Recurse on \([x, z^*]\) with the left \(n/2\) players, and on \([z^*, y]\) with the right \(n/2\) players
Even-Paz
Even-Paz: proportionality

• Claim: The Even-Paz protocol produces a proportional allocation

• Proof:
  • At stage 0, each of the $n$ players values the whole cake at 1
  • At each stage the players who share a piece of cake value it at least at $V_i([x,y])/2$
  • Hence, if at stage $k$ each player has value at least $1/2^k$ for the piece he’s sharing, then at stage $k + 1$ each player has value at least $\frac{1}{2^{k+1}}$
  • The number of stages is $\log n$
Even-Paz: complexity

Overall: \( n \log n \)
Complexity of proportionality

• Theorem [Edmonds and Pruhs 2006]: Any proportional protocol needs $\Omega(n \log n)$ operations in the RW model
• We will prove the theorem on Tuesday
• The Even-Paz protocol is provably optimal!
What about envy?
Selfridge-Conway

- **Stage 0**
  - Player 1 divides the cake into three equal pieces according to $V_1$
  - Player 2 trims the largest piece s.t. there is a tie between the two largest pieces according to $V_2$
  - Cake 1 = cake w/o trimmings, Cake 2 = trimmings

- **Stage 1 (division of Cake 1)**
  - Player 3 chooses one of the three pieces of Cake 1
  - If player 3 did not choose the trimmed piece, player 2 is allocated the trimmed piece
  - Otherwise, player 2 chooses one of the two remaining pieces
  - Player 1 gets the remaining piece
  - Denote the player $i \in \{2, 3\}$ that received the trimmed piece by $T$, and the other by $T'$

- **Stage 2 (division of Cake 2)**
  - $T'$ divides Cake 2 into three equal pieces according to $V_{T'}$
  - Players $T$, 1, and $T'$ choose the pieces of Cake 2, in that order
The complexity of EF

• Theorem [Brams and Taylor 1995]: There is an unbounded EF cake cutting algorithm in the RW model

• Theorem [P 2009]: Any EF algorithm requires $\Omega(n^2)$ queries in the RW model

• Theorem [Kurokawa, Lai, P, 2013]: EF cake cutting with piecewise uniform valuations is as hard as general case
The complexity of EF
The complexity of EF

• Theorem [Kurokawa, Lai, P, 2013]: EF cake cutting with piecewise linear valuations is polynomial in the number of breakpoints
**RW is for honest kids**

- EF protocol that uses $n$ queries
- $f = 1$-1 mapping from valuation functions to $[0,1]$
- The protocol asks each player $\text{cut}_i(0, 1/2)$
- Player $i$ replies with $y_i = f(V_i)$
- The protocol computes $V_i = f^{-1}(y_i)$
- We therefore need to assume that players are “honest”