On Characterizations of Truthful Mechanisms for Combinatorial Auctions and Scheduling (Addendum)

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Abstract

We show that the only social choice functions that are implementable in a multi-unit auctions setting with two bidders which always allocate all items are affine maximizers.

1 Introduction

This note is an addendum to [2]. One of the results of [2] is that the only implementable social choice functions for combinatorial auctions with subadditive bidders (or XOS) are affine maximizers. In this note we prove the analogue of this result in the multi-unit auctions setting. This result was implicit at [2] and uses no new techniques. We refer the reader to [2] for context for this note and discussion.

A function f is called *affine maximizer* if there exists a subset of the alternatives $\mathcal{A}'$ such that $f(v_1, \ldots, v_n) = \arg\max_{a \in \mathcal{A}' \subseteq \mathcal{A}} w_i v_i(a) + c_a$, where the w_i's and the c_a's are predetermined non-negative constants. The only simplification for the sake of presentation we make is that we consider only functions that are scalable. A function is *scalable* if for each $\alpha > 0$ we have that $f(v_1, \ldots, v_n) = f(\alpha \cdot v_1, \ldots, \alpha \cdot v_n)$. Very informally, the "currency" that we use does not matter. Notice that the scalability assumptions means that all the c_a's in the definition of affine maximizers are zero (such functions are also called *weighted welfare maximizers*). This assumption can be removed, but we do not do it here.

In a multi-unit auction we have m items and n bidders. Bidder's i's private information is his valuation function $v_i : [m] \to \mathbb{R}$. We assume that the v_i are monotone, and that $v_i(0) = 0$. The theorem we prove is that every (scalable) implementable function f must be an affine maximizer.

The theorem is a strengthening of [3] that proves an identical result, but using the extra condition that the f is *decisive*: if player i bids high enough for $v_i(m)$, he will indeed be assigned all the m items. Thus, our proof can also be seen as a simpler (and stronger) proof of this result from [3].

Let us say a word about the computational implications of this result. Let A be a deterministic truthful mechanism for multi-unit auctions that provides a $2 - \epsilon$ approximation ratio, for some $\epsilon > 0$, and always allocates all items. By the theorem, it must be an affine maximizer. However, by

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the result of [1], an affine maximizer that provides a better-than-2 approximation ratio must run in exponential time. Thus, any truthful mechanism *that always allocates all items* cannot provide a better-than-2 approximation ratio in polynomial time, although a non-truthful FPTAS exists. Removing the assumption that A always allocates all items, and thus proving a genuine separation between the power of polynomial time truthful mechanisms and polynomial time algorithms is a major open question in algorithmic mechanism design.

2 The Characterization

Theorem 2.1 *Let f be a scalable social choice function of a truthful multi-unit auction with 2 bidders that always allocates all items. Suppose that f has a range of size at least 3. Then, f must be an affine maximizer.*

Before proceeding to the proof, we require some notation. Let v, v' and u be valuations. A mechanism f is *weakly monotone* if for all valuations v, v' and u, if $f(v,u) = a$, and $f(v',u) = b$, then $v(a)-v(b) \leq v'(a)-v'(b)$. A mechanism is called *strongly monotone* if for all valuations it holds that $v(a) - v(b) < v'(a) - v'(b)$, if $a \neq b$. In domains where a player is interested in minimizing his cost rather than maximizing his value (like scheduling) the direction of the inequalities is reversed.

Note that every implementable social choice function satisfies weak monotonicity. Further, using the following lemma from [3], we can focus on social choice functions that satisfy strong monotonicity.

Lemma 2.2 *Suppose there exists a social choice function $f : V \rightarrow A$ that satisfies weak monotonicity, allocates all items and is not an affine maximizer. Then there exists a social choice function $f' : V \rightarrow A$ with the same range that satisfies strong monotonicity, allocates all items and is not an affine maximizer.*

Fix an arbitrary social choice function $f : V \rightarrow A$ that satisfies strong monotonicity, allocates all items and has at least three outcomes in its range. Since we have that f satisfies strong monotonicity, we do no have to assume that $v(0) = u(0) = 0$. These values can take arbitrary values (while keeping the monotonicity constraints) since the valuation can always be normalized by deducting $v(0)$ from all values of the valuation. The output of the mechanism is left unchanged, by strong monotonicity.

The main part of the proof is to show that f is an affine maximizer. Let the vectors v and u denote the valuations of bidders 1, 2 respectively. The outcome (n_1, n_2) allocates n_1 items to bidder 1 and n_2 items to bidder 2. We now define the sets $P(x, y)$ for any two allocations $(x, m-x), (y, m-y) \in \mathcal{A}$.

Definition 2.3 *Let $(x, m-x)$ and $(y, m-y)$ be two allocations in the range of f. $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ is in $P(x, y)$ if there exist two valuations v and u such that $f(v, u) = (x, m-x)$, and $\alpha_1 = v(x)-v(y)$ and $\alpha_2 = u(m-x) - u(m-y)$.*

The sets $P(x, y)$ have the following geometric interpretation. Suppose we plot $(\alpha_1, \alpha_2) = \alpha$ on the cartesian plane. We can partition the plane into three regions. $P(x, y)$, $\overline{P}(x, y)$, and an invalid region. We say that a point $\alpha = (\alpha_1, \alpha_2)$ is *invalid* if there are no two valuations v and u such that $\alpha_1 = v(x)-v(y)$ and $\alpha_2 = u(m-x)-u(m-y)$. Notice that monotonicity of valuations, implies for

$x > y$, bidder 1 prefers x over y and bidder 2 prefers $m - y$ over $m - x$; thus the only valid points are in the south-west quadrant. By definition, any $\alpha \in P(x, y)$ is valid. Finally, a point belongs to $\overline{P}(x, y)$ if it valid and not in $P(x, y)$. We use the notation $\dot{P}(x, y)$ to denote the interior of this set, which is defined in the usual topological meaning.

The proof structure is as follows. The next lemma shows the definition of the P's "makes sense": for any two valuations profiles (v, u) and (v', u') such that for each bidder, the relative preference of the outcome $(x, m - x)$ over the outcome $(y, m - y)$ in both valuation profiles is the same, the allocation chosen by the mechanism must satisfy a consistency property.

The motivation is as follows. If f is a weighted welfare maximizer, then there must exist a constants w_1, w_2 for the two players such that for any valuation v and $f(v) = (x, m - x)$, and for any other outcome $(y, m - y)$,

$$w_1 \cdot v_1(x) + w_2 \cdot v_2(m - x) \geq w_1 \cdot v_1(y) + w_2 \cdot v_2(m - y)$$

Rearranging, we have that

$$w_1 \cdot (v_1(x) - v_1(y)) \geq w_2 \cdot (v_2(m - y) - v_2(m - x))$$

This implies that if we plot the sets $P(x, y)$ and $\overline{P}(x, y)$ on the cartesian plane, they must be separated by a line (through the origin for weighted welfare maximizers). Further, the slopes of the line must be identical for all choices of x and y, $x \geq y$. This is exactly what the next two lemmas prove.

Lemma 2.5 proves that all the P's are identical in the interiors, for all x, y where $x \geq y$. Finally, Lemma 2.9 shows that $P(x, y)$ is separated from $\overline{P}(x, y)$ by a line. Recall that since all the interiors of the P's are identical, this line must be identical for all $P(x, y)$ too. Thus, since each P is separated from $\overline{P}$ by the same line, it must be the case that f is an affine maximizer.

Lemma 2.4 *Let v and u be two valuations such that $f(v, u) = (x, m - x)$. Let $\alpha = (v(x) - v(y), u(m - x) - u(m - y)$. Let v' and u' be two valuations where $v'(x) - v'(y) = v(x) - v(y)$ and $u'(x) - u'(y) = u(x) - u(y)$. Then, $f(v', u') \neq (y, m - y)$.*

Proof: Throughout the proof we assume that $x > y$. The other case where $x < y$ is symmetric. The proof is by contradiction, assume that $f(v', u') = y$. Define (v'', u'') as follows; intuitively, the relative preferences of the outcomes in (v'', u'') is α and in valuation profiles (v', u'), (v, u) and the valuation profile is 'flattened' so that all other outcomes are less attractive. Note that $\epsilon > 0$ is chosen to be small enough so that the valuation is monotone.

$$v''(t) = \begin{cases} \max(v(t), v'(t)) + c, & t = x, t = y \\ \max(v(x), v'(x)) + c + t \cdot \epsilon, & t > x \\ \max(v(y), v'(y)) + c + t \cdot \epsilon, & x > t > y \\ t \cdot \epsilon, & y > t. \end{cases}$$

$$u''(t) = \begin{cases} \max(u(t), u'(t)) + c, & t = m - x, t = m - y \\ \max(u(m - y), u'(m - y)) + c + t \cdot \epsilon, & t > m - y \\ \max(u(m - x), u'(m - x)) + c + t \cdot \epsilon, & m - y > t > m - x \\ t \cdot \epsilon, & m - x > t. \end{cases}$$

For small enough ϵ, by strong monotonicity and since all items are allocated: $f(v,u) = f(v'',u) = f(v'',u'') = (x, m-x)$. Similarly, $f(v',u') = f(v'',u') = f(v'',u'') = (y, m-y)$. As $x \neq y$, we have a contradiction. $\square$

In a sense, the lemma says that we can use the following processes of "pairwise elections": fix two valuations u and v, and select two allocations $(x, m-x)$ and $(y, m-y)$. Then, by looking only at the vector of differences $(v(x)-v(y), u(m-x, m-y)$, we can rule out the possibility that $f(v,u) = (x, m-x)$ or the possibility that $f(v,u) = (y, m-y)$. If we continue this pairwise elections process for all possible pairs of allocations, we are guaranteed to find $f(v,u)$ (notice that the order we conduct this election process does not matter).

Lemma 2.5 *Suppose that for each $P(x,y)$, $x > y$, and every $\epsilon > 0$ there exists $\beta \in \dot{P}(x,y)$, $|\beta| < \epsilon$. Let $x > y$, $w > z$. If $\alpha \in \dot{P}(x,y)$, then $\alpha \in \dot{P}(w,z)$.*

Proof: It is enough to prove the following two claims:

Claim 2.6 *Let $x > y > z$. $\alpha \in \dot{P}(x,y) \iff \alpha \in \dot{P}(x,z)$.*

Proof: In one direction, let $\alpha \in \dot{P}(x,y)$ and choose some $\beta \to 0$, $\beta \in P(y,z)$. By Lemma **??**, $\alpha + \beta \in P(x,y)$, and thus $\alpha \in \dot{P}(x,z)$.

As for the other direction, take some v, u, such that $f(v,u) = (x, m-x)$, and $\alpha = (v(x) - v(z), u(m-x) - u(m-z))$. For each t in the range of f where $x > t > z$, select some $\beta^t \in P(z,y)$, $\beta^t \to 0$. Define v' and u':

$$v'(t) = \begin{cases} v(z) - \beta_1^t, & x < t < z, \\ v(t), & \text{otherwise.} \end{cases}$$

$$u'(t) = \begin{cases} v(m-z) + \beta_2^t, & x < t < z, \\ v(t), & \text{otherwise.} \end{cases}$$

Notice that v' is indeed monotone for some small enough choices of the β^t's (since v is strictly increasing). Observe that $f(v,u) = f(v',u') = (x, m-x)$: by Lemma 2.4 the output is not $(t, m-t)$, for $x > t > z$. The output is also not $(t, m-t)$ for some $t > x$ or $t \leq z$, by using strong monotonicity and the fact that $f(v,u) \neq (t, m-t)$. Thus, $f(v',u') = (x, m-x)$. Finally observe that by choosing small enough β^t's we get that $(v'(x) - v'(y), u'(x) - u'(y)) \to \alpha$, hence the lemma. $\square$

Claim 2.7 *Let $x > y > z$. $\alpha \in \dot{P}(x,z) \iff \alpha \in \dot{P}(y,z)$.*

Proof: Both proofs are similar to the proof of the previous claim. For the proof that $\alpha \in \dot{P}(x,z) \implies \alpha \in \dot{P}(y,z)$ we choose small enough $\beta^t \in \dot{P}(y,t)$, for $x \geq t > y$. For the proof of $\alpha \in \dot{P}(y,z) \implies \alpha \in \dot{P}(x,z)$, we choose small enough $\beta^t \in \dot{P}(x,y)$, for $x > t \geq y$. In both cases continue as in the proof of the previous claim. $\square$

$\square$

The last lemma essentially proves that if $x > y$ and $w > z$ then $\dot{P}(x,y)$ and $\dot{P}(w,z)$ are equal. Therefore, if $x > y$ drop the index (x,y) from $P(x,y)$.

Lemma 2.8 (Closure) *Let $\alpha \in P(x,y)$. Let $\epsilon = (\epsilon_1, \epsilon_2)$. If $x > y$ then $\epsilon_1 \geq 0, \epsilon_2 \leq 0$, if $x < y$ then $\epsilon_1 \leq 0, \epsilon_2 \geq 0$. Then $\alpha + \epsilon \in P(x,y)$. As a corollary, if $\alpha \in \overline{P}(x,y)$ then $\alpha - \epsilon \in \overline{P}(x,y)$.*

Proof: Let v and u be two valuations where $\alpha = (v(x) - v(y), u(m - x) - u(m - y)$, and $f(v, u) = (x, m-x)$. Define $v'(t) = v(t)+\epsilon_1, u'(t) = u(t)+\epsilon_2$. Observe that by strong monotonicity $f(v, u) = f(v', u) = f(v', u') = (x, m - x)$, as needed. □

Lemma 2.9 *P and $\overline{P}$ are separated by a line.*

Proof: Let α be some point that is on the border of P and $\overline{P}$ (notice that such point must exist, otherwise either $(x, m - x)$ or $(y, m - y)$ are not in the range). Notice that by scalability $t \cdot \alpha$ is also on the border, for all t. Finally, observe that the closure lemma separates the plane into two separate regions: if β is above the $t \cdot \alpha$ line then it is in P, if it is below the line, then it is in $\overline{P}$. □

As we observed before, all the P's are equal, up to invalid points. By this and the previous lemma, all the P's are separated by the same line, up to invalid points. This is enough conclude the proof and claim that f is an affine maximizer.

References

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