

Paths, Cycles and Mechanism Design

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Contents

1	Network Flow Problem	7
1.1	Network Flow Problem	9
1.2	Flow Decomposition	13
1.3	The Shortest Path Polyhedron	15
2	Incentive Compatability	19
2.1	Dominant Strategy Incentive Compatability	19
2.1.1	2-cycle Conditions	24
2.1.2	Roberts' Theorem	27
2.2	Bayesian Incentive Compatability	32
2.3	Revenue Equivalence	41
3	Optimality	47
3.1	An Example	49
3.2	What is a Solution?	50
3.3	One Dimensional Types	51
3.3.1	A Formulation	53
3.3.2	The Myerson Case	55
3.4	Multidimensional Types	56
3.4.1	Wilson Example	56
3.4.2	Capacity Constrained Bidders	64
4	Rationalizability	75
4.1	The Quasi-linear Case	75
4.2	The General Case	76

Introduction

The incentive compatability¹ constraints of mechanism design (with quasi-linear utilities) can be interpreted as being the dual inequalities of a shortest path problem. I believe this interpretation provides an intuitive and powerful lens through which to view the implications of incentive compatability. This document provides an (incomplete) account of the main results in mechanism design from this point of view. They assume familiarity with mechanism design and basic linear programming and convex analysis. I welcome suggestions for additions, deletions and improvements to this material.

This point of view is the product of collaborations with many individuals. Amongst them Sushil Bikhchandani, Sven de Vries, Alexey Malakhov, Rudolf Müller, Mallesh Pai and James Schummer. However they are not responsible for any errors of commision or omission. The impetus for putting this down on paper in one place was an invitation from S. Pekec, L. Rigotti and P. Lopomo to talk about these matters at Fuqua.

¹Luca Rigotti noticed that this word was misspelt and surmised it was part of some secret code. ‘Google’ the word yourself and draw your own conclusion.

Chapter 1

Network Flow Problem

In this chapter a class of linear program's called network optimization problems and their properties are discussed.

A **graph** is a collection of two objects. The first is a finite set $V = \{1, \dots, n\}$ called **nodes**. The second is a set E of (unordered) pairs of nodes called **edges**. As an example, suppose $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (2, 3), (1, 3), (3, 4)\}$. A pictorial representation of this graph is shown in Figure 2.1.

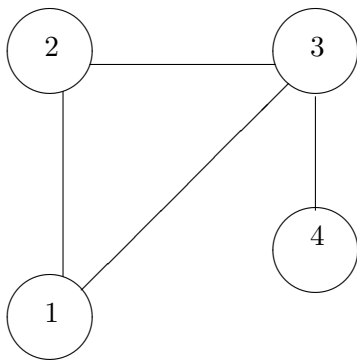


Figure 2.1

A graph is called **complete** if E consists of every pair of nodes in V . The **end points** of an edge $e \in E$ are the two nodes i and j that define that edge. In this case we write $e = (i, j)$. The **degree** of a node is the number of edges that contain it.

In the graph of Figure 2.1, the degree of node 3 is 3 while the degree of node 2 is 2. A pair $i, j \in V$ is called **adjacent** if $(i, j) \in E$.

Fix a graph $G = (V, E)$ and a sequence v^1, v^2, \dots, v^r of nodes in G . A path is a sequence of edges e_1, e_2, \dots, e_{r-1} in E such that $e_i = (v^i, v^{i+1})$. The node v^1 is the initial node on the path and v^r is the terminal node. An example of a path is the sequence $(1, 2), (2, 3), (3, 4)$ in Figure 2.1. A **cycle** is a path whose initial and terminal nodes are the same. The edges $\{(1, 2), (2, 3), (3, 1)\}$ form a cycle in Figure 2.1. A graph G is called **connected** if there is a path in G between every pair of nodes. The graph of Figure 2.1 is connected. The graph of Figure 2.2 below is disconnected. A subset T of edges is called a **spanning tree** if the graph (V, T) is connected and acyclic. In the graph of Figure 2.1, the set $\{(1, 2), (1, 3), (3, 4)\}$ is a spanning tree.

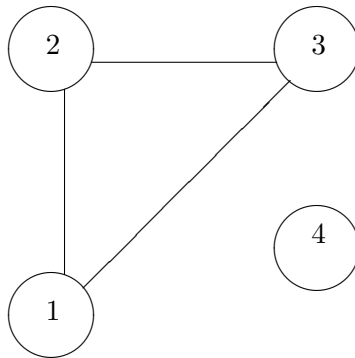


Figure 2.2

If the arcs of a graph are oriented, i.e., and edge (i, j) can be traversed from i to j but not the other way around, the graph is called **directed**. If a graph is directed, the edges are called **arcs**. Formally, a directed graph consists of a set V of nodes and set A of *ordered* pairs of nodes. As an example, suppose $V = \{1, 2, 3, 4\}$ and $A = \{(1, 2), (2, 3), (3, 1), (2, 4)\}$. A pictorial representation of this graph is shown in Figure 2.3 below.

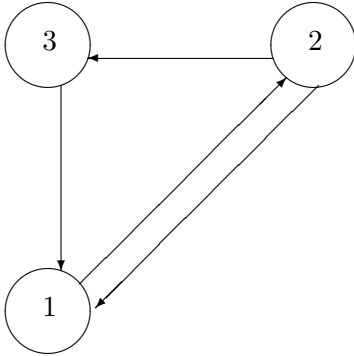


Figure 2.3

A path in a directed graph has the same definition as in the undirected case except now the orientation of each arc must be respected. To emphasize this it is common to call a path directed. In our example above, $1 \rightarrow 3 \rightarrow 2$ would not be a directed path, but $1 \rightarrow 2 \rightarrow 3$ would be. A cycle in a directed graph is defined in the same way as in the undirected case, but again the orientation of the arcs must be respected.

A directed graph is called **strongly connected** if there is a directed path between every ordered pair of nodes. It is easy to see that this is equivalent to requiring that there be a directed cycle through every pair of nodes. If each arc in a directed graph has a number (e.g. a cost, distance, or capacity) associated with it, the directed graph is termed a **network**.

1.1 Network Flow Problem

Single commodity network flow models seek to prescribe the movement of a homogeneous good, through a directed network from a set of source nodes to a set of destination nodes. Consider, for example, the network defined by $V = \{s, x, y, t\}$ and $A = \{(s, x), (s, y), (x, y), (x, t), (y, t)\}$. Figure 2.4 depicts this network. The number associated with each arc in Figure 2.4 is an arc length.

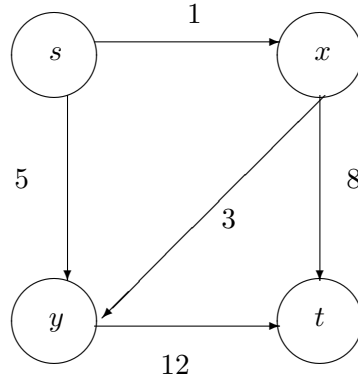


Figure 2.4

A network flow problem that will play an important role in what is to come is the **shortest path problem**. This problem is to find the minimum length (sum of arc lengths) path from node s to node t in the network. It is easy to see that in this case the shortest path from s to t proceeds from nodes s to x to t and has a total length of 9 units. The problem can be rephrased as finding the cheapest way to route one indivisible unit of flow from s to t .

To model the good's movement on a given directed network with node set V and arc set A , let x_{ij} , for each arc $(i, j) \in A$, denote the flow in an appropriately defined unit of measurement of the good on that arc. We assume conservation of flow requiring that the total flow out of any node i minus the total flow $\sum_j x_{ij}$ into that node must equal the net demand b_i at the node, i.e.

$$-\sum_{j:(i,j) \in A} x_{ij} + \sum_{k:(k,i) \in A} x_{ki} = b_i \quad \forall i \in V.$$

In addition to the conservation equations, network flow constraints also include lower bounds and capacities imposed upon arc flows:

$$l_{ij} \leq x_{ij} \leq u_{ij} \quad \forall (i, j) \in A.$$

These flow bounds might model physical capacities, contractual obligations, or simply operating ranges of interest. Frequently, the given lower bound capacities are all zero; $l_{ij} = -\infty$ and $u_{ij} = +\infty$ are possibilities as well.

For easy reference, we distinguish three types of nodes: the set $S \subseteq V$ of source, origin, or supply nodes i with $b_i < 0$, the set $T \subseteq V$ of terminal,

destination, or sink nodes i with $b_i > 0$, and the set $I \subset V$ of intermediate or transshipment nodes i with $b_i = 0$. Note that, by convention, at any terminal node i , $-b_i > 0$ is the net demand for flow.

Figure 2.5 illustrates a small network flow distribution system with 2 sources, 2 destinations, and 1 intermediate node. All lower bounds on flow are zero and, except for arc (3,4), are uncapacitated with upper bounds on flow equal to $+\infty$.

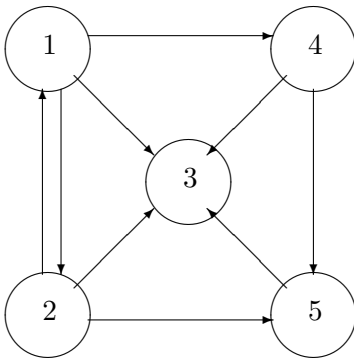


Figure 2.5

$V = \{1, 2, 3, 4, 5\}$, $A = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 5), (3, 4), (3, 5), (4, 5)\}$
 $S = \{1, 2\}$, $I = \{3\}$ and $T = \{4, 5\}$.

The flow constraints for this model are:

$$-x_{12} - x_{13} - x_{14} + x_{21} = -2$$

$$x_{12} - x_{21} - x_{23} - x_{25} = -3$$

$$x_{13} + x_{23} - x_{34} - x_{35} = 0$$

$$x_{14} + x_{34} - x_{45} = 4$$

$$x_{25} + x_{35} + x_{45} = 1$$

$$x_{ij} \geq 0 \quad \forall (i, j) \in A$$

$$x_{34} \leq 3$$

In order to represent network flow constraints more compactly we use a **node-arc** incidence matrix $\mathcal{N} = \{n_{ia}\}$ with one row for each node i of the network and one column for each arc $a \in A$. The entry $n_{ia} = -1$ if arc a

is directed out of node i , equal to $+1$ if arc a is directed into node i and zero otherwise. In other words \mathcal{N} is the matrix of the coefficients of the flow conservation constraints. As an example the node-arc incidence matrix for the network of Figure 2.5 would be:

(1, 2)	(1, 3)	(1, 4)	(2, 1)	(2, 3)	(2, 5)	(3, 4)	(3, 5)	(4, 5)	arc's
-1	-1	-1	-1	0	0	0	0	0	node 1
1	0	0	-1	-1	-1	0	0	0	node 2
0	1	0	0	1	0	-1	-1	0	node 3
0	0	1	0	0	0	1	0	-1	node 4
0	0	0	0	0	1	0	1	1	node 5

Letting l and u denote vectors of lower bounds and capacities on arc flows and letting b denote the vector of supplies/demands at the nodes, we see that in matrix form the constraints imposed upon the vector x or arc flows becomes

$$\mathcal{N}x = b, \quad l \leq x \leq u.$$

Call a vector x a **feasible flow** or is a feasible solution if it satisfies these flow balance and lower and upper bound conditions.

Each variable x_{ij} appears in two flow conservation equations, as an output from node i with a -1 coefficient and as an input to node j with a $+1$ coefficient, each column of \mathcal{N} contains exactly two nonzero coefficients, a $+1$ and a -1 . For now, let us make three observations.

Observation 1: Summing the conservation equations eliminates all the flow variables and gives

$$0 = \sum_{j \in V} b_j \Rightarrow \sum_{j \in S} -b_j = \sum_{j \in T} b_j.$$

Consequently, total supply must equal total demand if the conservation equations $\mathcal{N}x = b$ are to have a feasible solution.

Observation 2: If total supply equals total demand, then summing all conservation equations gives the zero equation $0 \cdot x = 0$. Equivalently, any equation is redundant being equal to minus the sum of all other equations.

Observation 3: Let μ be a cycle in G and x^μ its incidence vector. If x is a feasible flow, $\epsilon \in \mathbb{R}$ and $l \leq x + \epsilon x^\mu \leq u$, then $x + \epsilon x^\mu$ is a feasible flow as well. To see why, consider any node i on the cycle μ and suppose

$(p, i), (i, q) \in \mu$. We have to verify flow conservation at node j .

$$- \sum_{j:(i,j) \in A} [x_{ij} + \epsilon x_{ij}^\mu] + \sum_{k:(k,i) \in A} [x_{ki} + \epsilon x_{ki}^\mu] = - \sum_{j:(i,j) \in A} x_{ij} - \sum_{k:(k,i) \in A} x_{ki} - \epsilon [x_{iq}^\mu - x_{pi}^\mu] = b_i,$$

where the last equation follows from the fact that $x_{iq}^\mu = x_{pi}^\mu = 1$.

If to each feasible flow x there is an associated cost, profit, or some other measure of performance, we may wish to find a feasible flow that is best with respect to that measure. In particular, if the function g assigns a real number $g(x)$ to each feasible network flow x , we may wish to solve the following optimization problem:

$$\begin{aligned} & \min g(x) \\ & \text{s.t. } \mathcal{N}x = b \\ & \quad l \leq x \leq u \end{aligned}$$

Of particular importance is the case when g is a linear function of x . Note also that one can, without loss, assume that $l = 0$. This is because we can replace the variable x by $y + l$ where $0 \leq y \leq u - l$.

In the shortest path problem, $g(x) = \sum_{(i,j) \in A} c_{ij} x_{ij}$ where c_{ij} is the length of arc (i, j) . We set $l_{ij} = 0$ and $u_{ij} = +\infty$ for all $(i, j) \in A$. Finally, $b_s = 1$, $b_t = -1$ and $b_i = 0$ for all $i \notin \{s, t\}$. In addition we would require the flow variables x_{ij} to be integral. This integrality requirement introduces no added difficulties and can be relaxed, since, as we establish later in this chapter, the fact that all supplies and demands and all (finite) capacities are integer implies that there is an integral optimal solution.

1.2 Flow Decomposition

We have modeled the network flow problem in terms of arc flows x_{ij} . An alternate representation for the problem that is useful is to view the supply and demand requirements as being satisfied by flow along paths joining the source and sink nodes.

Let \mathcal{P} denote the set of directed paths joining the source and sink nodes and let \mathcal{C} denote the directed cycles in the network. Let h_ρ be the magnitude of the flow along path $\rho \in \mathcal{P}$ and g_μ be the magnitude of the flow along the cycle $\mu \in \mathcal{C}$. To convert these path and cycle flows into arc flows we add, for each arc, the flow on all paths and cycles through that arc. Formally,

$$x_{ij} = \sum_{\rho \in \mathcal{P}} \delta_{ij}(\rho) h_\rho + \sum_{\mu \in \mathcal{C}} \delta_{ij}(\mu) g_\mu.$$

Here $\delta_{ij}(\rho) = 1$ if $(i, j) \in \rho$ and zero otherwise. Similarly $\delta_{ij}(\mu) = 1$ if $(i, j) \in \mu$ and zero otherwise. When arc flows are expressed in terms of path and cycle flows in this way, we say that the arc flows are decomposed as the sum of path and cycle flows.

Of course, the arc flows x_{ij} corresponding to any given path and cycle flows need not satisfy the conservation equations $Nx = b$. They will, though, if the sum of the flows on paths originating from any source node equals its supply and the sum of the flows on paths terminating with any sink equals its demand. In this case, we say that the path and cycle flows satisfy supply and demand requirements.

The following result shows that network flow problem can be modeled in terms of arc flows or path and cycle flows. That is, not only can we define arc flows in terms of given path and cycle flows, but given an arc flow representation, we can find corresponding path and cycle flows.

Theorem 1.2.1 Flow Decomposition. *An assignment of path and cycle flows satisfying supply and demand requirements corresponds to a unique assignment of arc flows satisfying the conservation equations $Nx = b$. Any arc flows fulfilling the conservation equations can be decomposed as the sum of path and cycle flows for some assignment of path flows h_ρ and cycle flows g_μ that satisfy supply and demand requirements.*

Proof: The observations preceding the theorem establish the first statement. To establish the second, we describe a procedure for converting arc flows to path and cycle flows. The procedure successively reduces arc flows by converting them into path and cycle flows. We start in step (1) with all path and cycle flows set equal to zero and with arc flows at the values given to us.

1. If all arc flows as modified thus far by the procedure are equal to zero, then stop; the current path and cycle flows give the desired representation of arc flows. Otherwise, select some arc (i, j) with $x_{ij} > 0$, choosing, if possible, i as a source node with $b_i < 0$. Let ρ be a path with the single arc (i, j)
2. Add to path ρ one node and arc at a time, choosing at each step any arc (k, l) directed from the last node k on path ρ and with $x_{kl} > 0$. If, at any step, node k is a terminal node with $b_k > 0$, we go to step (3); if node k appears on path ρ twice, we go to step (4). Otherwise, by conservation of flow we can find an arc (k, l) with $x_{kl} > 0$ to extend the path.

3. ρ is a directed path from source node i to terminal node k and $x_{uv} > 0$ for every arc (u, v) on the path. Let h be the minimum of these arc flows and of $-b_i$ and b_k . Set $h_p = h$, redefine x_{uv} as $x_{uv} - h$ for every arc on ρ and redefine b_i as $b_i + h$ and b_k as $b_k - h$. Return to step (1) with this modified data.
4. The nodes and arcs from the first and second appearance of node k on ρ define a cycle μ along which all arc flows x_{uv} are positive. Let g be the minimum of these arc flows. Set $g_\mu = g$ redefine x_{uv} as $x_{uv} - g$ for every arc on μ , and return to step (1) with the modified data.

If the procedure terminates in step (1), the path and cycle flows satisfy supply and demand requirements and the arc flows are decomposed as the sum of these path and cycle flows. Each time step (3) or (4) is invoked either some source or terminal node becomes a transshipment node in the modified network, or some arc flow becomes zero. Consequently, since no transshipment node ever becomes a source or terminal node and since we never add flow to any arc, in a finite number of steps, we must terminate in step (1). ■

The theorem makes no assertion about the uniqueness of the path and cycle decomposition of given arc flows. As an illustration, there are two decompositions for the arc flows in Figure 2.5, namely path 1-2-4 and cycle 1-4-3-1 each with a flow of one unit, or path 1-4 and cycle 1-2-4-3-1 each with a flow of one unit.

1.3 The Shortest Path Polyhedron

Let \mathcal{N} be the node-arc incidence matrix of a network $G = (V, A)$ with source node s and sink node t . Assume there is at least one $s - t$ path. Let c_{ij} be the length of arc (i, j) . Let $b^{s,t}$ be the vector such that $b_i^{s,t} = 0$ for all $i \in V \setminus \{s, t\}$, $b_s^{s,t} = -1$ and $b_t^{s,t} = 1$. The shortest path polyhedron is $\{x : \mathcal{N}x = b^{s,t}, x \geq 0\}$.

Theorem 1.3.1 *Every extreme point of $\{x : \mathcal{N}x = b^{s,t}, x \geq 0\}$ is integral.*

Proof: Let x^* be an extreme point of the polyhedron. Hence there is a c such that $x^* \in \arg \min\{cx : \mathcal{N}x = b^{s,t}, x \geq 0\}$.

Since x^* is a feasible flow, by flow decomposition, we can express x^* as the sum of positive flows around a set of cycles, \mathcal{C} , and flows along a set of paths from s to t , \mathcal{P} , say. For each $\mu \in \mathcal{C}$ let g_μ be the quantity of flow

around the cycle c and for each $\rho \in \mathcal{P}$ let h_ρ be the quantity of flow on this path. Denote by $c(\mu)$ the sum of arc lengths around the cycle and by $c(\rho)$ the sum of arc lengths along the path ρ . Hence

$$cx^* = \sum_{\mu \in \mathcal{C}} c(\mu)g_\mu + \sum_{\rho \in \mathcal{P}} c(\rho)h_\rho.$$

We show that $c(\mu) = 0$ for all $\mu \in \mathcal{C}$. If $c(\mu) > 0$ for some $\mu \in \mathcal{C}$, reduce the flow around the cycle μ by $\epsilon > 0$. This results in a new feasible flow of lower cost contradicting the optimality of x^* . Feasibility follows from observation 3. If $c(\mu) < 0$, increase the flow around μ by ϵ so generating a new feasible flow with lower cost. Again, a contradiction.

Now suppose there exist two paths $\rho, \rho' \in \mathcal{P}$ with $c(\rho) < c(\rho')$. Decrease the flow along ρ' by ϵ and increase the flow on ρ by the same amount. This produces a new feasible flow with lower cost, a contradiction. Hence $c(\rho) = c(\rho')$. Let x^1 be formed by transferring ϵ units of flow from ρ' to ρ . Let x^2 be the formed by transferring ϵ units of flow from ρ to ρ' . Notice that x^1 and x^2 both lie in the shortest path polyhedron. It is easy to see that $x^* = \frac{1}{2}x^1 + \frac{1}{2}x^2$ contradicting the fact that x^* is an extreme point.

Therefore x^* can be decomposed into a flow of one unit along a single $s - t$ path, i.e., x^* is the incidence vector of the shortest $s - t$ path. Hence x^* is integral. ■

It is useful to compare this result with the resolution theorem. That theorem says that any point in a polyhedron can be expressed (decomposed) into a convex combination of the polyhedron's extreme points and a non-negative linear combination of its extreme rays. The import of Theorem 2.2.2 is that the extreme points of the shortest path polyhedron correspond to $s - t$ paths and extreme rays correspond to cycles in the network.

COROLLARY 1.3.2 *Let $G = (V, A)$ be a network with source $s \in V$, sink $t \in V$ and arc length vector c . A shortest $s - t$ path (wrt c) exists in G iff. G contains no negative length cycles.*

Proof: If a shortest path in G exists, its incidence vector x^* is an optimal extreme point of the corresponding shortest path polyhedron. Suppose G has a negative length cycle, μ . Let x^μ be the incidence vector of this cycle. Then $x^* + \epsilon x^\mu$ is a point in the shortest path polyhedron by observation 3. However $cx^* + \epsilon cx^\mu < cx^*$, contradicting optimality of x^* . ■

Since the shortest path problem on G with arc length vector c can be written as $\min\{cx : Nx = b^{s,t}, x \geq 0\}$, it has a dual:

$$\begin{aligned} \max y_t - y_s \\ \text{s.t. } y\mathcal{N} \leq c \end{aligned}$$

The dual variables are called node potentials. The dual has one constraint for each arc. From the structure of \mathcal{N} , we see that the typical dual constraint looks like:

$$y_j - y_i \leq c_{ij}$$

where $(i, j) \in A$. By observation 2, we know that any one of the primal constraints is redundant. Hence we can always set one of the dual variables to zero. If we set $y_s = 0$ the dual becomes $\max\{y_t : y\mathcal{N} \leq c, y_s = 0\}$. Let y^* be an optimal solution to the dual. Hence, y_t^* is, by the duality theorem, the length of the shortest path from s to t . For any other node i , y_i^* is the length of the shortest path from s to the node i . Indeed, given any feasible dual solution, y_i is bounded above by the length of the shortest $s - i$ path.

The dual of the shortest path problem is relevant in analyzing the feasibility of a certain class of inequalities. To describe it let V be a finite set of indices and A a set of *ordered* pairs of elements of V . Associated with each pair $(i, j) \in A$ is a number w_{ij} . The system is:

$$x_j - x_i \leq w_{ij} \quad \forall (i, j) \in A. \tag{1.1}$$

We can associate a network with (1.1) in the following way. Each element of V is a node, and to each ordered pair $(i, j) \in A$ we associate a directed arc from node i to node j . Each arc $(i, j) \in A$ is assigned a length of w_{ij} .

This system (1.1) is feasible iff. the associated network contains no negative length cycle.¹ Second, if the system is feasible, one solution is to set each x_j equal to the length of the shortest path from an arbitrarily chosen source node to node j .

¹If the direction of the inequality in (1.1) is reversed, then the system is feasible iff. the network has no positive length cycle.

Chapter 2

Incentive Compatability

Any direct mechanism can be decomposed into two parts: an allocation rule and a payment rule. The allocation rule determines the allocation of resources as a function of the profile of reported types. The payment rule determines the payment each agent must make as a function of the profile of reported types. In this chapter we show how to characterize dominant strategy as well as Bayesian incentive compatible mechanisms in environments with quasi-linear utilities and multi-dimensional types. Attention is confined to direct revelation mechanisms where types are private. With a change in notation the results would extend to the interdependent value setting and ex-post incentive compatibility.

2.1 Dominant Strategy Incentive Compatability

Let T be a set of types. Denote by T^n the set of all n -agent profiles of types¹. An element of T^n will usually be written as (t^1, t^2, \dots, t^n) or \mathbf{t} . Let Γ be the set of outcomes. An **allocation rule** is a function

$$g : T^n \mapsto \Gamma.$$

Let $R_\alpha = \{\mathbf{t} \in T^n : g(\mathbf{t}) = \alpha\} \forall \alpha \in \Gamma$.

A **payment rule** is a function

$$P : T^n \mapsto \mathbb{R}^n,$$

that is, if the reported profile is (t^1, \dots, t^n) agent i makes a payment of $P_i(t^1, \dots, t^n)$.

¹The type space need not be identical across agents. The assumption is for simplicity of notation only.

Utilities are quasi-linear. The value that agent i with type $t \in T$ assigns to an allocation $\alpha \in \Gamma$ is denoted $v^i(\alpha|t)$.

Here are three examples.

1. **Homogenous, multi-item auctions with additive valuations.** Suppose we have k units of a homogenous good to allocate. The type of an agent is a vector in \mathbb{R}_+^k whose j^{th} component is the marginal value for the j^{th} unit. Each $\alpha \in \Gamma$ can be represented by an integral vector in \mathbb{R}_+^k whose i^{th} component represents the quantity allocated to agent i and sum of components is k . The i^{th} component will be denoted α_i , and $v^i(\alpha|t) = \sum_{j=1}^k \alpha_j t_j$.
2. **Combinatorial Auctions.** We have a set M of distinct goods to allocate. The type of an agent is a vector in $\mathbb{R}_+^{2^{|M|}}$ with one component for each subset of M that corresponds to the value assigned to that subset. If the allocation α assigns the set $S \subseteq M$ to agent i with type t then $v^i(\alpha|t) = t_S$. The vector representation is given by the unit vectors in $\mathbb{R}^{2^{|M|}}$.
3. **Unrestricted Preferences.** Following the previous example, but now we want to allow for the possibility that an agents payoff depends not just on the goods assigned to him, but the goods assigned to other agents as well. In this case a type would be a vector with one component for each allocation, i.e. in $\mathbb{R}_+^{|\Gamma|}$.

An allocation rule g is **dominant strategy incentive compatible** (DSIC) if there exists a payment rule P such that for all agents i and all types $s \neq t$:

$$v^i(g(t, \mathbf{t}^{-i})|t) - P_i(t, \mathbf{t}^{-i}) \geq v^i(g(s, \mathbf{t}^{-i})|t) - P_i(s, \mathbf{t}^{-i}) \quad \forall \mathbf{t}^{-i}. \quad (2.1)$$

It would be more correct to refer to g as being implementable in dominant strategies, but this is a mouthful.

The constraints (2.1) have a network interpretation. To see this it is will be useful to rewrite (2.1) as

$$P_i(t, \mathbf{t}^{-i}) - P_i(s, \mathbf{t}^{-i}) \leq v^i(g(t, \mathbf{t}^{-i})|t) - v^i(g(s, \mathbf{t}^{-i})|t) \quad \forall \mathbf{t}^{-i}. \quad (2.2)$$

Which coincide with the constraints of the dual of the shortest path problem. Given this observation, a natural step is to associate with each i and \mathbf{t}^{-i} a network with one node for each type $t \in T$ and a directed arc

between each ordered pair of nodes. Define the length of the arc directed from type s to type t^i by

$$\ell(s, t|\mathbf{t}^{-i}) = v^i(g(t, \mathbf{t}^{-i})|t) - v^i(g(s, \mathbf{t}^{-i})|t).$$

Call this network $T_g(\mathbf{t}^{-i})$. Following Corollary 1.3.2, if this network has no negative length cycles, then shortest path lengths exist. Choose any one of the nodes as a source node and set $P_i(t, \mathbf{t}^{-i})$ equal to the length of the shortest path from the source to node t . A direct application of Corollary 1.3.2 is not possible because it assumes a finite number of nodes. In the present case the network has as many nodes as there are elements in T , which could be infinite.

In the sequel we fix an agent i and a profile of types for the other $n - 1$ agents. For this reason we suppress dependence on the index i and \mathbf{t}^{-i} in unless we say otherwise.

If we restrict our attention to agent i , we will always identify allocations between which agent i is indifferent. By this we can assume that for all $\alpha \neq \beta$ of Γ there exists a type t such that $v(\alpha|t) \neq v(\beta|t)$. Furthermore, we will for fixed \mathbf{t}^{-i} restrict to $\Gamma = \{\alpha|\exists t \in T \text{ such that } g(t, \mathbf{t}^{-i}) = \alpha\}$. Accordingly, from now on $R_\alpha = \{t|g(t, \mathbf{t}^{-i}) = \alpha\}$. Under this convention inequality (2.1) becomes

$$v(g(t)|t) - P(t) \geq v(g(s)|t) - P(s). \quad (2.3)$$

The constraints (2.2) become

$$P(t) - P(s) \leq v(g(t)|t) - v(g(s)|t). \quad (2.4)$$

Define the length of the arc directed from type s to type t by

$$\ell(s, t) = v(g(t)|t) - v(g(s)|t),$$

and call this network T_g .²

Theorem 2.1.1 (Rochet(1987)) *Let T be any type space, $n \geq 2$ be the number of agents with quasi-linear utilities over a set Γ of outcomes and $f : T^n \mapsto \Gamma$ an allocation rule. The following statements are equivalent:*

- (1) g is (DSIC).
- (2) For every agent i , for every report \mathbf{t}^{-i} , the corresponding graph T_g does not have a finite cycle of negative length.

²For technical reasons we allow for loops. But observe that $\ell(t, t) = 0$.

Proof: (2) \Rightarrow (1)

Let $t_0 \in T$ be an arbitrary, but fixed type. Define

$$P(t) = \inf \left\{ \sum_{i=0}^k \ell(t_i, t_{i+1}) \mid k \geq 0, t_1, \dots, t_{k+1} \in T, t_{k+1} = t \right\}.$$

Set $P(t_0) = 0$ and observe that $P(t)$ is larger than or equal to $-\ell(t, t_0)$, and thus larger than $-\infty$. If not T_g would have a negative cycle. Finally for every $t, t' \in T$:

$$P(t') \leq P(t) + \ell(t, t') = P(t) + v(g(t')|t') - v(g(t)|t).$$

(1) \Rightarrow (2)

Let $t_1, \dots, t_k, t_{k+1} = t_1$ be a finite cycle. Since g is dominant strategy incentive compatible there exists a payment function P such that:

$$\sum_{i=1}^k \ell(t_i, t_{i+1}) = \sum_{i=1}^k v(g(t_{i+1})|t_{i+1}) - v(g(t_i)|t_{i+1}) \geq \sum_{i=1}^k P(t_{i+1}) - P(t_i) = 0.$$

■

A straight forward consequence of Theorem 2.1.1 is what is called the **taxation principle**. If g is DSIC, then we know there exist appropriate payments. Notice also, if $g(t) = g(s)$ then $P(t) = P(s)$. For this reason we can express payments as a function of allocation rather than types, i.e., $P(t) = P^\alpha$ for all $t \in R_\alpha$. Revert to the notation involving the full profile of types we can state the taxation principle as the following.

Lemma 2.1.2 *If g is DSIC then there exists a payment function P such that:*

$$g(\mathbf{t}) \in \arg \max_{\alpha \in \Gamma} [t_\alpha^i - P_i^\alpha(t^{-i})] \quad \forall i.$$

The taxation principle allows us to interpret a mechanism as offering a menu (one for each \mathbf{t}^{-i}) consisting of a list of allocations and corresponding prices to be paid. Specifically, if agent i chooses allocation α she must pay $P_i^\alpha(t^{-i})$.

Theorem 2.1.1 is useful in that it places conditions on the allocation rule rather than on the utilities that the agents derive. It is problematic in that one must verify non-negativity of a rather large number of cycles. Here is one application of the result.

Theorem 2.1.3 *Suppose $T, \Gamma \subseteq \mathbb{R}^1$, $g(t) \in \mathbb{R}^1$, $v(x|t)$ is non-decreasing in x for each fixed t and satisfies strict increasing differences (supermodularity). Then g is DSIC iff g is non-decreasing.*

Proof: Suppose g is DSIC and for a contradiction is decreasing somewhere. Then there exists $r, s \in T$ such that $r \leq s$ and $g(r) > g(s)$. DSIC implies that

$$P(r) - P(s) \leq v(g(r)|r) - v(g(s)|r).$$

Increasing differences implies that the right hand side of the above is strictly less than $v(g(r)|s) - v(g(s)|s)$. Hence

$$P(r) - P(s) < v(g(r)|s) - v(g(s)|s),$$

which violated DSIC.

Now suppose that g is non-decreasing. We show that g is DSIC. Increasing differences implies that for all $t > s$ we have

$$\ell(s, t) = [v(g(t)|t) - v(g(s)|t)] \geq [v(g(t)|s) - v(g(s)|s)] = -\ell(t, s).$$

Hence $\ell(s, t) + \ell(t, s) \geq 0$, i.e., all cycles on pairs of nodes have non-negative length. To show that all cycles have non-negative length we use induction. Suppose all cycles with k or fewer nodes have non-negative length. For a contradiction suppose there is a cycle, C , of $k + 1$ nodes with negative length. Let that cycle consist of the nodes t_1, t_2, \dots, t_{k+1} . Without loss we may suppose that $t_{k+1} > t_j$ for all $j < k + 1$.

If we can show that $\ell(t_k, t_{k+1}) + \ell(t_{k+1}, t_1) \geq \ell(t_k, t_1)$, then the length of C is bounded below by the length of the cycle $t_1 \rightarrow t_2 \rightarrow \dots \rightarrow t_k \rightarrow t_1$. By the induction hypothesis this would contradict the fact that C is a negative cycle.

Suppose $\ell(t_k, t_{k+1}) + \ell(t_{k+1}, t_1) < \ell(t_k, t_1)$. Then

$$v(g(t_{k+1})|t_{k+1}) - v(g(t_k)|t_{k+1}) + v(g(t_1)|t_1) - v(g(t_{k+1})|t_1) < v(g(t_1)|t_1) - v(g(t_k)|t_1).$$

The right hand side of this inequality can be written as

$$v(g(t_1)|t_1) - v(g(t_{k+1})|t_1) + v(g(t_{k+1})|t_1) - v(g(t_k)|t_1).$$

Substituting it in we get

$$v(g(t_{k+1})|t_{k+1}) - v(g(t_k)|t_{k+1}) < v(g(t_{k+1})|t_1) - v(g(t_k)|t_1),$$

which cannot be by increasing differences. ■

The proof is instructive in that it suggests that sometimes it is sufficient to verify non-negativity of cycles on pairs of nodes. This is what we investigate next.

2.1.1 2-cycle Conditions

Reversing the roles of t and s in (2.3) implies

$$v(g(s)|s) - P(s) \geq v(g(t)|s) - P(t). \quad (2.5)$$

Adding (2.3) to (2.5) yields

$$v(g(t)|t) + v(g(s)|s) \geq v(g(s)|t) + v(g(t)|s).$$

Rewriting:

$$v(g(t)|t) - v(g(s)|t) \geq -[v(g(s)|s) - v(g(t)|s)]. \quad (2.6)$$

We call (2.6) a **2-cycle inequality**. An allocation rule that satisfies the 2-cycle inequality for every pair $s, t \in T$ is said to satisfy the 2-cycle conditions. Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan and Sen (2006) use the term **weak monotonicity** instead. That the 2-cycle condition holds is a necessary condition of dominant strategy incentive compatibility. We show that under certain conditions it is a sufficient condition.

First, we suppose that $|\Gamma|$ is finite. In this case we can assume that $T \subseteq \mathbb{R}^{|\Gamma|}$ and we interpret the i^{th} component of any $t \in T$ as the value that type t places on outcome $i \in \Gamma$. Therefore, if $g(t) = \alpha$ we interpret t_α to be $v(g(t)|t)$.

When Γ is finite it is more convenient to work with a different but related network, called the **allocation network**. We associate with each element $\alpha \in \Gamma$ a node. The length, $\ell(\alpha, \beta)$ of an arc directed from allocation α to allocation β is given by

$$\ell(\alpha, \beta) = \inf_{s \in R_\beta} [v(\beta|s) - v(\alpha|s)] = \inf_{s \in R_\beta} s_\beta - s_\alpha.$$

Symmetrically, we associate an arc directed from β to α with length:

$$\ell(\beta, \alpha) = \inf_{t \in R_\alpha} t_\alpha - t_\beta.$$

Denote the graph by Γ_g . Notice that if the 2-cycle condition holds,

$$\ell(\alpha, \beta) + \ell(\beta, \alpha) \geq 0 \quad \forall \alpha, \beta \in \Gamma. \quad (2.7)$$

From Rochet's theorem or Corollary 1.3.2 we obtain the following.

COROLLARY 2.1.4 *Let T be any type space, $n \geq 2$ be the number of agents with quasi-linear utilities over a set Γ of outcomes and $g : T^n \mapsto \Gamma$ an allocation rule. The following statements are equivalent:*

- (1) *g is DSIC.*
- (2) *For every agent, for every report \mathbf{t}^{-i} , the corresponding graph Γ_g does not have a finite cycle of negative length.*

If (2.7) holds the set R_α is contained in a polyhedron Q_α for all $\alpha \in \Gamma$, and these polyhedra can be chosen such that the intersection of T with the interior, $I(Q_\alpha)$, of Q_α is contained in R_α . Observe that for $t \in R_\alpha$ we have

$$t_\alpha - t_\beta \geq \inf_{x \in R_\alpha} [x_\alpha - x_\beta] = \ell(\beta, \alpha). \quad (2.8)$$

Thus R_α is a subset of

$$Q_\alpha = \{x \in \mathbb{R}^k : x_\alpha - x_\beta \geq \ell(\beta, \alpha) \forall \beta \neq \alpha\}.$$

Now assume $I(Q_\alpha) \neq \emptyset$ and consider a t of $T \cap I(Q_\alpha)$. We show that $t \in R_\alpha$. Observe that for all $\beta \neq \alpha$, $t \notin R_\beta$. Indeed, otherwise we get the contradiction³

$$t_\beta - t_\alpha < \ell(\alpha, \beta) < t_\beta - t_\alpha.$$

Notice that there is a one-to-one correspondence between the constraints of these polyhedra and arcs of Γ_g . Specifically, the constraint $x_\beta - x_\alpha \geq \ell(\alpha, \beta)$ corresponds to the arc (α, β) .

Theorem 2.1.5 (Saks and Yu (2005)) *Suppose $|\Gamma|$ is finite, $T \subseteq \mathbb{R}^{|\Gamma|}$ is convex and g is onto. Then, g is DSIC iff*

$$v(g(t)|t) - v(g(s)|t) \geq -[v(g(s)|s) - v(g(t)|s)] \forall t, s \in T.$$

Proof: We confine ourselves to the case when $|\Gamma| = 3$.⁴ Suppose $\Gamma = \{\alpha, \beta, \gamma\}$. Assume then that Γ_f has the negative cycle $\alpha \rightarrow \gamma \rightarrow \beta \rightarrow \alpha$. Let F be the feasible region associated with the system below.

$$x_\alpha - x_\beta > \ell(\beta, \alpha) \quad (2.9)$$

³We make use of $v(\alpha|\cdot) \neq v(\beta|\cdot)$ for $\alpha \neq \beta$. This ensures that inequality (2.8) has non-zero left-hand-side, and thus every point in the interior of Q_α satisfies (2.8) with strict inequality.

⁴This theorem subsumes earlier results due to Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan and Sen (2006) and Gui, Müller and Vohra (2004).

$$x_\gamma - x_\alpha > \ell(\alpha, \gamma) \quad (2.10)$$

$$x_\beta - x_\gamma > \ell(\gamma, \beta) \quad (2.11)$$

Since the network associated with (2.9, 2.10, 2.11) has no positive length cycle, $F \neq \emptyset$. Notice that (2.9) implies that $x \notin R_\alpha$, (2.10) implies that $x \notin R_\gamma$ and (2.11) implies that $x \notin R_\beta$. Therefore $F \cap T = \emptyset$, recall that $T = R_\alpha \cup R_\beta \cup R_\gamma$.

By the separating hyperplane theorem we can separate F from T by a hyperplane,

$$\sum_{\theta \in \Gamma} d_\theta x_\theta > C \quad (2.12)$$

say. All $x \in F$ satisfy (2.12) and all $x \in T$ violate (2.12).

Now (2.12) can be obtained or dominated by an inequality generated by taking a non-negative linear combination of the inequalities in (2.9, 2.10, 2.11). Specifically

$$w_1(x_\alpha - x_\beta) + w_2(x_\gamma - x_\alpha) + w_3(x_\beta - x_\gamma) > w_1\ell(\beta, \alpha) + w_2\ell(\alpha, \gamma) + w_3\ell(\gamma, \beta).$$

Any such inequality formed by taking all $w_i > 0$ is dominated by one where at most two of the w_i are positive. This is because the the left hand sides of (2.9-2.11) add up to zero while the right hand side adds up to something negative. Hence, wlog we can take $w_1, w_2 > 0$ and $w_3 = 0$. Thus, our separating hyperplane becomes:

$$w_1(x_\alpha - x_\beta) + w_2(x_\gamma - x_\alpha) > w_1\ell(\beta, \alpha) + w_2\ell(\alpha, \gamma). \quad (2.13)$$

By our assumption, every $x \in T$ must violate (2.13). Pick an ϵ sufficiently small and choose $x \in R_\gamma$ such that $x_\gamma - x_\alpha = \ell(\alpha, \gamma) + \epsilon$. We claim that $x_\alpha - x_\beta \geq \ell(\beta, \alpha)$. Suppose not. Then

$$\ell(\beta, \gamma) \leq x_\gamma - x_\beta = x_\gamma - x_\alpha + x_\alpha - x_\beta < \ell(\alpha, \gamma) + \epsilon + \ell(\beta, \alpha).$$

Since $\alpha \rightarrow \gamma \rightarrow \beta \rightarrow \alpha$ is a negative length cycle, for ϵ sufficiently small

$$0 > \ell(\beta, \alpha) + \ell(\alpha, \gamma) + \ell(\gamma, \beta) + \epsilon > \ell(\beta, \gamma) + \ell(\gamma, \beta) \geq 0,$$

a contradiction.

Since our chosen $x \in R_\gamma$ satisfies $x_\gamma - x_\alpha > \ell(\alpha, \gamma)$ and $x_\alpha - x_\beta \geq \ell(\beta, \alpha)$ it follows that x satisfies (2.13). This contradicts the existence of the negative cycle $\alpha \rightarrow \gamma \rightarrow \beta \rightarrow \alpha$. \blacksquare

The theorem is false when $|\Gamma|$ is not finite. An additional integrability condition is needed (see Rochet (1987)). We postpone discussion of this issue when we take up the case of Bayesian incentive compatibility. The assumption that g is onto is not without loss since it rules out allocation rules that could, for example, break ties between outcomes based on preferences over irrelevant outcomes. Convexity of the type space is essential, otherwise the theorem is false. For example, when the type space is discrete.

2.1.2 Roberts' Theorem

In this section we give a proof of a theorem of Kevin Roberts (1979) that gives a characterization of DSIC allocation rules as solutions to maximization problems. The proof is a reinterpretation of the original emphasizing the underlying network structure.⁵

Theorem 2.1.6 (Roberts' Theorem) *Suppose $|\Gamma|$ is finite and at least 3. Let $T = \mathbb{R}^{|\Gamma|}$ and g be DSIC and onto. Then there exists a non-zero, non-negative $w \in \mathbb{R}^n$ and $|\Gamma|$ real numbers $\{D_\gamma\}_{\gamma \in \Gamma}$ such that*

$$g(\mathbf{t}) \in \arg \max_{\gamma \in \Gamma} \sum_{i=1}^n w_i t_\gamma^i - D_\gamma.$$

The class of allocation rules described in the theorem above are called **affine maximizers**.

Fix a non-zero and nonnegative vector w . We associate a network with w called Γ_w . The network will have one node for each $\gamma \in \Gamma$. For each ordered pair (β, α) where $\beta, \alpha \in \Gamma$ introduce a directed arc from β to α of length

$$l_w(\beta, \alpha) = \inf_{\mathbf{t}: g(\mathbf{t})=\alpha} \sum_{i=1}^n w_i (t_\alpha^i - t_\beta^i).$$

If there is a choice of w for which Γ_w has no negative length cycles, Roberts theorem is proved. We will prove the following,

Theorem 2.1.7 *Let g be DSIC, onto and $|\Gamma| \geq 3$. Then there exists a non-zero, non-negative $w \in \mathbb{R}^n$ such that Γ_w has no negative length cycles.*

In what follows we can assume wlog that $w \geq 0$ and $\sum w_i = 1$. Such a vector w will be called feasible.

First we give an outline of the proof. Suppose a cycle $C = \alpha_1 \rightarrow \dots \rightarrow \alpha_k \rightarrow \alpha_1$ through the elements of Γ . From each α_j pick a profile $\mathbf{t}[j]$ such

⁵See also Lavi, Mu'alem and Nisan (2004) and Meyer-ter-Vehn and Moldovanu (2002).

that $f(\mathbf{t}[j]) = \alpha_j$. We associate with the cycle C a vector b whose i^{th} component is

$$b^i = (t_{\alpha_1}^i[1] - t_{\alpha_k}^i[1]) + (t_{\alpha_2}^i[2] - t_{\alpha_1}^i[2]) + \dots + (t_{\alpha_k}^i[k] - t_{\alpha_{k-1}}^i[k]).$$

Let $K \subseteq \mathbb{R}^n$ be the set of vectors that can be associated with some cycle through the elements of Γ . Theorem 2.1.7 asserts the existence of a feasible w such that $w \cdot b \geq 0$ for all $b \in K$. The major milestones of the proof are as follows.

1. If $b \in K$ is associated with the cycle $\alpha_1 \rightarrow \dots \rightarrow \alpha_k \rightarrow \alpha_1$ then b is associated with the cycle $\alpha_1 \rightarrow \alpha_k \rightarrow \alpha_1$. In words, each element of K is associated with a cycle through a pair of elements of Γ .
2. If $b \in K$ is associated with a cycle through (α, β) then b is associated with a cycle through (γ, θ) for all $(\gamma, \theta) \neq (\alpha, \beta)$. In words attention can be restricted to just one cycle.
3. Show that K is convex.
4. Finally, it suffices to note that K is disjoint from the negative orthant. Theorem 2.1.7 follows by invoking the separating hyperplane theorem.

Let $U(\beta, \alpha) = \{d \in \mathbb{R}^n : \exists \mathbf{t} \in T^n \text{ s.t. } g(\mathbf{t}) = \alpha, \text{ s.t. } d_i = t_{\alpha}^i - t_{\beta}^i \forall i\}$. Since g is onto, $U(\beta, \alpha) \neq \emptyset$ for all $\alpha, \beta \in \Gamma$. Notice that $l_w(\beta, \alpha) = \inf_{d \in U(\beta, \alpha)} w \cdot d$.

Lemma 2.1.8 *Suppose $g(\mathbf{t}) = \alpha$ and $s \in T^n$ such that $s_{\alpha}^i - s_{\beta}^i > t_{\alpha}^i - t_{\beta}^i$ for all i . Then $g(\mathbf{s}) \neq \beta$.*

Proof: This lemma is a consequence of the 2-cycle inequality and we leave its proof as an exercise. ■

A consequence of Lemma 2.1.8 is that $U(\beta, \alpha)$ is upper comprehensive for all $\alpha, \beta \in \Gamma$.

For every pair $\alpha, \beta \in \Gamma$ define

$$h(\beta, \alpha) = \inf_{\mathbf{t} \in T^n : g(\mathbf{t}) = \alpha} \max_i t_{\alpha}^i - t_{\beta}^i = \inf_{d \in U(\beta, \alpha)} \max_i d^i.$$

Lemma 2.1.9 *For every pair $\alpha, \beta \in \Gamma$, $h(\beta, \alpha)$ is finite.*

Proof: Suppose not. Fix a pair α and β for which the lemma is false. Then $h(\beta, \alpha)$ can be made arbitrarily small. Since $U(\beta, \alpha)$ is upper comprehensive this would imply that $U(\beta, \alpha) = \mathbb{R}^n$.

Choose $d \in U(\alpha, \beta)$. Then there is an $\mathbf{s} \in T^n$ such that $s_\beta^i - s_\alpha^i = d^i$ for all i and $g(\mathbf{s}) = \beta$. Since $U(\beta, \alpha) = \mathbb{R}^n$ there is a $\mathbf{t} \in T^n$ such that $t_\alpha^i - t_\beta^i < -d^i$ for all i and $g(\mathbf{t}) = \alpha$. Since $t_\beta^i - t_\alpha^i > s_\beta^i - s_\alpha^i$ for all i it follows from Lemma 2.1.8 that $g(\mathbf{t}) \neq \alpha$, a contradiction. ■

Since $U(\alpha, \beta) \neq \mathbb{R}^n$ it follows that the complement of $U(\alpha, \beta)$ must contain all vectors x such that $-x \in U(\beta, \alpha)$. In particular this means that $\text{int}[U(\alpha, \beta) + U(\beta, \alpha)]$ does not contain the origin. In addition, if for some $\mathbf{t} \in T^n$ we have that $t_\alpha^i - t_\beta^i < h(\beta, \alpha)$ for all i then $g(\mathbf{t}) \neq \alpha$.

Lemma 2.1.10 *For all $\alpha, \beta \in \Gamma$, $h(\alpha, \beta) + h(\beta, \alpha) = 0$.*

Proof: Suppose first that $h(\alpha, \beta) + h(\beta, \alpha) > 0$. Choose $\mathbf{t} \in T^n$ to satisfy

$$t_\alpha^i - t_\beta^i < h(\beta, \alpha) \quad \forall i \quad (2.14)$$

$$t_\beta^i - t_\alpha^i < h(\alpha, \beta) \quad \forall i \quad (2.15)$$

$$t_\gamma^i - t_\alpha^i < h(\alpha, \gamma) \quad \forall i \quad \forall \gamma \neq \alpha, \beta \quad (2.16)$$

It is easy to see that the network associated with this system has no negative length cycle. So, it has a solution. Now (2.14) implies that $g(\mathbf{t}) \neq \alpha$. Similarly (2.15) implies that $g(\mathbf{t}) \neq \beta$. Together with (2.16) we deduce that $g(\mathbf{t}) \notin \Gamma$ a contradiction.

Now suppose that $h(\alpha, \beta) + h(\beta, \alpha) < 0$. Choose $\mathbf{t} \in T^n$ to satisfy

$$t_\beta^i - t_\alpha^i > h(\alpha, \beta) \quad \forall i \quad (2.17)$$

$$t_\alpha^i - t_\beta^i > h(\beta, \alpha) \quad \forall i \quad (2.18)$$

$$t_\gamma^i - t_\alpha^i < h(\alpha, \gamma) \quad \forall i \quad \forall \gamma \neq \alpha, \beta \quad (2.19)$$

It is easy to see that the system has a solution. Now (2.19) implies that $g(\mathbf{t}) \in \{\alpha, \beta\}$. However (2.18) coupled with Lemma 2.1.8 means that $g(\mathbf{t}) \neq \alpha$. But, (2.17) and lemma 2.1.8 imply that $g(\mathbf{t}) \neq \beta$ a contradiction. ■

Denote the vector all of whose components equal 1 by \hat{u} .

Lemma 2.1.11 *If $d_{\gamma, \beta} \in U(\gamma, \beta)$, $d_{\beta, \alpha} \in U(\beta, \alpha)$ then for all $\epsilon > 0$*

$$d_{\gamma, \beta} + d_{\beta, \alpha} + \epsilon \hat{u} \in U(\gamma, \alpha).$$

Proof: Choose $\mathbf{t} \in T^n$ to satisfy the following.

$$t_\alpha^i - t_\beta^i = d_{\beta,\alpha}^i + \hat{u}\epsilon/2 \quad \forall i \quad (2.20)$$

$$t_\beta^i - t_\gamma^i = d_{\gamma,\beta}^i + \hat{u}\epsilon/2 \quad \forall i \quad (2.21)$$

$$t_\theta^i - t_\alpha^i < h(\alpha, \theta) \quad \forall \theta \notin \{\alpha, \beta, \gamma\}, \quad \forall i \quad (2.22)$$

Since the network associated with the system has no cycle the system has a solution. Inequality (2.22) combined with the definition of h implies that $g(\mathbf{t}) \in \{\alpha, \beta, \gamma\}$. Inequality (2.21) and Lemma 2.1.8 imply that $g(\mathbf{t}) \neq \gamma$. Inequality (2.20) combined with Lemma 2.1.8 imply that $g(\mathbf{t}) \neq \beta$. Hence $g(\mathbf{t}) = \alpha$. Observe that $t_\alpha^i - t_\gamma^i = d_{\beta,\alpha}^i + d_{\gamma,\beta}^i + \epsilon$ for all i and this proves the lemma. \blacksquare

If the length function l_w is well defined for all arcs of Γ_w the conclusion of Lemma 2.1.11 can be restated as

$$l_w(\alpha, \beta) \leq l_w(\alpha, \gamma) + l_w(\gamma, \beta) + \epsilon.$$

In words, l_w satisfies the triangle inequality.

Lemma 2.1.12 *Suppose for some feasible w we have that $l_w(\alpha, \beta) + l_w(\beta, \gamma) \geq 0$ for all $\alpha, \beta \in \Gamma$. Then Γ_w has no negative length cycles.*

Proof: The condition that $l_w(\alpha, \beta) + l_w(\beta, \gamma) \geq 0$ for all $\alpha, \beta \in \Gamma$ implies that the length function l_w is finite. To prove the lemma it suffices to show that l_w satisfies the triangle inequality. This follows from Lemma 2.1.11. \blacksquare

Lemma 2.1.13 *For any $d_{\gamma,\alpha} \in U(\gamma, \alpha)$, $d_{\alpha,\beta} \in U(\alpha, \beta)$ and $d_{\beta,\alpha} \in U(\beta, \alpha)$ we have*

$$d_{\gamma,\alpha} + d_{\alpha,\beta} + d_{\beta,\alpha} + 2\hat{u}\epsilon \in U(\gamma, \alpha).$$

Proof: Lemma 2.1.11 implies that $d_{\gamma,\alpha} + d_{\alpha,\beta} + \hat{u}\epsilon \in U(\gamma, \beta)$. Lemma 2.1.11 implies that

$$d_{\gamma,\alpha} + d_{\alpha,\beta} + \hat{u}\epsilon + d_{\beta,\alpha} + \hat{u}\epsilon \in U(\gamma, \alpha).$$

\blacksquare

Lemma 2.1.14 *$\text{int}[U(\alpha, \beta) + U(\beta, \alpha)]$ is convex for all $\alpha, \beta \in \Gamma$.*

Proof: Pick $d_{\alpha,\beta}^i \in U(\alpha, \beta)$ and $d_{\beta,\alpha}^i \in U(\beta, \alpha)$ for $i = 1, 2$. It suffices to prove that $(d_{\alpha,\beta}^1 + d_{\beta,\alpha}^1)/2 + ((d_{\alpha,\beta}^2 + d_{\beta,\alpha}^2))/2 \in \text{int}[U(\alpha, \beta) + U(\beta, \alpha)]$. Suppose not. Then, without loss, we may suppose that $(d_{\alpha,\beta}^1 + d_{\beta,\alpha}^1)/2 \notin \text{int}U(\alpha, \beta)$. Hence $-(d_{\alpha,\beta}^1 + d_{\beta,\alpha}^1)/2 \in U(\beta, \alpha)$. Therefore, by Lemma 2.1.13

$$h(\gamma, \alpha)\hat{u} + \epsilon + d_{\alpha,\beta}^1 - (d_{\alpha,\beta}^1 + d_{\beta,\alpha}^1)/2 = (d_{\alpha,\beta}^1 - d_{\beta,\alpha}^1)/2 + \hat{u}\epsilon + h(\gamma, \alpha)\hat{u} \in U(\gamma, \alpha).$$

Since $h(\alpha, \gamma)\hat{u} + \hat{u}\epsilon \in U(\alpha, \gamma)$ it follows from Lemma 2.1.10

$$(d_{\alpha,\beta}^1 - d_{\beta,\alpha}^1)/2 + 2\epsilon = (d_{\alpha,\beta}^1 - d_{\beta,\alpha}^1)/2 + \hat{u}\epsilon + h(\gamma, \alpha)\hat{u} + h(\alpha, \gamma)\hat{u} + \hat{u}\epsilon \in \text{int}[U(\alpha, \gamma) + U(\gamma, \alpha)].$$

By Lemma 2.1.13

$$d_{\alpha,\beta}^2 + (d_{\alpha,\beta}^1 - d_{\beta,\alpha}^1)/2 + 2\hat{u}\epsilon \in U(\alpha, \beta)$$

a contradiction. \blacksquare

Lemma 2.1.15 *If there is a feasible w and pair $\alpha, \beta \in \Gamma$ such that $l_w(\alpha, \beta) + l_w(\beta, \alpha) < 0$ then for all $\gamma, \theta \in \Gamma$ we have $l_w(\gamma, \theta) + l_w(\theta, \gamma) < 0$.*

Proof: It suffices to prove that $\text{int}[U(\alpha, \beta) + U(\beta, \alpha)] = \text{int}[U(\gamma, \delta) + U(\delta, \gamma)]$ for all $\alpha, \beta, \gamma, \delta \in \Gamma$. For any $d_{\alpha,\beta} \in U(\alpha, \beta)$ we have by Lemma 2.1.11 that $d_{\alpha,\beta} + h(\beta, \gamma)\hat{u} + \hat{u}\epsilon \in U(\alpha, \gamma)$. Similarly, for any $d_{\beta,\alpha} \in U(\beta, \alpha)$ we have that $d_{\beta,\alpha} + h(\gamma, \beta)\hat{u} + \hat{u}\epsilon \in U(\gamma, \alpha)$. Hence, by Lemma 2.1.10,

$$d_{\alpha,\beta} + h(\beta, \gamma)\hat{u} + \hat{u}\epsilon + d_{\beta,\alpha} + h(\gamma, \beta)\hat{u} + \hat{u}\epsilon = (d_{\alpha,\beta} + \hat{u}\epsilon) + (d_{\beta,\alpha} + \hat{u}\epsilon) \in U(\alpha, \gamma) + U(\gamma, \alpha).$$

Hence $\text{int}[U(\alpha, \beta) + U(\beta, \alpha)] \subseteq U(\alpha, \gamma) + U(\gamma, \alpha)$. Switching the roles of β and α we conclude that $\text{int}[U(\alpha, \beta) + U(\beta, \alpha)] = \text{int}[U(\alpha, \gamma) + U(\gamma, \alpha)]$. Repeating the argument again with α replaced by δ completes the proof. \blacksquare

We now prove Theorem 2.1.7. Suppose it is false. Then, by Lemma 2.1.15 for every feasible choice of w and pair $\alpha, \beta \in \Gamma$ we have $l_w(\alpha, \beta) + l_w(\beta, \alpha) < 0$. By Lemma 2.1.14 the set $\text{int}(U(\alpha, \beta)) + \text{int}(U(\beta, \alpha))$ is convex. Notice also that the set $\text{int}(U(\alpha, \beta)) + \text{int}(U(\beta, \alpha))$ cannot contain the origin. Therefore, by the separating hyperplane theorem there is a feasible w^* such that $w^* \cdot z \geq 0$ for all $z \in U(\alpha, \beta) + U(\beta, \alpha)$, a contradiction.

Theorem 2.1.7 is false when $|\Gamma| = 2$ and when T is a strict subset of $\mathbb{R}^{|\Gamma|}$.

Example 1 *We describe a counter example to Theorem 2.1.7 when $|\Gamma| = 2$. Suppose $\Gamma = \{\alpha, \beta\}$. Let $g(\mathbf{t}) = \alpha$ whenever $t_\alpha^i \geq t_\beta^i$ for all agents i otherwise*

$g(\mathbf{t}) = \beta$. Clearly, g cannot be expressed as an affine maximizer. To show that g is DSIC we verify that the 2-cycle condition holds. Fix \mathbf{t}^{-i} . We have two cases. In the first $t_\alpha^j \geq t_\beta^j$ for all $j \neq i$. Then the corresponding graph Γ_g consists of just two nodes; α and β . As long as $t_\alpha^i - t_\beta^i \geq 0$ we know that $f(t^i, \mathbf{t}^{-i}) = \alpha$. Hence $\ell(\beta, \alpha) = 0$. Similarly, $\ell(\alpha, \beta) = 0$. In the second case, $t_\alpha^j \leq t_\beta^j$ for at least one $j \neq i$. In this case $f(t^i, \mathbf{t}^{-i}) = \beta$ for all t^i . Hence, the corresponding graph Γ_g is a single vertex.

Example 2 We describe a counter example to Theorem 2.1.7 when T is a strict subset of $\mathbb{R}^{|\Gamma|}$. Consider two buyers and one good. There are three possible allocations: the good goes to agent 1, it goes to agent 2 or it is not allocated at all. An agents type records the value of each possible allocation. Suppose private values, i.e., agent i receives value of $v_i \in [0, 1]$ when she receives the good and zero for all other allocations. Hence T is a one dimensional subset of $[0, 1]^3$. Consider the following allocation rule. If $v_1, v_2 \in [0, 1/2)$ then the good is not allocated. When $v_1, v_2 \in (1/2, 1]$ the good is not allocated. In all other cases the good is allocated to the agent with the highest value. It is straightforward to verify that the rule is DSIC and not an affine maximizer.

2.2 Bayesian Incentive Compatability

This section is based on Müller, Perea and Wolf (2005). They have been kind enough to let me plagiarize at will.

Take agent i having true type t^i and reporting r^i while the others have true types \mathbf{t}^{-i} and report \mathbf{r}^{-i} . The value that agent i assigns to the resulting allocation is denoted by $v^i(g(r^i, \mathbf{r}^{-i}) | t^i, \mathbf{t}^{-i})$. Agents' types are independently distributed. Let π^i denote the density on T . The joint density π^{-i} on \mathbf{t}^{-i} is then given by

$$\pi^{-i}(\mathbf{t}^{-i}) = \prod_{\substack{j \in N \\ j \neq i}} \pi(t^j).$$

Assume that agent i believes all other agents to report truthfully. If agent i has true type t^i , then his expected utility for making a report r^i is given by

$$\begin{aligned} U^i(r^i | t^i) &= \int_{T^{n-1}} (v^i(g(r^i, \mathbf{t}^{-i}) | t^i, \mathbf{t}^{-i}) - P_i(r^i, \mathbf{t}^{-i})) \pi^{-i}(\mathbf{t}^{-i}) dt^{-i} \\ &= E_{-i} [v^i(g(r^i, \mathbf{t}^{-i}) | t^i, \mathbf{t}^{-i}) - P_i(r^i, \mathbf{t}^{-i})]. \end{aligned} \quad (2.23)$$

We assume $E_{-i} [v^i (g (r^i, \mathbf{t}^{-i}) | t^i, \mathbf{t}^{-i})]$ to be finite $\forall r^i, t^i \in T$.

An allocation rule g is **Bayes-Nash incentive compatible** (BNIC) if there exists a payment rule P such that $\forall i \in N$ and $\forall r^i, \tilde{r}^i \in T^i$:

$$E_{-i} [v^i (g (r^i, \mathbf{t}^{-i}) | r^i, \mathbf{t}^{-i}) - P_i (r^i, \mathbf{t}^{-i})] \geq E_{-i} [v^i (g (\tilde{r}^i, \mathbf{t}^{-i}) | r^i, \mathbf{t}^{-i}) - P_i (\tilde{r}^i, \mathbf{t}^{-i})]. \quad (2.24)$$

Symmetrically, we have also

$$E_{-i} [v^i (g (\tilde{r}^i, \mathbf{t}^{-i}) | \tilde{r}^i, \mathbf{t}^{-i}) - P_i (\tilde{r}^i, \mathbf{t}^{-i})] \geq E_{-i} [v^i (g (r^i, \mathbf{t}^{-i}) | \tilde{r}^i, \mathbf{t}^{-i}) - P_i (r^i, \mathbf{t}^{-i})]. \quad (2.25)$$

By adding (2.24) and (2.25) we get the expected utility version of the 2-cycle condition. An allocation rule g satisfies the 2-cycle condition if $\forall i \in N$ and $\forall r^i, \tilde{r}^i \in T^i$:

$$\begin{aligned} & E_{-i} [v^i (g (r^i, \mathbf{t}^{-i}) | r^i, \mathbf{t}^{-i}) - v^i (g (\tilde{r}^i, \mathbf{t}^{-i}) | r^i, \mathbf{t}^{-i})] \\ & \geq E_{-i} [v^i (g (r^i, \mathbf{t}^{-i}) | \tilde{r}^i, \mathbf{t}^{-i}) - v^i (g (\tilde{r}^i, \mathbf{t}^{-i}) | \tilde{r}^i, \mathbf{t}^{-i})]. \end{aligned}$$

Obviously, the 2-cycle condition is necessary for BNIC.

As before the constraints in (2.24) have a natural network interpretation. For each agent i we build a complete directed graph T_g^i . A node is associated with each type and a directed arc is inserted between each ordered pair of nodes. For agent i the length of an arc directed from r^i to \tilde{r}^i is denoted $l^i(r^i, \tilde{r}^i)$ and is defined as the *cost of manipulation*:

$$l^i (r^i, \tilde{r}^i) = E_{-i} [v^i (g (r^i, \mathbf{t}^{-i}) | r^i, \mathbf{t}^{-i}) - v^i (g (\tilde{r}^i, \mathbf{t}^{-i}) | r^i, \mathbf{t}^{-i})]. \quad (2.26)$$

Given our previous assumptions, the arc length is finite. For technical reasons we allow for loops. However, note that an arc directed from r^i to r^i has length $l^i(r^i, r^i) = 0$.

The following is now straightforward.

Theorem 2.2.1 *An allocation rule g is BNIC if and only if there is no finite, negative length cycle in T_g^i , $\forall i \in N$.*

The costs of manipulation are decomposition monotone if $\forall \underline{r}^i, \bar{r}^i \in T^i$ and $\forall r^i \in T^i$ s.t. $r^i = (1 - \alpha)\underline{r}^i + \alpha\bar{r}^i$, $\alpha \in (0, 1)$ we have

$$l^i (\underline{r}^i, \bar{r}^i) \geq l^i (\underline{r}^i, r^i) + l^i (r^i, \bar{r}^i).$$

If decomposition monotonicity holds then the arc between those nodes is at least as long as any path connecting the same two nodes via nodes lying on the line segment between them. Figure 3.1 gives an illustrative example.

Decomposition monotonicity implies that the arc from \underline{r}^i to \bar{r}^i is at least as long as the path $A = (\underline{r}^i, r_{**}^i, \bar{r}^i)$ and that A is at least as long as the path $\tilde{A} = (\underline{r}^i, r_*^i, r_{**}^i, r_{***}^i, \bar{r}^i)$.

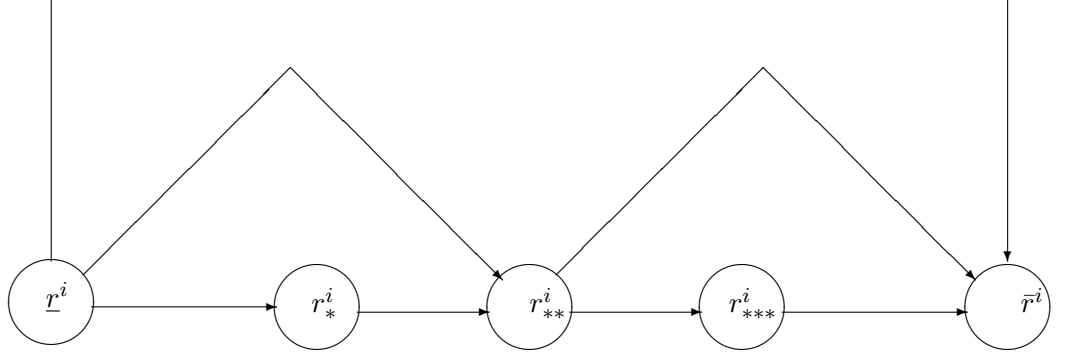


Figure 3.1

We now assume that T is convex for each agent i . Furthermore, we now assume that an agent's valuation function is linear in his own true type. So if agent i has true type t^i and reports r^i while the others have true types \mathbf{t}^{-i} and report \mathbf{r}^{-i} , his valuation for the resulting allocation is

$$v^i(g(r^i, \mathbf{r}^{-i}) | t^i, \mathbf{t}^{-i}) = \alpha^i(g(r^i, \mathbf{r}^{-i}) | \mathbf{t}^{-i}) + \beta^i(g(r^i, \mathbf{r}^{-i}) | \mathbf{t}^{-i}) t^i. \quad (2.27)$$

Note that $\alpha^i : \Gamma \times T^{n-1} \mapsto \mathbb{R}$ and $\beta^i : \Gamma \times T^{n-1} \mapsto \mathbb{R}^k$, i.e. α^i assigns to every $(\gamma, \mathbf{t}^{-i}) \in \Gamma \times T^{n-1}$ a value in \mathbb{R} , whereas β^i assigns to every $(\gamma, \mathbf{t}^{-i}) \in \Gamma \times T^{n-1}$ a vector in \mathbb{R}^k . Similarly, assuming he believes all other agents to report truthfully, agent i 's expected valuation for reporting r^i while having true type t^i is

$$E_{-i}[v^i(g(r^i, \mathbf{t}^{-i}) | t^i, \mathbf{t}^{-i})] = E_{-i}[\alpha^i(g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] + E_{-i}[\beta^i(g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] t^i. \quad (2.28)$$

Using (2.28), the 2-cycle condition becomes

$$E_{-i}[\beta^i(g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i}) - \beta^i(g(\tilde{r}^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] (r^i - \tilde{r}^i) \geq 0 \quad \forall i \in N, \forall r^i, \tilde{r}^i \in T. \quad (2.29)$$

In this restricted setting the 2-cycle condition implies that the costs of manipulation are decomposition monotone:

Lemma 2.2.2 *Suppose that every agent i has a valuation function which is linear in his true type: If g satisfies the 2-cycle condition then the costs of manipulation are decomposition monotone.*

Proof: Take some agent i and let $\underline{r}^i, \bar{r}^i \in T$. Let $r^i \in T^i$ such that $r^i = (1 - \alpha)\underline{r}^i + \alpha\bar{r}^i$ for some $\alpha \in (0, 1)$. The 2-cycle inequality implies that

$$E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i}) - \beta^i (g(\bar{r}^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] (r^i - \bar{r}^i) \geq 0.$$

Note that $\underline{r}^i - r^i$ is proportional to $r^i - \bar{r}^i$, specifically $\underline{r}^i - r^i = \frac{\alpha}{1-\alpha} (r^i - \bar{r}^i)$. Since $\alpha \in (0, 1)$, the above inequality implies that

$$E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i}) - \beta^i (g(\bar{r}^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] (\underline{r}^i - r^i) \geq 0.$$

Adding $E_{-i} [\beta^i (g(\underline{r}^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i}) - \beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] \underline{r}^i$ to both sides of the latter inequality and rearranging terms yields

$$\begin{aligned} & E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i}) - \beta^i (g(\bar{r}^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] \underline{r}^i \\ & + E_{-i} [\beta^i (g(\underline{r}^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i}) - \beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] \underline{r}^i \\ \geq & E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i}) - \beta^i (g(\bar{r}^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] r^i \\ & + E_{-i} [\beta^i (g(\underline{r}^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i}) - \beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] \underline{r}^i. \end{aligned}$$

The first and the last term on the left-hand side of the inequality cancel. Using (2.26), the above can be written as

$$l^i(\underline{r}^i, \bar{r}^i) \geq l^i(\underline{r}^i, r^i) + l^i(r^i, \bar{r}^i),$$

so the costs of manipulation are decomposition monotone. \blacksquare

Even when the type space is convex the 2-cycle condition is not sufficient to characterize BNIC. We give a counterexample. It is constructed based on the following insight: Suppose that the allocation function g and the mapping β^i are such that we can write

$$E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] = r^i B_i,$$

where B_i is some agent specific $k \times k$ matrix. The two cycle condition requires

$$(r^i - \bar{r}^i) B_i (r^i - \bar{r}^i)' \geq 0 \quad \forall r^i, \bar{r}^i \in T,$$

where $'$ denotes “transposed”. Note that

$$B_i = \frac{1}{2} (B_i + B_i') + \frac{1}{2} (B_i - B_i'),$$

that is, B_i can be decomposed into a symmetric part $\frac{1}{2}(B_i + B'_i)$ and an anti-symmetric part $\frac{1}{2}(B_i - B'_i)$. The two cycle condition is satisfied if the symmetric part of B_i is positive semi-definite. However, there are no finite, negative length cycles in T_g^i (and thus g is BNIC) if and only if B_i is symmetric and positive semi-definite (both results follow from Rockafellar, 1970, p.240).

Example 3 *For simplicity assume a single agent. Furthermore, take the mapping β^i in (2.27) to be linear and the mapping α^i to be equal to zero. The agent's type space is the convex hull of the simplex in \mathbb{R}^3 . Let $x = (1, 0, 0)$, $y = (0, 1, 0)$ and $z = (0, 0, 1)$. There are three outcomes, denoted a , b and c . If the agent is of type x , his valuations for these outcomes are given by the first column of the following matrix*

$$V = \begin{pmatrix} 2 & 0 & 3 \\ 3 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}.$$

The first element is his valuation for a , the second one for b and the third one for c . Similarly, if the agent is of type y or z , his valuations for the elementary outcomes are given by the second and the third column of V . The allocation rule g is a linear mapping associating each type report with a probability distribution over the three elementary outcomes. The outcome space Γ is the set of all possible probability distributions on $\{a, b, c\}$. Generic element $\gamma = (\gamma_a, \gamma_b, \gamma_c)$ indicates that a is achieved with probability γ_a , b with probability γ_b and c with probability γ_c .

The allocation rule works as follows: if the agent reports x as his type then g awards him with the second-best outcome according to this type, that is $g(x) = (1, 0, 0)$. Similarly, $g(y) = (0, 1, 0)$ and $g(z) = (0, 0, 1)$. In general we have $g(r) = rI$, where I denotes the 3×3 identity matrix.

Using the above, the agent's valuation function becomes $v(g(r) | t) = rVt'$. As can be easily checked (by verifying that the symmetric part $\frac{1}{2}(V + V')$ of V is positive definite), the 2-cycle inequality is satisfied, that is, $(r - \tilde{r})V(r - \tilde{r})' \geq 0$, $\forall r, \tilde{r} \in T$. Nevertheless, the 3-cycle $C = (x, y, z, x)$ has length $\ell(x, y) + \ell(y, z) + \ell(z, x) = -3$. The existence of a negative length cycle implies that g is not BNIC (see Theorem 2.2.1).

We show that that 2-cycle condition along with an integrability condition is sufficient to characterize BNIC. A special case of this result was first proved in Jehiel and Moldovanu (2001).

Definition 2.2.3 (Path Independence) Let $\psi: T^i \mapsto \mathbb{R}^k$ be a vector field. ψ is called path independent if for any two $\underline{r}^i, \bar{r}^i \in T^i$ the path integral of ψ from \underline{r}^i to \bar{r}^i

$$\int_{\underline{r}^i, S}^{\bar{r}^i} \psi$$

is independent of the path of integration S .

Note that $E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) \mid \mathbf{t}^{-i})]$ is a vector field $T \mapsto \mathbb{R}^k$.

Theorem 2.2.4 Suppose that every agent i has a convex type space and a valuation function which is linear in his true type. Then the following statements are equivalent:

- 1) g is BNIC.
- 2) g satisfies the 2-cycle condition and for every agent i , $E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) \mid \mathbf{t}^{-i})]$ is path independent.

Proof: (1) \Rightarrow (2): Assume that g is BNIC. Clearly the 2-cycle condition holds. Furthermore, from Theorem 2.2.1 it follows that for every agent i the graph T_g^i has no finite, negative length cycles. Let $C = (r_1^i, \dots, r_m^i, r_{m+1}^i = r_1^i)$ denote a finite cycle in T_g^i . Then,

$$\sum_{j=1}^m l^i(r_j^i, r_{j+1}^i) \geq 0,$$

which can be rewritten using (2.26) and (2.28) as

$$\sum_{j=1}^m E_{-i} [\beta^i (g(r_j^i, \mathbf{t}^{-i}) \mid \mathbf{t}^{-i}) - \beta^i (g(r_{j+1}^i, \mathbf{t}^{-i}) \mid \mathbf{t}^{-i})] r_j^i \geq 0.$$

This implies that

$$\sum_{j=1}^m E_{-i} [\beta^i (g(r_{j+1}^i, \mathbf{t}^{-i}) \mid \mathbf{t}^{-i})] (r_{j+1}^i - r_j^i) \geq 0.$$

Thus, $E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) \mid \mathbf{t}^{-i})]$ is cyclically monotone.⁶ From Rockafellar (1970), Theorem 24.8, it follows that there exists a convex function $\varphi: T^i \mapsto \mathbb{R}$ such that $E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) \mid \mathbf{t}^{-i})]$ is a selection from its subdifferential mapping, that is,

$$E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) \mid \mathbf{t}^{-i})] \in \partial\varphi(r^i), \forall r^i \in T.$$

⁶The notion of cyclical monotonicity was introduced by Rockafellar (1966).

This implies (see Krishna and Maenner, 2001, Theorem 1) that for any smooth path S in T joining \underline{r}^i and \bar{r}^i the following holds:

$$\int_{\underline{r}^i, S}^{\bar{r}^i} E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] = \varphi(\bar{r}^i) - \varphi(\underline{r}^i),$$

so $E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})]$ is path independent.

(2)⇒(1): Let us assume that g satisfies the 2-cycle condition and that for every agent i , $E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})]$ is path independent. Take any arc from T_g^i and denote its starting node \underline{r}^i and its ending node \bar{r}^i . Let L denote the line segment between \underline{r}^i and \bar{r}^i , i.e. $L = \{r^i \in T^i \mid r^i = (1 - \alpha)\underline{r}^i + \alpha\bar{r}^i, \alpha \in [0, 1]\}$. Choose any $r^i \in L$ and replace the original arc with the path $A = (\underline{r}^i, r^i, \bar{r}^i)$ which has length $l^i(\underline{r}^i, r^i) + l^i(r^i, \bar{r}^i)$. By Lemma 2.2.2 we have

$$l^i(\underline{r}^i, \bar{r}^i) \geq l^i(\underline{r}^i, r^i) + l^i(r^i, \bar{r}^i), \quad (2.30)$$

that is, the original arc is at least as long as the path A . By repeated substitution we can generate a new path $\tilde{A} = (r_1^i = \underline{r}^i, \dots, r_m^i, r_{m+1}^i = \bar{r}^i)$ where $r_j^i \in L, \forall j \in \{1, \dots, m+1\}$. Then (2.30) implies that the original arc is at least as long as \tilde{A} , that is,

$$l^i(\underline{r}^i, \bar{r}^i) \geq \sum_{j=1}^m l^i(r_j^i, r_{j+1}^i),$$

(see also the example given in Figure 3.2) Note that

$$\begin{aligned} \sum_{j=1}^m l^i(r_j^i, r_{j+1}^i) &= \sum_{j=1}^m E_{-i} [v^i(g(r_j^i, \mathbf{t}^{-i}) | r_j^i, \mathbf{t}^{-i}) - v^i(g(r_{j+1}^i, \mathbf{t}^{-i}) | r_j^i, \mathbf{t}^{-i})] \\ &= E_{-i} [v^i(g(r_1^i, \mathbf{t}^{-i}) | r_1^i, \mathbf{t}^{-i}) - v^i(g(r_{m+1}^i, \mathbf{t}^{-i}) | r_m^i, \mathbf{t}^{-i})] \\ &\quad + \sum_{j=1}^{m-1} E_{-i} [v^i(g(r_{j+1}^i, \mathbf{t}^{-i}) | r_{j+1}^i, \mathbf{t}^{-i}) - v^i(g(r_{j+1}^i, \mathbf{t}^{-i}) | r_j^i, \mathbf{t}^{-i})] \\ &= E_{-i} [v^i(g(r_1^i, \mathbf{t}^{-i}) | r_1^i, \mathbf{t}^{-i}) - v^i(g(r_{m+1}^i, \mathbf{t}^{-i}) | r_{m+1}^i, \mathbf{t}^{-i})] \\ &\quad + \sum_{j=1}^m E_{-i} [v^i(g(r_{j+1}^i, \mathbf{t}^{-i}) | r_{j+1}^i, \mathbf{t}^{-i}) - v^i(g(r_{j+1}^i, \mathbf{t}^{-i}) | r_j^i, \mathbf{t}^{-i})] \\ &= E_{-i} [v^i(g(\underline{r}^i, \mathbf{t}^{-i}) | \underline{r}^i, \mathbf{t}^{-i}) - v^i(g(\bar{r}^i, \mathbf{t}^{-i}) | \bar{r}^i, \mathbf{t}^{-i})] \\ &\quad + \sum_{j=1}^m E_{-i} [\beta^i(g(r_{j+1}^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] (r_{j+1}^i - r_j^i). \end{aligned}$$

The first equality follows from the definition of the arc length given in (2.26). The second equality follows from rearranging the terms of the summation. The third equality is derived by adding and subtracting $E_{-i} [v^i (g(r_{m+1}^i, \mathbf{t}^{-i}) | r_{m+1}^i, \mathbf{t}^{-i})]$. To derive the last equality we use (2.28) and that $r_1^i = \underline{r}^i$, $r_{m+1}^i = \bar{r}^i$. By repeated substitution we can generate paths with more and more arcs. In the limit the distance between neighboring nodes goes to zero and

$$\sum_{j=1}^m E_{-i} [\beta^i (g(r_{j+1}^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] (r_{j+1}^i - r_j^i) \rightarrow \int_{\underline{r}^i, L}^{\bar{r}^i} E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})].$$

Thus, the length of \tilde{A} goes to

$$E_{-i} [v^i (g(\underline{r}^i, \mathbf{t}^{-i}) | \underline{r}^i, \mathbf{t}^{-i}) - v^i (g(\bar{r}^i, \mathbf{t}^{-i}) | \bar{r}^i, \mathbf{t}^{-i})] + \int_{\underline{r}^i, L}^{\bar{r}^i} E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})], \quad (2.31)$$

as $m \rightarrow \infty$. Now, let $C = (r_1^i, \dots, r_m^i, r_{m+1}^i = r_1^i)$ denote a finite cycle in T_g^i . Furthermore, let L_j denote the line segment between r_j^i and r_{j+1}^i . The result in (2.31) and the path independence of $E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})]$ imply for the length of C that

$$\begin{aligned} & \sum_{j=1}^m l^i(r_j^i, r_{j+1}^i) \\ & \geq \sum_{j=1}^m E_{-i} [v^i (g(r_j^i, \mathbf{t}^{-i}) | r_j^i, \mathbf{t}^{-i}) - v^i (g(r_{j+1}^i, \mathbf{t}^{-i}) | r_{j+1}^i, \mathbf{t}^{-i})] \\ & \quad + \sum_{j=1}^m \int_{r_j^i, L_j}^{r_{j+1}^i} E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] \\ & = 0, \end{aligned}$$

that is, C has non-negative length. In order to see the equality relation, note the following: the terms of the first summation cancel each other out. Furthermore, the second summation describes an integral over a closed path in T^i which, due to path independence, equals zero. ■

The 2-cycle condition and path independence of $E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})]$ do not imply one another. Example 3 and Theorem 2.2.4 shows that the 2-cycle condition does not imply path independence. It can also be derived directly from Example 3. If we consider for example path A consisting of the line segment between x and y and path \tilde{A} consisting of the line segment

between x and z and the line segment between z and y , we find that

$$\int_{x,A}^y \beta(g(r)) = -\frac{3}{2} \quad \text{and} \quad \int_{x,\tilde{A}}^y \beta(g(r)) = 3.$$

So the path integral of $\beta(g(r))$ from x to y is not independent of the path of integration. That weak monotonicity of g does not imply path independence of $E_{-i}[\beta^i(g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})]$ depends upon the assumption of multi-dimensional types. If we would consider one-dimensional type spaces instead, then weak monotonicity would indeed imply path independence.

That path independence does not imply the 2-cycle condition is illustrated by the following example.

Example 4 *Let us consider the allocation of a single, indivisible object. For simplicity we assume that there exists only a single agent to possibly allocate to. He has a type $t \in T = [0, 1]$ which reflects the value of the object for him. Given a report r of the agent, the allocation rule $g: T \mapsto [0, 1]$ assigns to him a probability for getting the object. The agent's valuation for the resulting allocation is $v(g(r) | t) = g(r)t$. Specifically, we set $g(r) = -(2r - 1)^2 + 1$. Clearly, g is path independent but fails the 2-cycle condition.*

If g is BNIC, the corresponding payments can be constructed by using shortest path lengths (as described in the proof of Theorem 2.2.1). For each $i \in N$, let us pick some a^i as the source node in T_g^i . Thus, if agent i reports t^i , he has to make a payment

$$P_i(t^i) = \inf \sum_{j=1}^m l^i(r_j^i, r_{j+1}^i), \quad (2.32)$$

where the infimum is taken over all finite paths from t^i to a^i . Take any finite path $A = (r_1^i = t^i, \dots, r_{m+1}^i = a^i)$ in T_g^i . Let L_j denote the line segment between r_j^i and r_{j+1}^i , whereas L_t denotes the line segment between the source and t^i . Following the repeated substitution approach presented in the second part of the proof of Theorem 2.2.4, we can construct paths that are shorter (or as long) by letting them visit the same nodes as A and also additional nodes along the line segments in between. In the limit, as the number of nodes goes to infinity, the distance between neighboring nodes goes to zero and the length of the paths goes to

$$\begin{aligned} & \sum_{j=1}^m \left(E_{-i} [v^i(g(r_j^i, \mathbf{t}^{-i}) | r_j^i, \mathbf{t}^{-i}) - v^i(g(r_{j+1}^i, \mathbf{t}^{-i}) | r_{j+1}^i, \mathbf{t}^{-i})] \right. \\ & \left. + \int_{r_j^i, L_j}^{r_{j+1}^i} E_{-i} [\beta^i(g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] \right). \end{aligned} \quad (2.33)$$

Using path independence in (2.33) we have that⁷

$$\sum_{j=1}^m \int_{r_j^i, L_j}^{r_{j+1}^i} E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})] = \int_{t^i, L_t}^{a^i} E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})].$$

Applying the above to (2.32) yields

$$\begin{aligned} P_i(t^i) &= E_{-i} [v^i (g(t^i, \mathbf{t}^{-i}) | t^i, \mathbf{t}^{-i}) - v^i (g^i(a^i, \mathbf{t}^{-i}) | a^i, \mathbf{t}^{-i})] \\ &\quad - \int_{a^i, L_t}^{t^i} E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})], \end{aligned} \quad (2.34)$$

implying that the expected utility (see (2.23) for definition) for truthfully reporting t^i is⁸

$$U^i(t^i | t^i) = U^i(a^i | a^i) + \int_{a^i, L_t}^{t^i} E_{-i} [\beta^i (g(r^i, \mathbf{t}^{-i}) | \mathbf{t}^{-i})]. \quad (2.35)$$

2.3 Revenue Equivalence

Suppose T is a type space and g an allocation rule that is DSIC. **Revenue equivalence** is said to hold for g on T if for any pair of payment schemes P and \tilde{P} such that (g, P) and (g, \tilde{P}) are DSIC mechanisms and for each agent i and report \mathbf{t}^{-i} of the other agents, there exists $h_i(\mathbf{t}^{-i}) \in \mathbb{R}$ such that $\tilde{P}_i(t^i, \mathbf{t}^{-i}) = P_i(t^i, \mathbf{t}^{-i}) + h_i(\mathbf{t}^{-i})$. In other words any two payment rules differ by a constant.

Now let us revert again to the more compact notation where the dependence on \mathbf{t}^{-i} is suppressed. If (g, P) is DSIC, we have that for any pair of types $t, s \in T$ such that $g(t) = g(s) = \gamma$ for some $\gamma \in \Gamma$. Therefore, by the Taxation principle, payments must be equal, i.e. $P(t) = P(s)$. In particular, payments can be indexed by allocation rather than type. That is for all t such that $g(t) = \alpha$, $P(t) = P_\alpha$. In this notation revenue equivalence can be stated this way. Choose a $\alpha \in \Gamma$ and fix the value of P_α . Then the values of P_γ for all $\gamma \in \Gamma \setminus \alpha$ are uniquely determined.

The bulk of prior work on revenue equivalence has been devoted to identifying sufficient conditions on the type space for all allocation rules from a certain class to satisfy revenue equivalence. The papers by Green and

⁷The line segment L_t for the path of integration is picked for convenience. Due to path independence, it can be replaced with any other path connecting the source and t^i .

⁸In order to derive (2.35) one can use that by construction $P_i(a^i) = 0$ and thus add this term to the right-hand side of (2.34).

Laffont (1977) and Holmstrom (1979) restrict attention to allocation rules called ‘utilitarian maximizers’, that is, allocation rules that maximize the sum of the valuations of all agents. They show that when the type space is smoothly path connected then utilitarian maximizers satisfy revenue equivalence.

Myerson (1981) shows that revenue equivalence holds for every implementable rule in a setting where the type space is an interval of the real line, the outcome space is a lattice and an agent’s valuation for an outcome is continuous and supermodular in her type.

Krishna and Maenner (2001) derive revenue equivalence under two different hypotheses. In the first, agents’ type spaces must be convex and the valuation function of an agent is a convex function of the type of the agent. Under these conditions they show that every implementable rule satisfies revenue equivalence. The second hypothesis requires the allocation rule to satisfy certain differentiability and continuity conditions and the outcome space to be a subset of the Euclidean space. Furthermore, the valuation functions must be regular Lipschitzian and monotonically increasing in all arguments.

Milgrom and Segal (2002) show that revenue equivalence is a consequence of a particular envelope theorem in a setting where the type spaces are one-dimensional and the outcome space is arbitrary. An agent’s valuation function is assumed differentiable and absolutely continuous in the type of the agent and the partial derivative of the valuation function with respect to the type must satisfy a certain integrability condition. Their result can be applied to multi-dimensional type spaces as well. In this case the type spaces must be smoothly connected and the valuation functions must be differentiable with bounded gradient.

There are two papers that identify necessary as well as sufficient conditions for revenue equivalence to hold. If the outcome space is finite, Suijs (1996) characterizes type spaces and valuation functions for which utilitarian maximizers satisfy revenue equivalence. Chung and Olszewski (2006) characterize type spaces and valuation functions for which *every* implementable allocation rule satisfies revenue equivalence, again under the assumption of a finite outcome space. From their characterization, they derive sufficient conditions on the type spaces and valuation functions that generalize known results when the outcome space is countable or a probability distribution over a finite set of outcomes.

Here we describe conditions on both the allocation rule g and the type

space T that characterize when revenue equivalence holds.⁹ The characterization presented here differs from prior work in that a joint condition on the type spaces, the valuation functions *and* the implementable allocation rule is given that characterizes revenue equivalence. This characterization differs from the one in Chung and Olszewski (2006) in three ways. First, it holds for general outcome spaces. Second, it implies revenue equivalence in cases where their result does not apply. In fact, given agents' type spaces and valuation functions, several allocation rules may be implementable in dominant strategies, some of which satisfy revenue equivalence and some do not. In this case, the conditions on the type space and valuation functions from their paper obviously cannot hold. However, the present characterization can be used to determine which of the allocation rules do satisfy revenue equivalence. Third, the characterization in Chung and Olszewski (2006) is a corollary of the present one in the sense that their necessary and sufficient condition is naturally related to the graph theoretic interpretation of revenue equivalence.

Theorem 2.3.1 *Let $T \subseteq \mathbb{R}^\Gamma$ and g be DSIC. For every pair $\alpha, \beta \in \Gamma$ denote by $\pi(\alpha, \beta)$ the shortest path from α to β in Γ_g and by $\ell(\pi(\alpha, \beta))$ its length. Revenue equivalence holds for g on T iff, $\ell(\pi(\alpha, \beta)) + \ell(\pi(\beta, \alpha)) = 0$ for all $\alpha, \beta \in \Gamma$.*

Proof: Since g is DSIC, the network Γ_g has no negative length cycles. Therefore, shortest path lengths in Γ_g are well defined. Choose a $\theta \in \Gamma$ and set P_γ^θ equal to the length of the shortest path from θ to $\gamma \in \Gamma$. Notice that (g, P^θ) is DSIC for all choices of $\theta \in \Gamma$. Suppose revenue equivalence holds. Then for each $\mu, \nu \in \Gamma$, there is a constant c such that $P_\gamma^\mu - P_\gamma^\nu = c$ for all $\gamma \in \Gamma$. In particular, $c = P_\mu^\mu - P_\mu^\nu = -P_\mu^\nu$. Substituting $\mu = \alpha$ and $\nu = \theta$ we deduce that

$$P_\beta^\alpha = P_\beta^\theta - P_\alpha^\theta \quad (2.36)$$

for some $\theta \neq \alpha, \beta$. Similarly,

$$P_\alpha^\beta = P_\alpha^\theta - P_\beta^\theta. \quad (2.37)$$

Adding equation (2.36) to (2.37) we obtain:

$$\ell(\pi(\alpha, \beta)) + \ell(\pi(\beta, \alpha)) = P_\beta^\alpha + P_\alpha^\beta = 0.$$

Now suppose that $\ell(\pi(\alpha, \beta)) + \ell(\pi(\beta, \alpha)) = 0$ for all $\alpha, \beta \in \Gamma$. Consider the shortest path tree, T , rooted at node θ . This is the union of shortest

⁹It is based on Heydenreich, Müller, Uetz and Vohra (2007).

paths from θ to all $\gamma \in \Gamma$ in Γ_g . Pick any node α such that the arc $(\theta, \alpha) \in T$. Recall that $P_\alpha - P_\theta \leq \ell(\theta, \alpha)$. Similarly $P_\theta - P_\alpha \leq \pi(\alpha, \theta)$. Since $P_\theta = 0$ we have that $-\ell(\pi(\alpha, \theta)) \leq P_\alpha \leq \ell(\theta, \alpha)$. Since $\ell(\pi(\alpha, \beta)) + \ell(\pi(\beta, \alpha)) = 0$ it follows that $\ell(\theta, \alpha) + \ell(\pi(\alpha, \theta)) = 0$. Therefore $-\ell(\pi(\alpha, \theta)) = \ell(\theta, \alpha)$. Hence the value of P_α can only be $\ell(\theta, \alpha)$. Now repeat the argument for any β such that the arc $(\alpha, \beta) \in T$ and so on. Thus, there is only one payment rule P such that $P_\theta = 0$ and (g, P) is DSIC. ■

Theorem 2.3.1 imposes conditions on the pair (g, T) to conclude that revenue equivalence holds for g . Chung and Olszewski (2006) on the other hand characterize the type space for which *every* DSIC allocation rule satisfies revenue equivalence. It is an easy consequence of Theorem 2.3.1.

So that a comparison can be made we describe the condition on Type spaces introduced in Chung and Olszewski (2006). Let B_1, B_2 be disjoint subsets of Γ and $r : B_1 \cup B_2 \rightarrow \mathbb{R}$. For every $\epsilon > 0$, let

$$V_1(\epsilon) = \cup_{b \in B_1} \{t \in T : \forall_{a \in B_2} t_b - t_a > r(b) - r(a) + \epsilon\}$$

and

$$V_2(\epsilon) = \cup_{a \in B_2} \{t \in T : \forall_{b \in B_1} t_b - t_a < r(b) - r(a) - \epsilon\}.$$

Finally, $V_i = \cup_{\epsilon > 0} V_i(\epsilon)$. Observe that $V_1 \cap V_2 = \emptyset$. Call a set T **splittable** if there are B_1, B_2 and r such that T is a subset of $V_1 \cup V_2$ where $T \cap V_i \neq \emptyset$ for $i = 1, 2$.

Theorem 2.3.2 *Let g be DSIC and Γ finite. Then, the following two statements are equivalent.*

1. g satisfies revenue equivalence.
2. T is not splittable.

In Theorem 2.3.1 no assumption on the cardinality of Γ is made, whereas in Theorem 2.3.2, Γ is assumed finite. When Γ is not finite but of cardinality less than the continuum it is shown in Chung and Olszewski (2006) that item 2 of Theorem 2.3.2 implies revenue equivalence.

Now we show how Theorem 2.3.1 can be used to derive a sufficiency result. First a definition.

Call Γ_g **two-cycle connected** if for every partition $\Gamma_1 \cup \Gamma_2 = \Gamma$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1, \Gamma_2 \neq \emptyset$, there are $\alpha_1 \in \Gamma_1$ and $\alpha_2 \in \Gamma_2$ with $\ell(\alpha_1, \alpha_2) + \ell(\alpha_2, \alpha_1) = 0$.

Theorem 2.3.3 *Let g be DSIC. If Γ_g is two-cycle connected then g satisfies revenue equivalence.*

Proof: First, we show that if Γ_g is two-cycle connected, then any two nodes $\alpha, \beta \in \Gamma$ are connected in Γ_g by a finite path with nodes $\alpha = \alpha_0, \alpha_1, \dots, \alpha_k = \beta$ such that $\ell(\alpha_i, \alpha_{i+1}) + \ell(\alpha_{i+1}, \alpha_i) = 0$ for $i = 0, \dots, k-1$. Call such a path a zero-path. Suppose not. Then, there is a node $\alpha \in \Gamma$ that is not connected to all nodes in Γ by a zero-path with the described property. Define Γ_1 to be the set containing all nodes β that can be reached from α by a zero-path. Let $\Gamma_2 = \Gamma \setminus \Gamma_1$. By assumption $\Gamma_2 \neq \emptyset$. Since Γ_g is two-cycle connected, there are $\alpha_1 \in \Gamma_1$ and $\alpha_2 \in \Gamma_2$ with $\ell(\alpha, \alpha_2) + \ell(\alpha_2, \alpha_1) = 0$ contradicting the fact that $\alpha_2 \in \Gamma_2$.

Now, for any $\alpha, \beta \in \Gamma$, $\ell(\pi(\alpha, \beta))$ is bounded above by the length of a zero-path from α to β . Hence $\ell(\pi(\alpha, \beta)) \leq 0$. Similarly, $\ell(\pi(\beta, \alpha)) \leq 0$. Therefore $\ell(\pi(\alpha, \beta)) + \ell(\pi(\beta, \alpha)) \leq 0$. Since g is DSIC, all cycles in Γ_g have non-negative length. Therefore $\ell(\pi(\alpha, \beta)) + \ell(\pi(\beta, \alpha)) = 0$. The theorem now follows from Theorem 2.3.1. ■

Theorem 2.3.4 *Let Γ be finite and T a (topologically) connected subset of \mathbb{R}^k . Each agent's valuation function $v_i(a, \cdot)$ is a continuous function in the type of the agent for every $\gamma \in \Gamma$. Then, every onto, DSIC allocation rule $g: T^n \rightarrow \Gamma$ satisfies revenue equivalence.*

Proof: It is enough to show that Γ_g is two-cycle connected. We use the following fact from topology:

Let $T \subseteq \mathbb{R}^k$ be a connected set. Then any partition of T into subsets $T_1, T_2 \neq \emptyset$, $T_1 \cup T_2 = T$, $T_1 \cap T_2 = \emptyset$ satisfies $\overline{T_1} \cap \overline{T_2} \neq \emptyset$, where $\overline{T_i}$ is the closure of T_i in T .

Consider a single agent with type space T and valuation function v . Regard g as a function on T as before. Let $\Gamma_1 \cup \Gamma_2 = \Gamma$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1, \Gamma_2 \neq \emptyset$ be a partition of Γ . Then, $T = g^{-1}(\Gamma_1) \cup g^{-1}(\Gamma_2)$, $g^{-1}(\Gamma_1) \cap g^{-1}(\Gamma_2) = \emptyset$ is a partition of T and $g^{-1}(\Gamma_1), \overline{g^{-1}(\Gamma_2)} \neq \emptyset$, since g is onto. According to the fact above, there exists $t \in \overline{g^{-1}(\Gamma_1)} \cap \overline{g^{-1}(\Gamma_2)}$. Hence, there are sequences $(t_1^n) \subseteq g^{-1}(\Gamma_1)$ and $(t_2^n) \subseteq g^{-1}(\Gamma_2)$ with $\lim_{n \rightarrow \infty} t_1^n = \lim_{n \rightarrow \infty} t_2^n = t$. As Γ is finite, there must be $\alpha_1 \in \Gamma_1$ and $\alpha_2 \in \Gamma_2$ and subsequences $(t_1^{n_k}) \subseteq (t_1^n)$ and $(t_2^{n_m}) \subseteq (t_2^n)$ with $g(t_1^{n_k}) = \alpha_1$ for all k and $g(t_2^{n_m}) = \alpha_2$ for all m . Since v is continuous in the type,

$$\begin{aligned} 0 &= v(\alpha_2, t) - v(\alpha_1, t) - v(\alpha_2, t) + v(\alpha_1, t) \\ &= \lim_{n \rightarrow \infty} (v(\alpha_2, t_2^{n_m}) - v(\alpha_1, t_2^{n_m}) + v(\alpha_1, t_1^{n_k}) - v(\alpha_2, t_1^{n_k})). \end{aligned}$$

According to the definition of the arc length in Γ_g , the latter can be bounded from below as follows.

$$\lim_{n \rightarrow \infty} (v(\alpha_2, t_2^{n_m}) - v(\alpha_1, t_2^{n_m}) + v(\alpha_1, t_1^{n_k}) - v(\alpha_2, t_1^{n_k})) \geq \ell(\alpha_1, \alpha_2) + \ell(\alpha_2, \alpha_1) \geq 0.$$

The last inequality is true, since Γ_g has no negative cycles. Hence, all inequalities are equalities and $\ell(\alpha_1, \alpha_2) + \ell(\alpha_2, \alpha_1) = 0$. Consequently, Γ_g is two-cycle connected. ■

Notice that we cannot completely relax the continuity assumption, as the following example demonstrates.

Example 5 *Let there be one agent with type $t \in [0, 1]$ and two outcomes $A = \{a, b\}$. Let the agent's valuation be $v(a, t) = 1$, if $t < 1/2$ and $v(a, t) = 0$, if $t \geq 1/2$. Let $v(b, t) = 1/2$ for all t . That is, $v(a, \cdot)$ is discontinuous at $t = 1/2$. Let the allocation rule be the efficient one, i.e., $f(t) = a$ for $t < 1/2$ and $f(t) = b$ otherwise. Then dominant strategy incentive compatibility is equivalent to $1 - \pi_a \geq 1/2 - \pi_b$ and $1/2 - \pi_b \geq -\pi_a$, which is satisfied whenever $|\pi_a - \pi_b| \leq 1/2$. For instance, $\pi_a = \pi_b = 0$ or $\pi'_a = 1/2$, $\pi'_b = 0$ are two payment schemes that make f truthful, but π and π' do not differ by a constant.*

Chapter 3

Optimality

Here we focus on finding a BNIC mechanism that maximizes the expected revenue to the designer. Let \mathcal{F} be the set of feasible allocations of the resources amongst the agents and the auctioneer. Let T be a finite set of an agent's types (possibly multi-dimensional). The type of agent i will be denoted t^i . A collection of types one for each (of n agents) will be called a profile. Let T^n be the set of all profiles. A profile in which attention is to be focused on agent i will sometimes be written as (t^i, \mathbf{t}^{-i}) . If $\alpha \in \mathcal{F}$ then agent i assigns monetary value $u(\alpha|t^i)$ to the allocation $\alpha \in \mathcal{F}$.

Assume, for convenience, that types are selected independently from T according to a common distribution. Denote by f_t the probability that an agent i has type $t \in T$, i.e. $t^i = t$. Denote the probability of a profile $\mathbf{t}^{-i} \in T^{n-1}$ being realized by $\pi(\mathbf{t}^{-i})$.

Let P_{t^i} be the expected payment that an agent who announces type t^i must make. An allocation rule assigns to each member of T^n an element of \mathcal{F} .¹ If α is an allocation rule, write $\alpha_i[t^i, \mathbf{t}^{-i}]$ to denote the allocation to an agent i with type $t^i \in T$. The expected utility of an agent i with type t^i under this rule will be (under the assumption that agents announce truthfully)

$$E_{\mathbf{t}^{-i}}[u(\alpha_i[t^i, \mathbf{t}^{-i}]|t^i)] = \sum_{\mathbf{t}^{-i}} u(\alpha_i[t^i, \mathbf{t}^{-i}]|t^i)\pi(\mathbf{t}^{-i}).$$

The expected utility of agent with type t^i announcing \hat{t}_i ($\neq t^i$) under allocation rule α is

¹Strictly speaking one should allow an allocation rule to be randomized. We omit this possibility.

$$E_{\mathbf{t}^{-i}}[u(\alpha_i[\hat{t}_i, \mathbf{t}^{-i}]|t^i)] = \sum_{\mathbf{t}^{-i}} u(\alpha_i[\hat{t}_i, \mathbf{t}^{-i}]|t^i)\pi(\mathbf{t}^{-i}).$$

To ensure that an agent will report truthfully we impose BNIC for each agent i with type t^i :

$$E_{\mathbf{t}^{-i}}[u(\alpha_i[t^i, \mathbf{t}^{-i}]|t^i)] - P_{t^i} \geq E_{\mathbf{t}^{-i}}[u(\alpha_i[\hat{t}_i, \mathbf{t}^{-i}]|t^i)] - P_{\hat{t}_i} \quad \forall t^i, \hat{t}_i \in T.$$

To ensure that each agent has the incentive to participate we impose the (interim) individual rationality (IR) constraint:

$$E_{\mathbf{t}^{-i}}[v(\alpha_i[t^i, \mathbf{t}^{-i}]|t^i)] - P_{t^i} \geq 0 \quad \forall t^i \in T.$$

If we add a dummy type, t_0 which assigns utility 0 to all allocations, and set $P_{t_0} = 0$, we can fold the IR constraint into the BNIC constraint. So, from now on T contains the dummy type t_0 .

The auctioneer's problem is to maximize expected revenue subject to BNIC and IR. Expected revenue (normalized for population) is $\sum_{t^i \in T} f_{t^i} P_{t^i}$ (we can pick an arbitrary i in an anonymous auction). Fix an allocation rule α and let

$$R(\alpha) = \max_{P_{t^i}} \sum_{t^i \in T} f_{t^i} P_{t^i}$$

$$\text{s.t. } E_{\mathbf{t}^{-i}}[u(\alpha_i[t^i, \mathbf{t}^{-i}]|t^i)] - P_{t^i} \geq E_{\mathbf{t}^{-i}}[u(\alpha_i[\hat{t}_i, \mathbf{t}^{-i}]|t^i)] - P_{\hat{t}_i} \quad \forall t^i, \hat{t}_i \in T.$$

Call this program LP_α . If LP_α is infeasible set $R(\alpha) = -\infty$. Thus the auctioneer's problem is to find the allocation rule α that maximizes $R(\alpha)$.

One way to solve this optimization problem is to fix the allocation rule α . Then we can rewrite the BNIC constraint as follows:

$$P_t - P_{\hat{t}} \leq E_{\mathbf{t}^{-i}}[u(\alpha_i[t, \mathbf{t}^{-i}]|t)] - E_{\mathbf{t}^{-i}}[u(\alpha_i[\hat{t}, \mathbf{t}^{-i}]|t)] \quad \forall t \neq \hat{t}.$$

If there is a feasible solution to this system of inequalities then we can find payments that implement α in an incentive compatible way.

The inequality system has a natural network interpretation. Introduce one node for each type (the node corresponding to the dummy type will be the source) and to each directed edge (\hat{t}, t) , assign a length of $E_{\mathbf{t}^{-i}}[u(\alpha_i[t, \mathbf{t}^{-i}]|t)] - E_{\mathbf{t}^{-i}}[u(\alpha_i[\hat{t}, \mathbf{t}^{-i}]|t)]$. Then P_t is upper bounded by the length of the shortest path from the source to vertex i . The optimal choice of P_t would be to set it equal to the length of this shortest path. For fixed α , the optimization problem reduces to determining the shortest path tree (union of all shortest paths from source to all nodes) in this network. Edges on the shortest path tree correspond to binding BNIC constraints.

If the network has a negative cycle, it follows from Rochet's theorem that our original problem is infeasible. Thus, there is no set of payments to make the allocation rule α incentive compatible. To summarize, given an allocation rule we have a way of checking whether it can be implemented in an incentive compatible way. If it can, then the length of the shortest path to a node/type is an upper bound on the payment that can be charged that type while preserving incentive compatibility.

3.1 An Example

Consider the allocation of one good among two agents with types $\{0, 1, 2\}$, here 0 is the dummy type. Let $f_1 = f_2 = 1/2$ and $\pi(1) = \pi(2) = 1/2$ be the probabilities. Choose as the allocation rule the following: assign the object to the agent with highest type, in case of ties randomize the allocation '50-50'. The possible allocations are: agent 1 wins, agent 2 wins, agent 1 and 2 get 1/2 of the item, seller keeps it. For the valuations we have: an agent of type t^i who wins the item values it at t^i ; if she tied with her competitor and gets only a half she derives value $t^i/2$ if finally she loses or the seller keeps it, she gets value 0. Now to the computation of expected utility when honest. If $t^i = 1$ then

$$E_{\mathbf{t}^{-i}}[u(\alpha[1, \mathbf{t}^{-i}]|1)] = \sum_{\mathbf{t}^{-i} \in \{0,1,2\}} u(\alpha[1, \mathbf{t}^{-i}]|1)\pi(\mathbf{t}^{-i}) = \frac{1}{4},$$

and is equal to $\frac{3}{2}$ if $t^i = 2$.

Similarly we obtain for $E_{\mathbf{t}^{-i}}[u(\alpha[t, \mathbf{t}^{-i}]|t^i)]$:

$t^i \setminus t^j$	0	1	2
0	0	0	0
1	0	$\frac{1}{4}$	$\frac{3}{4}$
2	0	$\frac{1}{2}$	$\frac{3}{2}$

So we obtain for the optimization problem

$$\begin{aligned} \max \quad & \frac{1}{2}P_2 + \frac{1}{2}P_2 \\ \text{s.t.} \quad & P_1 - P_0 \leq \frac{1}{4}, \quad P_0 - P_1 \leq 0, \quad P_0 - P_2 \leq 0, \\ & P_2 - P_0 \leq \frac{3}{2}, \quad P_2 - P_1 \leq 1, \quad P_1 - P_2 \leq -\frac{1}{2}, \\ & P_0 = 0. \end{aligned}$$

The corresponding network is depicted in Figure 3.1. The dashed edges form the shortest path tree. Reading off the shortest path distances to nodes 1 and 2 yields $P_1 = 1/4$ and $P_2 = 5/4$. So the auctioneer can realize revenue of $3/4$ with this allocation rule.

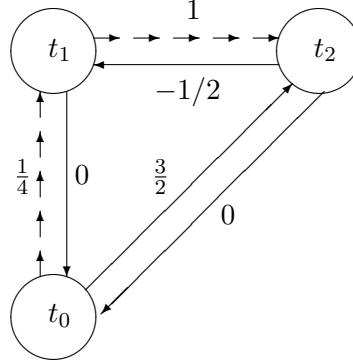


Figure 4.1

3.2 What is a Solution?

What exactly does it mean to solve an optimal auction problem? In our abstract set up, the problem of finding the optimal auction can be written as:

$$\begin{aligned} & \max_{P_{t^i}, \alpha} \sum_{t^i \in T} f_{t^i} P_{t^i} \\ \text{s.t. } & E_{\mathbf{t}^{-i}}[u(\alpha_i[t^i, \mathbf{t}^{-i}]|t^i)] - P_{t^i} \geq E_{\mathbf{t}^{-i}}[u(\alpha_i[\hat{t}_i, \mathbf{t}^{-i}]|t^i)] - P_{\hat{t}_i} \quad \forall t^i, \hat{t}_i \in T. \\ & \alpha(\mathbf{t}) \in \mathcal{F} \quad \forall \mathbf{t} \in T^n \end{aligned}$$

Call this problem (OAP). If membership in \mathcal{F} can be described by a set of linear inequalities, we have a linear program and its solution is the desired optimal auction. If this is what is meant by solving an optimal auction problem, the exercise is trivial. To formulate the problem in a non-trivial way, consider the problem of finding the optimal auction when the designer knows each agents type. Call this the **full information** problem. In this case, the BNIC constraints can be ignored. Only the IR and feasibility constraints matter. Given an allocation rule α we maximize expected revenue by charging each agent the value of what they receive, i.e.,

$$E_{\mathbf{t}^{-i}}[u(\alpha_i[t^i, \mathbf{t}^{-i}]|t^i)] = P_{t^i}.$$

Thus, in the full information case the problem of finding the optimal auction reduces to solving

$$\max_{\alpha} \sum_{t^i \in T} f_{t^i} E_{\mathbf{t}^{-i}} [u(\alpha_i[t^i, \mathbf{t}^{-i}] | t^i)]$$

$$\alpha(\mathbf{t}) \in \mathcal{F} \quad \forall \mathbf{t} \in T^n$$

The full information problem involves optimizing some function of α over \mathcal{F} , a potentially simpler problem.

We will say that problem (OAP) is solved if it can be reduced to an optimization problem that is of the same complexity as the full information problem.

3.3 One Dimensional Types

We will suppose that \mathcal{F} , the space of feasible allocation rules, is a subset of Euclidean space.² The type space is 1 dimensional and since we take it to be discrete, $T = \{1, 2, \dots, m\}$.

If \mathbf{a} is the allocation rule and \mathbf{t} a profile of types denote by $\mathbf{a}_i(t)$ the allocation to a type i in profile \mathbf{t} . Write \mathcal{A}_i to be the expected allocation that an agent with type i receives, i.e.,

$$\mathcal{A}_i = \sum_{\mathbf{t}^{-i} \in T^{n-1}} \mathbf{a}_i[i, \mathbf{t}^{-i}] \pi(\mathbf{t}^{-i}).$$

In an abuse of notation we write $v(\mathcal{A}_i|i)$ to mean the value that an agent of type i receives with a lottery over allocations with expected value \mathcal{A}_i . The approach taken is to identify inequalities that the \mathcal{A}_i 's must satisfy. This will yield a relaxation of the underlying optimization problem with the \mathcal{A}_i 's as decision variables. Subsequently we identify an allocation rule that will generate the expected allocations identified in the solution to the relaxation.

BNIC implies:

$$v(\mathcal{A}_i|i) - P_i \geq v(\mathcal{A}_j|i) - P_j.$$

We associate a network in the usual way with these BNIC constraints. Introduce a vertex i for each type i . Between every ordered pair of vertices (j, i) a directed edge of length $v(\mathcal{A}_i|i) - v(\mathcal{A}_j|i)$. The allocation rule will be BNIC iff. this network has no negative length cycle. if this network has no

²We can generalize a little further. For example, \mathcal{F} is a lattice, but overhead in notation is not worthwhile.

negative length cycles we can choose the P_i 's to be the length of the path from an arbitrarily chosen root vertex, r , to vertex i in the shortest path tree rooted at r . If in addition we want the IR constraint to hold, we would choose the root, r to be the vertex corresponding to the dummy type t_0 .

Theorem 3.3.1 *Suppose that v satisfies increasing differences. An allocation rule \mathbf{a} is BNIC iff it is monotonic. That is if $r \leq s$ then $\mathcal{A}_r \leq \mathcal{A}_s$.*

Proof: Same as Theorem 2.1.3. ■

If $i \geq j$ we refer to

$$v(\mathcal{A}_i|i) - P_i \geq v(\mathcal{A}_j|i) - P_j$$

as a downward BNIC constraint. If $i < j$ it is called an upward BNIC constraint. Next we show that ‘adjacent’ BNIC constraints suffice.

Theorem 3.3.2 *Suppose that v satisfies increasing differences. All BNIC constraints are implied by the following:*

$$v(\mathcal{A}_i|i) - P_i \geq v(\mathcal{A}_{i-1}|i) - P_{i-1} \quad \forall i = 1, \dots, m \quad (\text{BNIC}_i^d)$$

$$v(\mathcal{A}_i|i) - P_i \geq v(\mathcal{A}_{i+1}|i) - P_{i+1} \quad \forall i = 1, \dots, m-1 \quad (\text{BNIC}_i^u)$$

Proof: This also follows from the absence of negative cycles. If the network has no negative length cycles, then the length of the edge from i to $i+2$ must be at least as large as the length of $(i, i+1)$ plus the length of $(i+1, i+2)$.

■

In view of the above, our network can be depicted as shown in Figure 4.2.

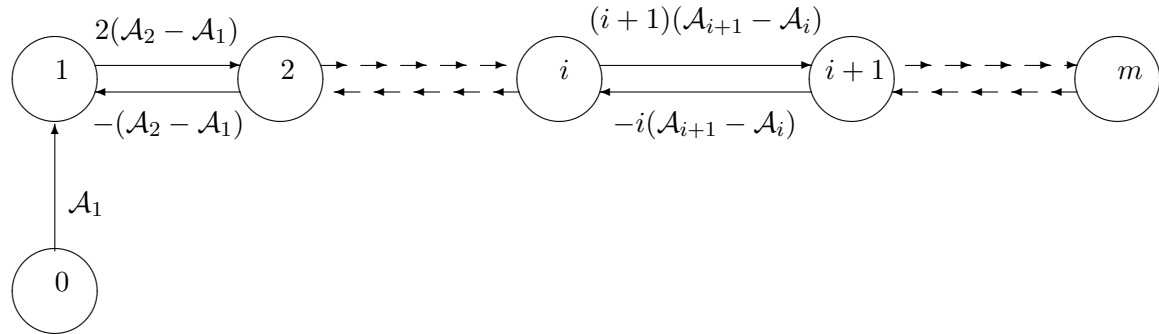


Figure 4.2

Suppose now that \mathcal{A} is BNIC. We know that the network of Figure 4.2 has no negative length cycles. It is easy to see that the shortest path tree

rooted at dummy vertex ‘0’ must be $0 \rightarrow 1 \rightarrow 2 \rightarrow \dots \rightarrow m$. Algebraically, we have set $\mathcal{A}_0 = 0$, and $P_0 = 0$ for the dummy type and

$$P_i = \sum_{r=1}^i [v(\mathcal{A}_r|r) - v(\mathcal{A}_{r-1}|r)]. \quad (3.1)$$

Notice that $P_i - P_{i-1} = v(\mathcal{A}_i|i) - v(\mathcal{A}_{i-1}|i)$ for the above expected payment schedule, hence all downward BNIC constraints are satisfied and bind, i.e.

$$v(\mathcal{A}_i|i) - P_i = v(\mathcal{A}_{i-1}|i) - P_{i-1} \quad \forall i = 1, \dots, m.$$

It is easy to see that the upward BNIC constraints all hold.

We now summarize our conclusions.

Theorem 3.3.3 *For any monotonic allocation rule \mathbf{a} there exists an expected payment schedule $\{P_i\}_{i=0}^m$, such that all the adjacent BNIC constraints are satisfied.*

Proof: Set $\mathcal{A}_r = 0$, and $P_0 = 0$ for the dummy type. Then set

$$P_i = \sum_{r=1}^i [v(\mathcal{A}_r|r) - v(\mathcal{A}_{r-1}|r)].$$

Notice that $P_i - P_{i-1} = v(\mathcal{A}_i|i) - v(\mathcal{A}_{i-1}|i)$ for the above expected payment schedule $\{P_i\}_{i=0}^m$, hence all (BNIC^d) are satisfied and bind, i.e.

$$v(\mathcal{A}_i|i) - P_i = v(\mathcal{A}_{i-1}|i) - P_{i-1} \quad \forall i = 1, \dots, m.$$

Hence all (BNIC^u) are satisfied, and thus by Theorem 3.3.2 the allocation rule \mathbf{a} is incentive compatible. ■

3.3.1 A Formulation

The problem of finding the optimal auction can be formulated as:

$$Z_1 = \max_{\{\mathbf{a}\}} \sum_{i=0}^m n f_i P_i \quad (OPT1)$$

$$\text{s.t. } v(\mathcal{A}_i|i) - P_i \geq v(\mathcal{A}_{i-1}|i) - P_{i-1} \quad \forall i = 1, \dots, m \quad (\text{BNIC}_i^d)$$

$$v(\mathcal{A}_i|i) - P_i \geq v(\mathcal{A}_{i+1}|i) - P_{i+1} \quad \forall i = 1, \dots, m-1 \quad (\text{BNIC}_i^u)$$

$$0 \leq \mathcal{A}_1 \leq \dots \leq \mathcal{A}_i \leq \dots \leq \mathcal{A}_m \quad (\text{M})$$

$$\mathcal{A}_i = \sum_{\mathbf{t}^{-i} \in T^{n-1}} \mathbf{a}_i[i, \mathbf{t}^{-i}] \pi(\mathbf{t}^{-i}) \quad (\text{A})$$

$$\mathbf{a}(\mathbf{t}) \in \mathcal{F} \quad \forall \mathbf{t} \in T^n \quad (\text{C})$$

An upper bound on each P_i is the length of the shortest path from the dummy node ‘0’ to vertex i in the network of figure 4.2. This is stated below.

Lemma 3.3.4 *All downward constraints ($BNIC_i^d$) bind in a solution to the [OPT1] problem.*

The essence of the previous result is that once the allocation rule is chosen, equation (3.1) pins down the payments necessary to ensure incentive compatibility. Our problem reduces to finding the optimal allocation rule.

Given equation (3.1), $Z_1 = \sum_{i=1}^m n f_i \sum_{r=1}^i [v(\mathcal{A}_r|r) - v(\mathcal{A}_{r-1}|r)]$. Write $F(i) = \sum_{r=1}^i f_r$ then $F(m) = 1$. Then

$$Z_1 = \sum_{i=1}^m n \{ f_i v(\mathcal{A}_i|i) + (1 - F(i)) [v(\mathcal{A}_i|i) - v(\mathcal{A}_i|i + 1)] \}.$$

We interpret $v(\mathcal{A}_i|m + 1)$ to be zero.

Let

$$\mu(\mathcal{A}_i) = v(\mathcal{A}_i|i) - \frac{1 - F(i)}{f_i} [v(\mathcal{A}_i|i + 1) - v(\mathcal{A}_i|i)].$$

The function μ is what Myerson (1981) calls the virtual valuation.

Problem [OPT1] becomes

$$Z_1 = \max_{\{\mathbf{a}\}} \sum_{i=1}^m n f_i \mu(\mathcal{A}_i)$$

$$\text{s.t. } 0 \leq \mathcal{A}_1 \leq \dots \leq \mathcal{A}_i \leq \dots \leq \mathcal{A}_m$$

$$\mathcal{A}_i = \sum_{\mathbf{t}^{-i} \in T^{n-1}} \mathbf{a}_i[i, \mathbf{t}^{-i}] \pi(\mathbf{t}^{-i})$$

$$\mathbf{a}(\mathbf{t}) \in \mathcal{F} \quad \forall \mathbf{t} \in T^n$$

3.3.2 The Myerson Case

Myerson(1981) investigates the optimal auction for the sale of a single unit to buyers with constant marginal values and where the distribution over types satisfies a monotone hazard rate condition. Specifically,

1. $v(q|i) = iq$, and
2. $\frac{1-F(i)}{f_i}$ is non-decreasing in i .

This second assumption is the monotone hazard condition. With these additional assumptions $\mu(\mathcal{A}_i) = \mathcal{A}_i \left(i - \frac{1-F(i)}{f_i} \right)$.

In this setup, \mathcal{A}_i is the expected quantity (or probability) of the good received by an agent who reports type i . Let $n_i(\mathbf{t})$ be the number of agents with type i in profile \mathbf{t} . The problem of finding the optimal auction can be formulated as:

$$\begin{aligned} Z_1 &= \max_{\{\mathbf{a}\}} \sum_{i=1}^m f_i \mathcal{A}_i \left(i - \frac{1-F(i)}{f_i} \right) \\ \text{s.t. } & 0 \leq \mathcal{A}_1 \leq \dots \leq \mathcal{A}_i \leq \dots \leq \mathcal{A}_m \\ & \mathcal{A}_i = \sum_{\mathbf{t}^{-i} \in T^{n-1}} \mathbf{a}_i[i, \mathbf{t}^{-i}] \pi(\mathbf{t}^{-i}) \\ & \sum_i n_i(\mathbf{t}) \mathbf{a}_i(\mathbf{t}) \leq 1 \quad \forall \mathbf{t} \in T^n \end{aligned}$$

We can rewrite this program to read:

$$\begin{aligned} & \max_{\{\mathbf{a}\}} \sum_i \sum_{\mathbf{t}^{-i} \in T^{n-1}} n f_i \mathbf{a}_i[t^i, \mathbf{t}^{-i}] \left(i + \frac{1-F(i)}{f_i} \right) \pi(\mathbf{t}^{-i}) \\ \text{s.t. } & 0 \leq \sum_{\mathbf{t}^{-i} \in T^{n-1}} \mathbf{a}_1[1, \mathbf{t}^{-i}] \pi(\mathbf{t}^{-i}) \leq \dots \leq \sum_{\mathbf{t}^{-i} \in T^{n-1}} \mathbf{a}_m[m, \mathbf{t}^{-i}] \pi(\mathbf{t}^{-i}) \leq 1 \\ & \sum_i n_i(\mathbf{t}) \mathbf{a}_i(\mathbf{t}) \leq K \quad \forall \mathbf{t} \in T^n \end{aligned}$$

If we ignore the monotonicity constraints, this problem can be decomposed into $|T^n|$ subproblems one for each profile of types:

$$\begin{aligned} & \max_{\{\mathbf{a}\}} \sum_i n_i(\mathbf{t}) \left(i - \frac{1-F(i)}{f_i} \right) \mathbf{a}_i(\mathbf{t}) \\ \text{s.t. } & \sum_i n_i(\mathbf{t}) \mathbf{a}_i(\mathbf{t}) \leq 1 \end{aligned}$$

This is an instance of a continuous knapsack problem. Its solution well known. Select any index $r \in \arg \max_{i: n_i(\mathbf{t}) > 0} \left\{ \frac{n_i(\mathbf{t})(i - \frac{1-F(i)})}{f_i} \right\}$ and set $a_r(\mathbf{t}) = 1/n_r(\mathbf{t})$. The monotone hazard condition ensures that the largest index is always chosen. Thus the solution to the program is monotonic, i.e. $\mathbf{a}_{i+1}(\mathbf{t}) \geq \mathbf{a}_i(\mathbf{t})$ for all i and profiles t . It follows from this that the ignored monotonicity constraints on expected allocations are satisfied.

The monotonicity constraints are useful because they allow us to restrict the space of possible allocation rules. However, their presence prevents us from decomposing the optimization problem into separate problems over profiles. The hazard rate condition is one sufficient condition for such a decomposition to be possible.

3.4 Multidimensional Types

In this section we show how the network approach can be used to identify optimal auctions when types are multidimensional. This section is based on joint work with Alexey Malakhov.

3.4.1 Wilson Example

One case where it is possible to find the optimal paths is in a discrete 2D analog of a continuous model first solved by Wilson (1993, Chapter 13). In Wilson's model customers are uniformly distributed on $\Omega = \{t \in \mathbb{R}_+^2, t_1 + t_2 \leq 1\}$, with utility $v(q|t) = q \cdot t$, and the seller has the cost $C(q) = \frac{\|q\|^2}{2}$. The objective is to maximize the seller's profit in a direct mechanism,

$$\max_{q, P} \int_{\Omega} P(t) - C(q(t)) dt \quad (\text{OPT-W})$$

$$\text{s.t. } v(q(t)|t) - P(t) \leq v(q(s)|t) - P(s) \quad \forall t, s \in \Omega \quad (\text{BNIC})$$

The solution to Wilson's problem is given by

$$q^*(t) = \frac{1}{2} \max \left(0, 3 - \frac{1}{\|t\|^2} \right) t. \quad (3.2)$$

Discrete Approach to the Problem

Let's solve the [OPT-W] problem in discrete polar coordinates using the network representation. Consider a discrete grid in polar coordinates (r, φ) ,

i.e.

$$\begin{aligned} t_1 &= r \cos \varphi, \\ t_2 &= r \sin \varphi. \end{aligned}$$

where $r \in \{r_1, \dots, r_n\}$, where $r_i = \frac{i}{n}$ and $\varphi \in \{0, \frac{\pi}{2k}, \frac{2\pi}{2k}, \dots, \frac{\pi}{2}\}$. Consider the direct mechanism approach with the allocation schedule given by

$$\begin{aligned} q_1(r, \varphi) &= R(r, \varphi) \cos \theta(r, \varphi), \\ q_2(r, \varphi) &= R(r, \varphi) \sin \theta(r, \varphi). \end{aligned}$$

The BNIC constraints then are

$$\begin{aligned} &R(r, \varphi) \cos \theta(r, \varphi) r \cos \varphi + R(r, \varphi) \sin \theta(r, \varphi) r \sin \varphi - P(r, \varphi) \geq \\ &\geq R(r', \varphi') \cos \theta(r', \varphi') r \cos \varphi + R(r', \varphi') \sin \theta(r', \varphi') r \sin \varphi - P(r', \varphi'). \end{aligned}$$

The cost function is given by

$$C(q) = \frac{R(r, \varphi)^2}{2}.$$

Our approach will be to conjecture that the optimal paths must be radial and then compute an optimal allocation for such a conjecture. This amounts to relaxing some of the BNIC constraints. We complete the argument by showing that the solution found satisfies the BNIC constraints that were relaxed.

Lemma 3.4.1 *If a payment $P(r_i, \varphi)$ is determined by a radial path $(0, \varphi) \longrightarrow (r_1, \varphi) \longrightarrow \dots \longrightarrow (r_i, \varphi)$, then the optimal allocations are given by*

$$\begin{aligned} q_1(r_i, \varphi) &= R(r_i, \varphi) \cos \varphi, \\ q_2(r_i, \varphi) &= R(r_i, \varphi) \sin \varphi, \end{aligned}$$

and the profit-maximizing payment $P(r_i, \varphi)$ is given by

$$P(r_i, \varphi) = \sum_{j=1}^i [r_j R(r_j, \varphi) - r_j R(r_{j-1}, \varphi)]. \quad (3.3)$$

Proof: The proof is done by induction in $r_i \in \{r_1, \dots, r_{n+1}\}$.

First, consider the case of $i = 1$, and $\varphi_j \in \{0, \frac{\pi}{2k}, \frac{2\pi}{2k}, \dots, \frac{\pi}{2}\}$.

If the payment is set through a radial path, i.e. $(0, \varphi_j) \longrightarrow (r_1, \varphi_j)$, then

$$P(r_1, \varphi_j) = R(r_1, \varphi_j) \cos \theta(r_1, \varphi_j) (r_1 \cos \varphi_j) + R(r_1, \varphi_j) \sin \theta(r_1, \varphi_j) (r_1 \sin \varphi_j),$$

and the profit is

$$\Pi_j = R(r_1, \varphi_j) \cos \theta(r_1, \varphi_j) (r_1 \cos \varphi_j) + R(r_1, \varphi_j) \sin \theta(r_1, \varphi_j) (r_1 \sin \varphi_j) - \frac{1}{2} R^2(r_1, \varphi_j).$$

Notice that the above profit along the path is maximized in θ when $\theta(r, \varphi_j) = \varphi_j$ (indeed, it does not affect the cost, while maximizing the revenue), hence

$$\begin{aligned} q_1(r_1, \varphi_j) &= R(r_1, \varphi_j) \cos \varphi_j, \\ q_2(r_1, \varphi_j) &= R(r_1, \varphi_j) \sin \varphi_j, \\ \Pi_j &= r_1 R(r_1, \varphi_j) - \frac{1}{2} R^2(r_1, \varphi_j), \end{aligned}$$

and

$$P(r_1, \varphi_j) = r_1 R(r_1, \varphi_j).$$

Now let's do the transition from i to $i + 1$. Assuming that

$$\begin{aligned} q_1(r_i, \varphi_j) &= R(r_i, \varphi_j) \cos \varphi_j, \\ q_2(r_i, \varphi_j) &= R(r_i, \varphi_j) \sin \varphi_j, \end{aligned}$$

$$P(r_i, \varphi) = \sum_{j=1}^i r_j R(r_j, \varphi) - r_j R(r_{j-1}, \varphi),$$

and that $P(r_{i+1}, \varphi_j)$ is determined by the path $(0, \varphi_j) \longrightarrow \dots \longrightarrow (r_i, \varphi_j) \longrightarrow (r_{i+1}, \varphi_j)$, we conclude

$$\begin{aligned} P(r_{i+1}, \varphi_j) &= R(r_{i+1}, \varphi_j) \cos \theta(r_{i+1}, \varphi_j) (r_{i+1} \cos \varphi_j) + R(r_{i+1}, \varphi_j) \sin \theta(r_{i+1}, \varphi_j) (r_{i+1} \sin \varphi_j) - \\ &\quad - r_{i+1} R(r_i, \varphi_j) + P(r_i, \varphi_j), \end{aligned}$$

and the profit along the path is

$$\begin{aligned} \Pi_j &= R(r_{i+1}, \varphi_j) \cos \theta(r_{i+1}, \varphi_j) (r_{i+1} \cos \varphi_j) + R(r_{i+1}, \varphi_j) \sin \theta(r_{i+1}, \varphi_j) (r_{i+1} \sin \varphi_j) - \\ &\quad - r_{i+1} R(r_i, \varphi_j) + P(r_i, \varphi_j) - \frac{1}{2} R^2(r_{i+1}, \varphi_j) + \\ &\quad + \sum_{l=0}^i \left[P(r_l, \varphi_j) - \frac{1}{2} R^2(r_l, \varphi_j) \right] \end{aligned}$$

By the same argument as in the case of r_1 , we conclude that the above profit is maximized when $\theta(r_{i+1}, \varphi_j) = \varphi_j$, hence

$$\begin{aligned} q_1(r_{i+1}, \varphi_j) &= R(r_{i+1}, \varphi_j) \cos \varphi_j, \\ q_2(r_{i+1}, \varphi_j) &= R(r_{i+1}, \varphi_j) \sin \varphi_j, \end{aligned}$$

and

$$P(r_{i+1}, \varphi) = \sum_{j=1}^{i+1} r_j R(r_j, \varphi) - r_j R(r_{j-1}, \varphi). \quad \blacksquare$$

Lemma 3.4.2 *If all profit-maximizing payments $P(r_i, \varphi)$ are determined by radial paths*

$(0, \varphi) \longrightarrow (r_1, \varphi) \longrightarrow \dots \longrightarrow (r_i, \varphi)$, *then optimal allocations are given by*

$$q_1(r_i, \varphi) = R(r_i) \cos \varphi, \quad (3.4)$$

$$q_2(r_i, \varphi) = R(r_i) \sin \varphi, \quad (3.5)$$

and payments $P(r_i, \varphi)$ do not depend on φ , i.e.

$$P(r_i, \varphi) = P(r_i) = \sum_{j=1}^i [r_j R(r_j) - r_j R(r_{j-1})]. \quad (3.6)$$

Proof: The proof is done by induction in $r_i \in \{r_1, \dots, r_{n+1}\}$.

First, consider the case of $i = 1$, and $\varphi \in \{0, \frac{\pi}{2k}, \frac{2\pi}{2k}, \dots, \frac{\pi}{2}\}$.

If the payments are set through radial paths, i.e. $(0, \varphi) \longrightarrow (r_1, \varphi)$, then Lemma 3.4.1 gives

$$q_1(r_1, \varphi) = R(r_1, \varphi) \cos \varphi,$$

$$q_2(r_1, \varphi) = R(r_1, \varphi) \sin \varphi,$$

$$P(r_1, \varphi) = r_1 R(r_1, \varphi),$$

and the profit is

$$\Pi = \sum_{j=0}^k r_1 R(r_1, \varphi_j) - \frac{1}{2} \sum_{j=0}^k R^2(r_1, \varphi_j).$$

Notice that the above profit is maximized in $R(r_1, \varphi)$ when

$$R(r_1, \varphi_j) = R(r_1) = \frac{1}{k+1} \sum_{j=1}^k R(r_1, \varphi_j), \quad (3.7)$$

since the allocation rule determined by (3.7) does not affect the revenue, while minimizing the cost..Hence

$$q_1(r_1, \varphi_j) = R(r_1) \cos \varphi_j,$$

$$q_2(r_1, \varphi_j) = R(r_1) \sin \varphi_j,$$

$$\Pi = \sum_{j=0}^k r_1 R(r_1) - \frac{1}{2} \sum_{j=0}^k R^2(r_1),$$

and

$$P(r_1, \varphi) = r_1 R(r_1).$$

Now let's do the transition from i to $i+1$. By the assumption of induction we have that

$$\begin{aligned} q_1(r_i, \varphi_j) &= R(r_i) \cos \varphi_j, \\ q_2(r_i, \varphi_j) &= R(r_i) \sin \varphi_j, \end{aligned}$$

$$P(r_i, \varphi) = \sum_{j=1}^i r_j R(r_j) - r_j R(r_{j-1}).$$

The assumption that $P(r_{i+1}, \varphi)$ is determined by the path $(0, \varphi) \longrightarrow \dots \longrightarrow (r_i, \varphi) \longrightarrow (r_{i+1}, \varphi)$, along with Lemma 3.4.1 give

$$\begin{aligned} q_1(r_{i+1}, \varphi) &= R(r_{i+1}, \varphi) \cos \varphi, \\ q_2(r_{i+1}, \varphi) &= R(r_{i+1}, \varphi) \sin \varphi, \end{aligned}$$

$$P(r_{i+1}, \varphi_i) = r_{i+1} R(r_{i+1}, \varphi_i) - r_{i+1} R(r_i) + P(r_i),$$

and the profit is

$$\begin{aligned} \Pi &= \sum_{l=0}^i \left[\sum_{j=0}^k P(r_l, \varphi) - \frac{1}{2} \sum_{j=0}^k R^2(r_l, \varphi_j) \right] + \\ &+ \sum_{j=0}^k [r_{i+1} R(r_{i+1}, \varphi_j) - r_{i+1} R(r_i, \varphi_j) + P(r_i, \varphi_j)] - \frac{1}{2} \sum_{j=0}^k R^2(r_{i+1}, \varphi_j). \end{aligned}$$

By the same argument as in the case of r_1 , we conclude that the above profit is maximized when

$$R(r_{i+1}, \varphi_j) = R(r_{i+1}) = \frac{1}{k+1} \sum_{j=1}^k R(r_{i+1}, \varphi_j),$$

hence

$$\begin{aligned} q_1(r_{i+1}, \varphi_j) &= R(r_{i+1}) \cos \varphi_j, \\ q_2(r_{i+1}, \varphi_j) &= R(r_{i+1}) \sin \varphi_j, \end{aligned}$$

and

$$P(r_i, \varphi) = P(r_i) = \sum_{j=1}^i [r_j R(r_j) - r_j R(r_{j-1})]. \quad \blacksquare$$

Lemma 3.4.3 *All profit-maximizing payments $P(r_i, \varphi)$ are determined by radial paths*

$$(0, \varphi) \longrightarrow (r_1, \varphi) \longrightarrow \dots \longrightarrow (r_i, \varphi).$$

Proof: The proof is done by induction in r_n .

First, consider the case of $i = 1$, and $\varphi \in \{0, \frac{\pi}{2k}, \frac{2\pi}{2k}, \dots, \frac{\pi}{2}\}$. Denote payments that are determined by radial paths $(0, \varphi) \longrightarrow (r_1, \varphi)$ as $P_r(r_1, \varphi)$, and payments that are determined by nonradial paths $(0, \varphi') \longrightarrow (r_1, \varphi') \longrightarrow (r_1, \varphi)$ as $P_{nr}(r_1, \varphi)$. Then

$$P_r(r_1, \varphi) = R(r_1, \varphi) \cos \theta(r_1, \varphi) (r_1 \cos \varphi) + R(r_1, \varphi) \sin \theta(r_1, \varphi) (r_1 \sin \varphi), \quad (3.8)$$

and

$$P_{nr}(r_1, \varphi) = R(r_1, \varphi) \cos \theta(r_1, \varphi) (r_1 \cos \varphi) + R(r_1, \varphi) \sin \theta(r_1, \varphi) (r_1 \sin \varphi) - R(r_1, \varphi') \cos \theta(r_1, \varphi') (r_1 \cos \varphi) - R(r_1, \varphi') \sin \theta(r_1, \varphi') (r_1 \sin \varphi) + P_r(r_1, \varphi').$$

Lemma 3.4.1 gives that $P_r(r_1, \varphi') = r_1 R(r_1, \varphi')$, hence

$$P_{nr}(r_1, \varphi) = R(r_1, \varphi) \cos \theta(r_1, \varphi) (r_1 \cos \varphi) + R(r_1, \varphi) \sin \theta(r_1, \varphi) (r_1 \sin \varphi) - R(r_1, \varphi') \cos \theta(r_1, \varphi') (r_1 \cos \varphi) - R(r_1, \varphi') \sin \theta(r_1, \varphi') (r_1 \sin \varphi) + r_1 R(r_1, \varphi'). \quad (3.9)$$

Combining (3.8) and (3.9) we obtain

$$P_{nr}(r_1, \varphi) = P_r(r_1, \varphi) + r_1 R(r_1, \varphi') (1 - \cos \theta(r_1, \varphi') \cos \varphi - \sin \theta(r_1, \varphi') \sin \varphi).$$

Finally, since $\cos \theta \cos \varphi + \sin \theta \sin \varphi < 1$ for $\forall \theta \neq \varphi$, we conclude that

$$P_{nr}(r_1, \varphi) \geq P_r(r_1, \varphi),$$

which proves that the radial path is the shorter one, and since payments are determined by shortest paths, profit-maximizing payments $P(r_1, \varphi)$ are determined by radial paths.

Now let's do the transition from i to $i + 1$. Assuming that $P(r_i, \varphi)$ are determined by radial paths by Lemma 3.4.2 we have

$$q_1(r_i, \varphi_j) = R(r_i) \cos \varphi_j, \quad (3.10)$$

$$q_2(r_i, \varphi_j) = R(r_i) \sin \varphi_j, \quad (3.11)$$

$$P(r_i, \varphi) = P(r_i) = \sum_{j=1}^i [r_j R(r_j) - r_j R(r_{j-1})]. \quad (3.12)$$

We now need to show that the payment $P_r(r_{i+1}, \varphi)$, determined by the radial path $(r_i, \varphi) \rightarrow (r_{i+1}, \varphi)$ is smaller than payments $P_{nr1}(r_{i+1}, \varphi)$ and $P_{nr2}(r_{i+1}, \varphi)$, that are determined by paths $(r_i, \varphi') \rightarrow (r_{i+1}, \varphi') \rightarrow (r_{i+1}, \varphi)$ and $(r_i, \varphi') \rightarrow (r_{i+1}, \varphi)$ respectively. Then using (3.10), (3.11), and (3.12), and without assuming anything about allocations at (r_{i+1}, φ) , we get

$$\begin{aligned} P_r(r_{i+1}, \varphi) &= R(r_{i+1}, \varphi) \cos \theta(r_{i+1}, \varphi) (r_{i+1} \cos \varphi) + R(r_{i+1}, \varphi) \sin \theta(r_{i+1}, \varphi) (r_{i+1} \sin \varphi) \\ &\quad - r_{i+1} R(r_i) + P_r(r_i). \end{aligned}$$

For the $P_{nr1}(r_{i+1}, \varphi)$ we have

$$\begin{aligned} P_{nr1}(r_{i+1}, \varphi) &= R(r_{i+1}, \varphi) \cos \theta(r_{i+1}, \varphi) (r_{i+1} \cos \varphi) + R(r_{i+1}, \varphi) \sin \theta(r_{i+1}, \varphi) (r_{i+1} \sin \varphi) - \\ &\quad - R(r_{i+1}, \varphi') \cos \theta(r_{i+1}, \varphi') (r_{i+1} \cos \varphi) - R(r_{i+1}, \varphi') \sin \theta(r_{i+1}, \varphi') (r_{i+1} \sin \varphi) + P_r(r_{i+1}, \varphi'). \end{aligned} \quad (3.13)$$

Now notice that for the $P_{nr1}(r_{i+1}, \varphi)$ to be determined by the shortest path, $P_r(r_{i+1}, \varphi')$ must be determined by the radial path, hence by Lemma 3.4.1

$$P_r(r_{i+1}, \varphi') = r_{i+1} R(r_{i+1}, \varphi') - r_{i+1} R(r_i) + P_r(r_i). \quad (3.14)$$

Combining (3.13) and (3.14) we obtain

$$\begin{aligned} P_{nr1}(r_{i+1}, \varphi) &= R(r_{i+1}, \varphi) \cos \theta(r_{i+1}, \varphi) (r_{i+1} \cos \varphi) + R(r_{i+1}, \varphi) \sin \theta(r_{i+1}, \varphi) (r_{i+1} \sin \varphi) - \\ &\quad - R(r_{i+1}, \varphi') \cos \theta(r_{i+1}, \varphi') (r_{i+1} \cos \varphi) - R(r_{i+1}, \varphi') \sin \theta(r_{i+1}, \varphi') (r_{i+1} \sin \varphi) + \\ &\quad + r_{i+1} R(r_{i+1}, \varphi') - r_{i+1} R(r_i) + P_r(r_i). \end{aligned} \quad (3.15)$$

Finally, (3.13) and (3.15) give

$$P_{nr1}(r_{i+1}, \varphi) = P_r(r_{i+1}, \varphi) + r_{i+1} R(r_{i+1}, \varphi') (1 - \cos \theta(r_{i+1}, \varphi') \cos \varphi - \sin \theta(r_{i+1}, \varphi') \sin \varphi),$$

and since $\cos \theta \cos \varphi + \sin \theta \sin \varphi < 1$ for $\forall \theta \neq \varphi$, we conclude that

$$P_{nr1}(r_{i+1}, \varphi) \geq P_r(r_{i+1}, \varphi). \quad (3.16)$$

For the $P_{nr2}(r_{i+1}, \varphi)$ we have

$$P_{nr1}(r_{i+1}, \varphi) = R(r_{i+1}, \varphi) \cos \theta(r_{i+1}, \varphi) (r_{i+1} \cos \varphi) + R(r_{i+1}, \varphi) \sin \theta(r_{i+1}, \varphi) (r_{i+1} \sin \varphi) -$$

$$-R(r_i, \varphi') \cos \theta(r_i, \varphi') (r_{i+1} \cos \varphi) - R(r_i, \varphi') \sin \theta(r_i, \varphi') (r_{i+1} \sin \varphi) + P_r(r_i, \varphi'). \quad (3.17)$$

Then using (3.10), (3.11), and (3.12), and without assuming anything about allocations at (r_{i+1}, φ) , we get

$$P_{nr1}(r_{i+1}, \varphi) = R(r_{i+1}, \varphi) \cos \theta(r_{i+1}, \varphi) (r_{i+1} \cos \varphi) + R(r_{i+1}, \varphi) \sin \theta(r_{i+1}, \varphi) (r_{i+1} \sin \varphi) - r_{i+1} R(r_i) + P_r(r_i). \quad (3.18)$$

Finally, (3.13) and (3.18) give

$$P_{nr1}(r_{i+1}, \varphi) = P_{nr1}(r_{i+1}, \varphi). \quad (3.19)$$

This inequality (3.16) and equality (3.18) prove that the radial path is the shortest one, and since payments are determined by shortest paths, all profit-maximizing payments $P(r_{i+1}, \varphi)$ are determined by radial paths. ■

Theorem 3.4.4 *Optimal allocations are given by*

$$\begin{aligned} q_1(r_i, \varphi) &= R(r_i) \cos \varphi, \\ q_2(r_i, \varphi) &= R(r_i) \sin \varphi, \end{aligned}$$

and payments $P(r_i, \varphi)$ do not depend on φ , i.e.

$$P(r_i, \varphi) = P(r_i) = \sum_{j=1}^i [r_j R(r_j) - r_j R(r_{j-1})].$$

Proof: Follows from Lemmas 3.4.2 and 3.4.3. ■

Theorem 3.4.4 allows us to successfully solve Wilson's optimization problem

in polar coordinates. The uniform probability density function changes from $g(t_1, t_2) = \frac{4}{\pi}$ to $g(r, \varphi) = 2r$ in continuous case, and to $g(i) = \frac{2i}{n(n+1)}$ and $F(i) = \frac{i(i+1)}{n(n+1)}$ in discrete case after the switch to the polar coordinates.

The problem [OPT-W] is now reduced to a standard one-dimensional profit maximization problem, which can be successfully solve by following Myerson's (1981) approach in a discrete case. Recall the general expression for the virtual valuation:

$$\mu(\mathcal{A}_i) = v(\mathcal{A}_i|i) - \frac{1 - F(i)}{f_i} [v(\mathcal{A}_i|i+1) - v(\mathcal{A}_i|i)],$$

which in our case looks like

$$\begin{aligned}\mu(R(i)) &= R(i) \frac{i}{n} - \frac{1 - F(i)}{g(i)} \left[R(i) \frac{i+1}{n} - R(i) \frac{i}{n} \right], \\ \mu(R(i)) &= R(i) \left(\frac{i}{n} - \frac{1 - F(i)}{ng(i)} \right).\end{aligned}$$

Hence the profit maximizing problem could be written as

$$\Pi = \max_{\{R(i)\}_{i=1}^n} \sum_{i=1}^n g(i) \left[R(i) \left(\frac{i}{n} - \frac{1 - F(i)}{ng(i)} \right) - C(R(i)) \right],$$

and can be solved type by type in the following formulation:

$$\begin{aligned}\max_{R(i)} R(i) \left(\frac{i}{n} - \frac{1 - F(i)}{ng(i)} \right) - C(R(i)), \\ \max_{R(i)} R(i) \left(\frac{i}{n} - \frac{1 - \frac{i(i+1)}{n(n+1)}}{\frac{2i}{(n+1)}} \right) - \frac{R^2(i)}{2}.\end{aligned}\tag{OPT-W'}$$

Solving [OPT-W'] we obtain the expression for the optimal choice of $R(i)$:

$$\begin{aligned}R(i) &= \max \left(0, \frac{2i^2 + i(i+1) - n(n+1)}{2in} \right), \\ R(i) &= \frac{n+1}{2i} \max \left(0, \frac{2i}{n+1} r_i + \frac{i+1}{n+1} r_i - 1 \right).\end{aligned}\tag{3.20}$$

The expression (3.20) is the exact discrete analog of Wilson's continuous solution to [OPT-W] given by (3.2).

3.4.2 Capacity Constrained Bidders

The type of an agent is a pair of numbers (a, b) . The first is her marginal value the second her capacity. Let the range of a be $R = \{1, 2, \dots, r\}$ and the range of b be $K = \{1, \dots, k\}$. Let f_{ij} be the probability that an agent has type (i, j) . Types are assumed to be independent. The value that an agent of type (i, j) assigns to q units will be written $v(q|i, j) = i(q, j)^-$. Observe that $v(q|i, j)$ satisfies increasing differences. That is, if $q' \geq q$ and $(i', j') \geq (i, j)$ then

$$v(q'|i', j') - v(q|i', j') \geq v(q'|i, j) - v(q|i, j).$$

Without loss of generality we can assume that the amount assigned to an agent who reports type (i, j) will be at most j . If \mathcal{A}_{ij} is the expected amount assigned to an agent who reports (i, j) then because this agent receives at most j in any allocation, his expected payoff is $i\mathcal{A}_{ij} = v(\mathcal{A}_{i,j}|i, j)$. Let P_{ij} be his expected payment if he reports (i, j) . Next, we discuss the relevant incentive compatibility constraints and identify the ones that are redundant.

The BNIC Constraints

There are 8 types of BNIC constraints listed below.

1. Horizontal Upward BNIC (HUBNIC)

Type (i, j) reporting (i', j) where $i' > i$:

$$v(\mathcal{A}_{i,j}|i, j) - P_{i,j} \geq v(\mathcal{A}_{i',j}|i, j) - P_{i',j}.$$

2. Horizontal Downward BNIC (HDBNIC)

Type (i, j) reporting (i', j) where $i' < i$:

$$v(\mathcal{A}_{i,j}|i, j) - P_{i,j} \geq v(\mathcal{A}_{i',j}|i, j) - P_{i',j}.$$

3. Vertical Upward BNIC (VUBNIC)

Type (i, j) reporting (i, j') where $j' > j$:

$$v(\mathcal{A}_{i,j}|i, j) - P_{i,j} \geq v(\mathcal{A}_{i,j'}|i, j) - P_{i,j'}.$$

4. Vertical Downward BNIC (VDBNIC)

Type (i, j) reporting (i, j') where $j' < j$:

$$v(\mathcal{A}_{i,j}|i, j) - P_{i,j} \geq v(\mathcal{A}_{i,j'}|i, j) - P_{i,j'}.$$

5. Diagonal Downward BNIC (DDBNIC)

Type (i, j) reporting (i', j') where $i' > i$ and $j' < j$:

$$v(\mathcal{A}_{i,j}|i, j) - P_{i,j} \geq v(\mathcal{A}_{i',j'}|i, j) - P_{i',j'}.$$

6. Diagonal Upward BNIC (DUBNIC)

Type (i, j) reporting (i', j') where $i' < i$ and $j' > j$:

$$v(\mathcal{A}_{i,j}|i, j) - P_{i,j} \geq v(\mathcal{A}_{i',j'}|i, j) - P_{i',j'}.$$

7. Leading Diagonal Downward BNIC (LDDBNIC)

Type (i, j) reporting (i', j') where $i' < i$ and $j' < j$:

$$v(\mathcal{A}_{i,j}|i, j) - P_{i,j} \geq v(\mathcal{A}_{i',j'}|i, j) - P_{i',j'}.$$

8. Leading Diagonal Upward BNIC (LDUBNIC)

Type (i, j) reporting (i', j') where $i' > i$ and $j' > j$:

$$v(\mathcal{A}_{i,j}|i, j) - P_{i,j} \geq v(\mathcal{A}_{i',j'}|i, j) - P_{i',j'}.$$

The new wrinkle that multi-dimensionality adds are the diagonal BNIC constraints. One might wish they are redundant. In general, they are not.

Simplification of Incentive Constraints

We now invoke the assumption that no agent is able to inflate their capacity. Under the no inflation assumption the VUBNIC, DUBNIC and LDUBNIC constraints can be thrown out.

Theorem 3.4.5 *An allocation rule satisfies HUBNIC and HDBNIC iff it is monotonic in the ‘i’ component. That is, for all $i \geq i'$ we have $\mathcal{A}_{ij} \geq \mathcal{A}_{i',j}$.*

The proof is standard and omitted. The absence of VUBNIC means that a similar monotonicity result does not hold for the ‘j’ argument. In other words it is not true that $\mathcal{A}_{ij} \geq \mathcal{A}_{i,j'}$ when $j \geq j'$. This is because both upward and downward incentive constraints are needed to ensure monotonicity of an allocation rule. However, we will assume monotonicity in both components of the allocation rule.

We now show that the two remaining diagonal BNIC constraints are redundant.

Theorem 3.4.6 *The LDDBNIC constraints relating type (i', j') to (i, j) where $(i', j') \geq (i, j)$:*

$$v(\mathcal{A}_{i',j'}|i', j') - P_{i',j'} \geq v(\mathcal{A}_{i,j}|i', j') - P_{i,j}, \quad (3.21)$$

are implied by HDBNIC, VDBNIC and monotonicity of the allocation rule.

Proof: To see that (3.21) is implied by HDBNIC and VDBNIC consider:

$$v(\mathcal{A}_{i',j'}|i', j') - P_{i',j'} \geq v(\mathcal{A}_{i,j'}|i', j') - P_{i,j'}$$

and

$$v(\mathcal{A}_{i,j'}|i, j') - P_{i,j'} \geq v(\mathcal{A}_{i,j}|i, j') - P_{i,j}.$$

Adding these two inequalities together yields:

$$v(\mathcal{A}_{i',j'}|i', j') - P_{i',j'} + v(\mathcal{A}_{i,j'}|i, j') \geq v(\mathcal{A}_{i,j'}|i', j') + v(\mathcal{A}_{i,j}|i, j') - P_{i,j}.$$

Rearranging:

$$\begin{aligned} & v(\mathcal{A}_{i',j'}|i',j') - v(\mathcal{A}_{i,j}|i',j') - [P_{i',j'} - P_{i,j}] \geq \\ & \geq [v(\mathcal{A}_{i,j'}|i',j') - v(\mathcal{A}_{i,j}|i',j')] - [v(\mathcal{A}_{i,j'}|i,j') - v(\mathcal{A}_{i,j}|i,j')]. \end{aligned}$$

The increasing differences property and the monotonicity of the allocation imply that

$$[v(\mathcal{A}_{i,j'}|i',j') - v(\mathcal{A}_{i,j}|i',j')] - [v(\mathcal{A}_{i,j'}|i,j') - v(\mathcal{A}_{i,j}|i,j')] \geq 0,$$

and hence we have that

$$v(\mathcal{A}_{i',j'}|i',j') - v(\mathcal{A}_{i,j}|i',j') \geq P_{i',j'} - P_{i,j}$$

which is equivalent to (3.21), and proves the claim. \blacksquare

Theorem 3.4.7 *Under the assumption that no agent can inflate their capacity the DDBNIC constraints are redundant.*

Proof: Let $i' \geq i$ and $j' \geq j$. Consider the following DDBNIC constraint:

$$v(\mathcal{A}_{i,j'}|i,j') - P_{i,j'} \geq v(\mathcal{A}_{i',j}|i',j') - P_{i',j}.$$

When we substitute in our expression for v we obtain:

$$i\mathcal{A}_{i,j'} - P_{i,j'} \geq i\mathcal{A}_{i',j} - P_{i',j}. \quad (3.22)$$

We show that it is implied by the addition of the following VDBNIC and HUBNIC constraints:

$$v(\mathcal{A}_{i,j'}|i,j') - P_{i,j'} \geq v(\mathcal{A}_{i,j}|i,j') - P_{i,j}, \quad (3.23)$$

$$v(\mathcal{A}_{i,j}|i,j) - P_{i,j} \geq v(\mathcal{A}_{i',j}|i,j) - P_{i',j}. \quad (3.24)$$

Adding (3.23) and (3.24) yields:

$$v(\mathcal{A}_{i,j'}|i,j') - P_{i,j'} + v(\mathcal{A}_{i,j}|i,j) \geq v(\mathcal{A}_{i,j}|i,j') + v(\mathcal{A}_{i',j}|i,j) - P_{i',j}. \quad (3.25)$$

Substitute in our expression for v :

$$i\mathcal{A}_{i,j'} - P_{i,j'} + i\mathcal{A}_{i,j} \geq i\mathcal{A}_{i,j} + i\mathcal{A}_{i',j} - P_{i',j}.$$

Cancelling common terms yields (3.22). \blacksquare

Theorem 3.4.8 *Only the adjacent downward constraints w.r.t. $(i + 1, j)$ and (i, j) and w.r.t. $(i, j + 1)$ and (i, j) matter out of all horizontal and vertical downward constraints. Only the adjacent upward constraints w.r.t. (i, j) and $(i + 1, j)$, w.r.t. (i, j) and $(i, j + 1)$ matter out of all horizontal and vertical upward constraints.*

The proof is standard and omitted.

Theorem 3.4.9 *If an adjacent HDBNIC or VDBNIC constraint binds, the corresponding adjacent upward BNIC constraint are satisfied.*

Proof: Given that the corresponding downward adjacent BNIC constraint binds, i.e. then upward BNIC constraints are satisfied. Indeed, if

$$v(\mathcal{A}_{i+1,j}|i+1, j) - v(\mathcal{A}_{i,j}|i+1, j) = P_{i+1,j} - P_{i,j},$$

then by increasing differences and monotonicity of the allocation rule:

$$v(\mathcal{A}_{i+1,j}|i, j) - v(\mathcal{A}_{i,j}|i, j) \leq P_{i+1,j} - P_{i,j},$$

which is the corresponding upward constraint $v(\mathcal{A}_{i,j}|i, j) - P_{i,j} \geq v(\mathcal{A}_{i+1,j}|i, j) - P_{i+1,j}$. So the corresponding adjacent upward BNIC constraint is satisfied. The argument is exactly the same w.r.t. the j dimension. ■

When an adjacent HDBNIC constraint does not bind, the corresponding adjacent HUBNIC constraint is not automatically satisfied. Also some of the adjacent downward BNIC constraints will be slack since not all arcs are likely to be used in a shortest path tree.

Summarizing, the only BNIC constraints that matter are the adjacent HUBNIC, HDBNIC and VDBNIC.

Optimal Auction Formulation and Solution

Denote by $\alpha_{ij}[\mathbf{t}]$ the *actual* allocation that a type (i, j) will receive under allocation rule \mathcal{A} when the announced profile is \mathbf{t} . In the case when type (i, j) does not appear in the profile \mathbf{t} we take $\alpha_{ij}[\mathbf{t}] = 0$. We will have cause to study how the allocation for an agent with a given type, (i, j) say, will change when the types of the other $n - 1$ agents change. In these cases we will write $\alpha_{ij}(\mathbf{t})$ as $\alpha_{ij}[(i, j), \mathbf{t}^{-i}]$. Then $\mathcal{A}_{ij} = \sum_{\mathbf{t}^{-i}} \pi(\mathbf{t}^{-i}) \alpha_{ij}[(i, j), \mathbf{t}^{-i}]$ where $\pi(\mathbf{t}^{-i})$ is the probability of profile \mathbf{t}^{-i} being realized and the sum is over all possible profiles. Let $n_{ij}(\mathbf{t})$ denote the fraction of bidders in the profile \mathbf{t} with type (i, j) .

We now formulate the problem of finding the revenue maximizing monotone mechanism as a linear program in the following way. We omit all diagonal BNIC constraints as well as all the upward vertical and horizontal BNIC constraints. The HUBNIC constraints are the only ones we have not shown to be redundant. However, Theorem 3.4.9 ensures that HUBNIC will be satisfied. The problem we study is $[OPT]$ is:

$$\begin{aligned}
Z &= \max_{P_{ij}} \sum_{i \in R} \sum_{j \in K} f_{ij} P_{ij} \\
\text{s.t. } & v(\mathcal{A}_{ij}|i, j) - P_{ij} \geq v(\mathcal{A}_{i-1, j}|i, j) - P_{i-1, j} \\
& v(\mathcal{A}_{ij}|i, j) - P_{ij} \geq v(\mathcal{A}_{i, j-1}|i, j) - P_{i, j-1} \\
& \mathcal{A}_{ij} \geq \mathcal{A}_{i'j'} \quad \forall (i, j) \geq (i', j') \\
& \mathcal{A}_{ij} = \sum_{\mathbf{t}^{-i}} \pi(\mathbf{t}^{-i}) \alpha_{ij}[(i, j), \mathbf{t}^{-i}] \quad \forall (i, j) \\
& \sum_{i \in R} \sum_{j \in K} n n_{ij}(\mathbf{t}) \alpha_{ij}[\mathbf{t}] \leq Q \quad \forall \mathbf{t} \\
& \alpha_{ij}[\mathbf{t}] \leq j \quad \forall i, j \quad \forall \mathbf{t}
\end{aligned}$$

To describe the network representation of this linear program fix the \mathcal{A}_{ij} 's. For each type (i, j) introduce a node including the dummy type $(0, 0)$. For each pair $(i, j), (i+1, j)$ introduce a directed arc from (i, j) to $(i+1, j)$ with length $v(\mathcal{A}_{i+1, j}|i+1, j) - v(\mathcal{A}_{ij}|i+1, j) = (i+1)\mathcal{A}_{i+1, j} - (i+1)\mathcal{A}_{ij}$. Similarly a directed arc from (i, j) to $(i, j+1)$ of length $i\mathcal{A}_{i, j+1} - i\mathcal{A}_{ij}$. Then P_{ij} will be the length of the shortest path from the dummy type $(0, 0)$ to (i, j) . We show that the shortest path from $(1, 1)$ to (i, j) is $(1, 1) \rightarrow (1, 2) \rightarrow (1, 3) \dots \rightarrow (1, j) \rightarrow (2, j) \dots \rightarrow (i, j)$. An example of such paths is provided in figure 4.3 below:

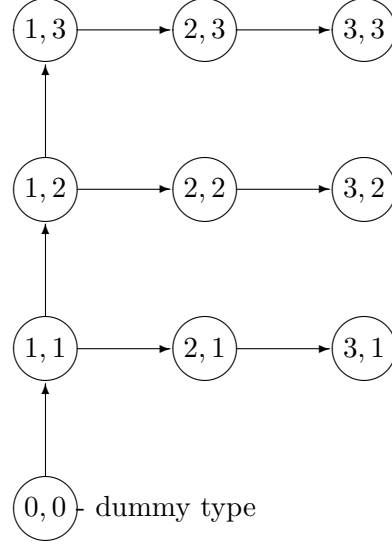


Figure 4.3

Theorem 3.4.10

$$P_{ij} = v(\mathcal{A}_{ij}|i, j) - \sum_{r=1}^{i-1} [v(\mathcal{A}_{rj}|r+1, j) - v(\mathcal{A}_{rj}|r, j)] = \sum_{r=1}^i r(\mathcal{A}_{rj} - \mathcal{A}_{r-1,j}) - \mathcal{A}_{11} + P_{11}.$$

Proof: It suffices to show that the shortest path from $(1, 1)$ to (i, j) is straight up and across. The proof is by induction. It is clearly true for nodes $(1, 2)$ and $(2, 1)$. Consider the node $(2, 2)$. The length of $(1, 1) \rightarrow (2, 1) \rightarrow (2, 2)$ is

$$2\mathcal{A}_{22} - 2\mathcal{A}_{21} + 2\mathcal{A}_{21} - 2\mathcal{A}_{11} + P_{11} = 2\mathcal{A}_{22} - 2\mathcal{A}_{11} + P_{11}.$$

The length of the path $(1, 1) \rightarrow (1, 2) \rightarrow (2, 2)$ is

$$2\mathcal{A}_{22} - 2\mathcal{A}_{1,2} + \mathcal{A}_{1,2} - \mathcal{A}_{11} + P_{11} = 2\mathcal{A}_{22} - \mathcal{A}_{12} - \mathcal{A}_{11} + P_{11}.$$

The difference in length between the first and the second path is

$$(2\mathcal{A}_{22} - 2\mathcal{A}_{11}) - (2\mathcal{A}_{22} - \mathcal{A}_{12} - \mathcal{A}_{11}) = \mathcal{A}_{12} - \mathcal{A}_{11} \geq 0,$$

where the last inequality follows by monotonicity of the \mathcal{A} 's.

Now suppose the claim is true for all nodes (i, j) where $i, j \leq n-1$. The shortest path from $(1, 1)$ to $(1, n)$ is clearly up the top. A similar argument to

the previous one shows that the shortest path from $(1, 1)$ to $(2, n)$ is also up the top and across. Consider now node $(3, n)$. There are two candidates for a shortest path from $(1, 1)$ to $(3, n)$. One is $(1, 1) \rightarrow (1, n-1) \rightarrow (3, n-1) \rightarrow (3, n)$. This path has length

$$3\mathcal{A}_{3n} - 3\mathcal{A}_{3,n-1} + 3\mathcal{A}_{3,n-1} - 3\mathcal{A}_{2,n-1} + 2\mathcal{A}_{2,n-1} - 2\mathcal{A}_{1,n-1} + P_{1,n-1} = 3\mathcal{A}_{3n} - \mathcal{A}_{2,n-1} - 2\mathcal{A}_{1,n-1} + P_{1,n-1}.$$

The other path, $(1, 1) \rightarrow (1, n) \rightarrow (3, n)$ has length

$$3\mathcal{A}_{3n} - 3\mathcal{A}_{2n} + 2\mathcal{A}_{2n} - 2\mathcal{A}_{1,n} + \mathcal{A}_{1,n} - \mathcal{A}_{1,n-1} + P_{1,n-1} = 3\mathcal{A}_{3n} - \mathcal{A}_{2n} - \mathcal{A}_{1,n} - \mathcal{A}_{1,n-1} + P_{1,n-1}.$$

The difference in length between the first and second path is

$$\mathcal{A}_{2n} + \mathcal{A}_{1n} + \mathcal{A}_{1,n-1} - \mathcal{A}_{2,n-1} - 2\mathcal{A}_{1,n-1} = \mathcal{A}_{2n} + \mathcal{A}_{1n} - \mathcal{A}_{2,n-1} - \mathcal{A}_{1,n-1} \geq 0.$$

Again the last inequality follows by monotonicity of the \mathcal{A} 's.

Proceeding inductively in this way we can establish the claim for nodes of the form (i, n) where $i \leq n-1$ and for (n, j) where $j \leq n-1$. It remains then to prove the claim for node (n, n) . One path is $(1, n-1) \rightarrow (n, n-1) \rightarrow (n, n)$ and has length

$$\begin{aligned} n\mathcal{A}_{nn} - n\mathcal{A}_{n,n-1} + n\mathcal{A}_{n,n-1} - n\mathcal{A}_{n-1,n-1} + (n-1)\mathcal{A}_{n-1,n-1} - (n-1)\mathcal{A}_{n-2,n-1} + \dots + P_{1,n-1} \\ = n\mathcal{A}_{nn} - \mathcal{A}_{n-1,n-1} - \mathcal{A}_{n-2,n-1} - \dots + P_{1,n-1}. \end{aligned}$$

The length of the other path, $(1, 1) \rightarrow (1, n) \rightarrow (n, n)$ is

$$n\mathcal{A}_{nn} - n\mathcal{A}_{n-1,n} + (n-1)\mathcal{A}_{n-1,n} - (n-1)\mathcal{A}_{n-2,n} + \dots + \mathcal{A}_{1n} - \mathcal{A}_{1,n-1} + P_{1,n-1}.$$

Again, by the monotonicity of the \mathcal{A} 's the second path is shorter than the first. \blacksquare

Substituting the expression for P_{ij} derived into the objective function for problem [OPT] we get

$$\sum_{j=1}^k \sum_{i=1}^n \{f_{ij}v(\mathcal{A}_{ij}|i, j) + (1 - F_j(i))[v(\mathcal{A}_{ij}|i, j) - v(\mathcal{A}_{ij}|i + 1, j)]\}$$

where $F_j(i) = \sum_{t=1}^i f_{tj}$. The expression within the summation term can be rewritten as

$$f_{ij}\{v(\mathcal{A}_{ij}|i, j) + \frac{1 - F_j(i)}{f_{ij}}[v(\mathcal{A}_{ij}|i, j) - v(\mathcal{A}_{ij}|i + 1, j)]\}.$$

he particular functional form of v allows us to rewrite this as:

$$f_{ij} \mathcal{A}_{ij} \left(i - \frac{1 - F_j(i)}{f_{ij}} \right).$$

Following Myerson, we can think of the term $\left(i - \frac{1 - F_j(i)}{f_{ij}} \right)$ as the virtual valuation conditional on wanting to consume at most j units. Problem [OPT] becomes:

$$\begin{aligned} Z &= \max_{\{\mathbf{a}\}} \sum_i \sum_j f_{ij} \mathcal{A}_{ij} \left(i - \frac{1 - F_j(i)}{f_{ij}} \right) \\ \text{s.t. } &\mathcal{A}_{ij} \geq \mathcal{A}_{i'j'} \quad \forall (i, j) \geq (i', j') \\ \mathcal{A}_{ij} &= \sum_{\mathbf{t}^{-i}} \pi(\mathbf{t}^{-i}) \alpha_{ij}[(i, j), \mathbf{t}^{-i}] \quad \forall (i, j) \\ \sum_{i \in R} \sum_{j \in K} n n_{ij}(\mathbf{t}) \alpha_{ij}[\mathbf{t}] &\leq Q \quad \forall \mathbf{t} \\ \alpha_{ij}[\mathbf{t}] &\leq j \quad \forall i, j \quad \forall \mathbf{t} \end{aligned}$$

Substituting out the \mathcal{A}_{ij} variables yields:

$$\begin{aligned} Z &= \max_{\{\mathbf{a}\}} \sum_i \sum_j f_{ij} \sum_{\mathbf{t}^{-i}} \pi(\mathbf{t}^{-i}) \alpha_{ij}[(i, j), \mathbf{t}^{-i}] \left(i - \frac{1 - F_j(i)}{f_{ij}} \right) \\ \text{s.t. } &\sum_{\mathbf{t}^{-i}} \pi(\mathbf{t}^{-i}) \alpha_{ij}[(i, j), \mathbf{t}^{-i}] \geq \sum_{\mathbf{t}^{-i'}} \pi(\mathbf{t}^{-i'}) \alpha_{i'j'}[(i', j'), \mathbf{t}^{-i'}] \quad \forall (i, j) \geq (i', j') \\ &\sum_{i \in R} \sum_{j \in K} n n_{ij}(\mathbf{t}) \alpha_{ij}[(i, j), \mathbf{t}^{-i}] \leq Q \quad \forall \mathbf{t} \\ &\alpha_{ij}[\mathbf{t}] \leq j \quad \forall i, j \quad \forall \mathbf{t} \end{aligned}$$

Theorem 3.4.11 *Suppose the conditional virtual values are monotone, i.e.,*

$$i - \frac{1 - F_j(i)}{f_{ij}} \geq i' - \frac{1 - F_{j'}(i')}{f_{i'j'}} \quad \forall (i, j) \geq (i', j').$$

Then the following procedure describes an optimal solution to problem [OPT]. Select the pair (i, j) which maximizes $i - \frac{1 - F_j(i)}{f_{ij}}$ and increase α_{ij} until it reaches its upper bound or the supply is exhausted, whichever comes first. If the supply is not exhausted repeat.

Proof: If we ignore the monotonicity condition

$$\sum_{\mathbf{t}^{-i}} \pi(\mathbf{t}^{-i}) \alpha_{ij}[(i, j), \mathbf{t}^{-i}] \geq \sum_{\mathbf{t}^{-i}} \pi(\mathbf{t}^{-i}) \alpha_{i'j'}[(i', j'), \mathbf{t}^{-i}],$$

the optimization problem reduces to a collection of optimization problems one for each profile \mathbf{t} :

$$\begin{aligned} Z(\mathbf{t}) &= n \max_{\{\mathbf{a}\}} \sum_i \sum_j n_{ij}(\mathbf{t}) \alpha_{ij}[\mathbf{t}] \left(i - \frac{1 - F_j(i)}{f_{ij}} \right) \\ \text{s.t. } &\sum_{i \in R} \sum_{j \in K} n n_{ij}(\mathbf{t}) \alpha_{ij}[\mathbf{t}] \leq Q \\ &\alpha_{ij}[\mathbf{t}] \leq j \quad \forall i, j \quad \forall \mathbf{t} \end{aligned}$$

This is an instance of a continuous knapsack problem with upper bound constraints on the variables which can be solved in the usual greedy manner. In each profile allocate as much as possible to the agents with highest conditional virtual values. Once they are saturated, proceed to the next highest and so on. Monotonicity of the conditional virtual values ensures that the resulting solution satisfies the omitted monotonicity constraint. ■

Requiring the conditional virtual values to be monotone is clearly a more demanding requirement than the analogous requirement when types are one dimensional. If $\frac{1 - F_j(i)}{f_{ij}}$ is non-increasing in (i, j) then the conditional virtual values are monotone. This condition can be interpreted as a type of affiliation. One example of a distribution on types that yields monotone conditional virtual values is when each component of the type is independent of the other and the i component is drawn from a distribution that satisfies the monotone hazard condition while the j component is drawn from a uniform distribution. When the conditional virtual values are not monotone, one can solve the problem using ‘ironed’ conditional virtual values. The details are omitted. It is easy to see that solution to the continuous case is exactly as described in Theorem 3.4.11.

Monotonicity and the Conditional Virtual Values

The Theorems above identify the optimal mechanism from amongst a restricted class of mechanisms. One where the associated allocation rule is monotone. In this section we drop this restriction. When the conditional

virtual values are monotone, we show that there is an optimal mechanism whose associated allocation rule is monotone.

The argument is in two steps. In the first step we upper bound the value of Z in $[OPT]$ by using a relaxation.

Lemma 3.4.12 *Fix an allocation rule \mathcal{A} . The expected revenue from the allocation rule \mathcal{A} is bounded above by*

$$\sum_{i \in R} \sum_{j \in K} f_{ij} \left(i - \frac{1 - F_j(i)}{f_{ij}} \right) \mathcal{A}_{ij}.$$

Proof: Consider the following relaxation of $[OPT]$, called $[rOPT]$, where the choice of \mathcal{A} is fixed.

$$\begin{aligned} Z(\mathcal{A}) &= \max_{P_{ij}} \sum_{i \in R} \sum_{j \in K} f_{ij} P_{ij} \\ \text{s.t. } v(\mathcal{A}_{ij}|i, j) - P_{ij} &\geq v(\mathcal{A}_{i-1,j}|i, j) - P_{i-1,j} & (3.26) \\ \mathcal{A}_{ij} &= \sum_{\mathbf{t}^{-i}} \pi(\mathbf{t}^{-i}) \alpha_{ij}[(i, j), \mathbf{t}^{-i}] \quad \forall (i, j) \\ \sum_{i \in R} \sum_{j \in K} n n_{ij}(\mathbf{t}) \alpha_{ij}[\mathbf{t}] &\leq Q \quad \forall \mathbf{t} \\ \alpha_{ij}[\mathbf{t}] &\leq j \quad \forall i, j \quad \forall \mathbf{t} \end{aligned}$$

To prove the lemma it suffices to show that $Z(\mathcal{A}) = \sum_{i \in R} \sum_{j \in K} f_{ij} \left(i - \frac{1 - F_j(i)}{f_{ij}} \right) \mathcal{A}_{ij}$. We do so by induction.

Fix a j and let i be the smallest number such that $\mathcal{A}_{ij} > 0$. Clearly, in an optimal solution to $[rOPT]$, $P_{ij} = i$. The induction hypothesis is that $P_{i+k,j} = (i+k)\mathcal{A}_{i+k,j} - \sum_{s=1}^{i+k-1} \mathcal{A}_{sj}$.

In an optimal solution to $[rOPT]$, the constraint (3.26) linking type $(i+1, j)$ to (i, j) must bind. Otherwise increase the value of $P_{i+1,j}$. Therefore

$$(i+k+1)\mathcal{A}_{i+k+1,j} - P_{i+k+1,j} = (i+k+1)\mathcal{A}_{i+k,j} - P_{i+k,j} = \sum_{s=i}^{i+k} \mathcal{A}_{sj}.$$

To complete the proof we mimic the calculation immediately following the proof of Theorem 3.4.10. \blacksquare

It is now easy to see that when the conditional virtual values are monotone, the mechanism of Theorem 3.4.11 achieves the upper bound implied by Lemma 3.4.12 and so, must be optimal.

Chapter 4

Rationalizability

Afriat's (1967) theorem is an answer to the question of when a sequence of purchase decisions is consistent with the purchaser maximizing a concave utility function $u(\cdot)$. As noted by Rochet (1987) we can think of this as an 'inverse' to the problem of characterizing DSIC. To see why, suppose a sequence of purchase decisions (p_i, x_i) , $i = 1, \dots, n$, where $p_i \in \mathbb{R}_+^n$ and $x_i \in \mathbb{R}_+^n$ are price vectors and purchased quantity vectors respectively. Via the taxation principle we can think of this sequence of price-quantity pairs as a menu offered by a mechanism. Thus, the mechanism is given and we ask for what preferences is this mechanism incentive compatible.

We begin with the case when the utility is assumed to be quasi-linear.

4.1 The Quasi-linear Case

A sequence of purchase decisions (p_i, x_i) , $i = 1, \dots, n$ is **rationalizable** by a concave quasi-linear utility function $u : \mathbb{R}_+^n \mapsto \mathbb{R}$ if for some budget B and for all i

$$x_i \in \arg \max \{u(x) + s_i : p_i \cdot x + s_i = B\}.$$

To determine if a sequence is rationalizable we suppose it is and ask what conditions it must satisfy and then check to see if those conditions are indeed sufficient.

If the sequence $\{p_i, x_i\}_{i=1}^n$ is rationalizable, it must be the case that at price p_i , x_j delivers less utility than x_i . Hence

$$\begin{aligned} u(x_i) + B - p_i \cdot x_i &\geq u(x_j) + B - p_i \cdot x_j \\ \Rightarrow u(x_j) - u(x_i) &\leq p_i \cdot (x_j - x_i) \end{aligned}$$

Thus, given the sequence $\{p_i, x_i\}_{i=1}^n$ we formulate the following system:

$$y_j - y_i \leq p_i \cdot (x_j - x_i), \quad \forall i, j \quad (4.1)$$

If this system is feasible we can use an appropriate feasible choice of $\{y_j\}_{j=1}^n$ to construct a concave utility function that rationalizes the sequence $\{p_i, x_i\}_{i=1}^n$. Determining feasibility of (4.1) is straightforward. We associate a network with (4.1) in the usual way. One node for each i and for each ordered pair (i, j) an arc with length $p_i \cdot (x_j - x_i)$. The system (4.1) is feasible iff. the associated network has no negative length cycles. For example, if $i_1 \rightarrow i_2 \rightarrow i_3 \rightarrow \dots \rightarrow i_k \rightarrow i_1$ is a cycle then

$$p_{i_1} \cdot (x_{i_2} - x_{i_1}) + \dots + p_{i_k} \cdot (x_{i_1} - x_{i_k}) \geq 0.$$

Suppose our associated network has no negative length cycles. Pick node 1 to be the source and set $u(x_i)$ to be the length of the shortest path from node 1 to node i . For any other $x \in \mathbb{R}_+^n$ set

$$u(x) = \min\{u(x_1) + p_1 \cdot (x - x_1), u(x_2) + p_2 \cdot (x - x_2) + \dots + u(x_n) + p_n \cdot (x - x_n)\}.$$

Since $u(x)$ is the minimum of a collection of linear functions with positive slope, it is easy to see that u is concave. It is easy to see from the argument that the absence of negative lengths cycles is a necessary and sufficient condition for the sequence to be rationalizable by concave quasi-linear utility function.

4.2 The General Case

A sequence of purchase decisions $\{p_i, x_i\}_{i=1}^n$ is **rationalizable** by a concave utility function $u : \mathbb{R}_+^n \mapsto \mathbb{R}$ if for some budget B and for all i

$$x_i \in \arg \max\{u(x) : p_i \cdot x \leq B\}.$$

As before we identify conditions that a rationalizable sequence must satisfy.

Suppose the sequence $\{p_i, x_i\}_{i=1}^n$ is indeed rationalizable. Since

$$x_i \in \arg \max\{u(x) : p_i \cdot x \leq B\}$$

each x_i must satisfy the usual first order conditions of optimality. In particular, if $s_i > 0$ is the optimal Lagrange multiplier it must be that

$$x_i \in \arg \max\{u(x) + s_i(B - p_i \cdot x)\}.$$

Hence, for all $j \neq i$

$$u(x_j) + s_i(B - p_i \cdot x_j) \leq u(x_i) + s_i(B - p_i \cdot x_i) \Rightarrow u(x_j) \leq u(x_i) + s_i p_i \cdot (x_j - x_i).$$

If we replace each $u(x_i)$ by y_i we obtain the system $\ell(A)$:

$$y_j \leq y_i + s_i a_{ij} \quad \forall i \neq j, \quad 1 \leq i, j \leq n$$

$$s_i > 0 \quad \forall 1 \leq i \leq n$$

Given a feasible solution to $\ell(A)$ we construct a concave utility function $u(\cdot)$ consistent with the sequence of purchase decisions (p_i, x_i) by setting:

$$u(x) = \min\{y_1 + s_1 p_1(x - x_1), y_2 + s_2 p_2(x - x_2), \dots, y_n + s_n p_n(x - x_n)\}.$$

Here is a different condition that is motivated by purely economic considerations. If $p_i \cdot (x_j - x_i) \leq 0$, the utility function u must satisfy $u(x_j) \leq u(x_i)$, otherwise, with purchase price of p_i , bundle x_j costs less but provides higher utility. If we have a sequence of decisions $(p_i, x_i), (p_j, x_j), (p_k, x_k), \dots, (p_r, x_r)$, with

$$p_i \cdot (x_j - x_i) \leq 0, \quad p_j \cdot (x_k - x_j) \leq 0, \dots, \quad p_r \cdot (x_i - x_r) \leq 0,$$

then $u(x_i) = u(x_j) = \dots = u(x_r)$, and

$$p_i \cdot (x_j - x_i) = 0, \quad p_j \cdot (x_k - x_j) = 0, \dots, \quad p_r \cdot (x_i - x_r) = 0.$$

The above necessary condition for rationalizability can be described in graph theoretic terms as follows. Let A be a $n \times n$ matrix where $a_{ij} = p_i \cdot (x_j - x_i)$. Associate with the matrix A a directed graph $D(A)$ as follows: introduce a vertex for each index and for each ordered pair (i, j) an edge with length a_{ij} . The matrix A is said to satisfy the Afriat condition (AC) if every negative length cycle in $D(A)$ contains at least one edge of positive weight.

We now state Afriat's theorem:

Theorem 4.2.1 $\ell(A)$ is feasible iff. $D(A)$ satisfies AC.

A number of proofs of the theorem exist.¹ Here we give a proof that makes explicit the network structure inherent in $\ell(A)$.²

¹Afriat's original proof assumed that $a_{ij} \neq 0 \quad \forall i \neq j$. This was relaxed by Diewert (1973) and Varian (1982).

²This based on Chung Piaw and Vohra (2003). The structure appears to have been overlooked (but used implicitly) in previous proofs. For example, Fostel, Scarf and Todd (2003).

To each $s \in \mathbb{R}_+^n$ and matrix A with zeros on the diagonals, we associate a directed graph $D(A, s)$ as follows: introduce a vertex for each index and for each ordered pair (i, j) an edge with length $s_i a_{ij}$. Notice that $D(A) = D(A, e)$ where e is the n -vector of all 1's.

Now fix $s \in \mathbb{R}_+^n$. Then feasibility of $\ell(A)$ reduces to identifying $y \in \mathbb{R}^n$ such that $y_j - y_i \leq s_i a_{ij}$ for all $i \neq j$. This system is feasible iff. $D(A, s)$ contains no negative cycles. Assuming feasibility, we can choose the y 's as follows: set $y_1 = 0$ and y_j to be the length of the shortest path from 1 to j in $D(A, s)$. Afriat's theorem can be rephrased as;

Theorem 4.2.2 *There is an $s \in \mathbb{R}_+^n$ such that $D(A, s)$ contains no negative cycles iff. $D(A, e)$ satisfies AC.*

Proof: If there is an $s \in \mathbb{R}_+^n$ such that $D(A, s)$ contains no negative cycles, the system of inequalities $\ell(A)$ (with s fixed) is feasible. So we can construct a utility function $u(\cdot)$ consistent with the sequence of purchase decisions. Therefore, $D(A, e)$ must satisfy AC. We next prove the non-trivial direction. Suppose $D(A, e)$ satisfies AC. We prove there exists $s \in \mathbb{R}_+^n$ such that $D(A, s)$ has no negative cycles.

Let $S = \{(i, j) : a_{i,j} < 0\}$, $E = \{(i, j) : a_{i,j} = 0\}$, and $T = \{(i, j) : a_{i,j} > 0\}$. Consider the weighted digraph G with edges in $S \cup E$, where arcs $(i, j) \in S$ are given weight $w_{ij} = -1$, and arcs $(i, j) \in E$ are given weight $w_{ij} = 0$. Since $D(A, e)$ satisfies AC, G does not contain a negative length cycle. Hence there exists a set of potentials $\{\phi_j\}$ on the nodes such that

$$\phi_j \leq \phi_i + w_{ij}, \quad \forall (i, j) \in E(G).$$

Without loss of generality, we relabel the vertices so that $\phi_n \leq \phi_{n-1} \leq \dots \leq \phi_1$. Choose $\{s_i\}$ non-decreasing so that

$$s_i \times \min_{(i,j) \in T} a_{ij} \geq (n-1) \times s_{i-1} \max_{(i,j) \in S} (-a_{ij}) \text{ if } \phi_i < \phi_{i-1},$$

and

$$s_i = s_{i-1} \text{ if } \phi_i = \phi_{i-1}$$

for all $i > 2$, with $s_1 = 1$.

For any cycle C in the digraph $D(A, s)$, let (v, u) be an edge in C such that (i) v has the smallest potential among all vertices in C , and (ii) $\phi_u > \phi_v$. Such an edge exists, otherwise ϕ_i is identical for all vertices i in C . In this case, all edges in C have non-negative edge weight in $D(A, s)$.

By selection, $\phi_u > \phi_v$. If $(v, u) \in S \cup E$, then we have $\phi_u \leq \phi_v + w_{vu} \leq \phi_v$, a contradiction. Hence $(v, w) \in T$. Now, note that all vertices q in C with the same potential as v must be incident to an edge (q, t) in C such that $\phi_t \geq \phi_q$. Hence the edge (q, t) must have non-negative length. i.e., $a_{q,t} \geq 0$. Let p denote a vertex in C with the second smallest potential. Now, C has length

$$s_v a_{vu} + \sum_{(k,l) \in C, (k,l) \neq (v,u)} s_k a_{k,l} \geq s_v a_{v,u} + s_p (n-1) \max_{(i,j) \in S} \{a_{ij}\} \geq 0,$$

i.e., C has non-negative length.

Since $D(A, s)$ is a digraph without any negative length cycles, $\ell(A)$ is feasible. ■

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