Weak Monotonicity Suffices for Truthfulness on Convex Domains

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ABSTRACT

Weak monotonicity is a simple necessary condition for a social choice function to be implementable by a truthful mechanism. Roberts [10] showed that it is sufficient for all social choice functions whose domain is unrestricted. Lavi, Mu'alem and Nisan [6] proved the sufficiency of weak monotonicity for functions over order-based domains and Gui, Muller and Vohra [5] proved sufficiency for order-based domains with range constraints and for domains defined by other special types of linear inequality constraints. Here we show the more general result, conjectured by Lavi, Mu'alem and Nisan [6], that weak monotonicity is sufficient for functions defined on any convex domain.

Categories and Subject Descriptors

J.4 [**Social and Behavioral Sciences**]: Economics; K.4.4 [**Computers and Society**]: Electronic Commerce—*payment schemes*

General Terms

Theory, Economics

Keywords

Dominant strategy, mechanism design, strategyproof, truthful, weak monotonicity

1. INTRODUCTION

Social choice theory centers around the general problem of selecting a single *outcome* out of a set A of *alternative outcomes* based on the individual preferences of a set P of *players*. A method for aggregating player preferences to select one outcome is called a *social choice function*. In this paper we assume that the range A is finite and that each player's preference is expressed by a *valuation function* which assigns to each possible outcome a real number representing the "benefit" the player derives from that outcome. The ensemble of player valuation functions is viewed as a *valuation matrix* with rows indexed by players and columns by outcomes.

A major difficulty connected with social choice functions is that players can not be required to tell the truth about their preferences. Since each player seeks to maximize his own benefit, he may find it in his interest to misrepresent his valuation function. An important approach for dealing with this problem is to augment a given social choice function with a *payment function*, which assigns to each player a (positive or negative) payment as a function of all of the individual preferences. By carefully choosing the payment function, one can hope to entice each player to tell the truth. A social choice function augmented with a payment function is called a *mechanism* [1] and the mechanism is said to *implement* the social choice function. A mechanism is *truthful* (or to be *strategyproof* or to have a *dominant strategy*) if each player's best strategy, knowing the preferences of the others, is always to declare his own true preferences. A social choice function is *truthfully implementable*, or *truthful* if it has a truthful implementation. (The property of truthful implementability is sometimes called *dominant strategy incentive compatibility*). This framework leads naturally to the question: which social choice functions are truthful?

This question is of the following general type: given a class of functions (here, social choice functions) and a property that holds for some of them (here, truthfulness), "characterize" the property. The definition of the property itself provides a characterization, so what more is needed? Here are some useful notions of characterization:

- *Recognition algorithm.* Give an algorithm which, given an appropriate representation of a function in the class, determines whether the function has the property.
- *Parametric representation.* Give an explicit parametrized family of functions and show that each function in the

[*]This work was supported in part by NSF grant CCR-9988526.

[†]This work was supported in part by NSF grant CCR-9988526 and DIMACS.

EC'05, June 5–8, 2005, Vancouver, British Columbia, Canada.

[1]The usual definition of mechanism is more general than this (see [8] Chapter 23.C or [9]); the mechanisms we consider here are usually called *direct revelation mechanisms.*

family has the property, and that every function with the property is in the family.

A third notion applies in the case of *hereditary properties* of functions. A function g is a *subfunction* of function f, or f *contains* g, if g is obtained by restricting the domain of f. A property $\mathcal{P}$ of functions is hereditary if it is preserved under taking subfunctions. Truthfulness is easily seen to be hereditary.

- *Sets of obstructions.* For a hereditary property $\mathcal{P}$, a function g that does not have the property is an *obstruction* to the property in the sense that any function containing g doesn't have the property. An obstruction is *minimal* if every proper subfunction has the property. A set of obstructions is *complete* if every function that does not have the property contains one of them as a subfunction. The set of all functions that don't satisfy $\mathcal{P}$ is a complete (but trivial and uninteresting) set of obstructions; one seeks a set of small (ideally, minimal) obstructions.

We are not aware of any work on recognition algorithms for the property of truthfulness, but there are significant results concerning parametric representations and obstruction characterizations of truthfulness. It turns out that the domain of the function, i.e., the set of allowed valuation matrices, is crucial. For functions with *unrestricted* domain, i.e., whose domain is the set of all real matrices, there are very good characterizations of truthfulness. For general domains, however, the picture is far from complete. Typically, the domains of social choice functions are specified by a system of constraints. For example, an *order constraint* requires that one specified entry in some row be larger than another in the same row, a *range constraint* places an upper or lower bound on an entry, and a *zero constraint* forces an entry to be 0. These are all examples of *linear inequality constraints* on the matrix entries.

Building on work of Roberts [10], Lavi, Mu'alem and Nisan [6] defined a condition called *weak monotonicity* (W-MON). (Independently, in the context of multi-unit auctions, Bikhchandani, Chatterji and Sen [3] identified the same condition and called it *nondecreasing in marginal utilities* (NDMU).) The definition of W-MON can be formulated in terms of obstructions: for some specified simple set $\mathcal{F}$ of functions each having domains of size 2, a function satisfies W-MON if it contains no function from $\mathcal{F}$. The functions in $\mathcal{F}$ are not truthful, and therefore W-MON is a necessary condition for truthfulness. Lavi, Mu'alem and Nisan [6] showed that W-MON is also sufficient for truthfulness for social choice functions whose domain is *order-based*, i.e., defined by order constraints and zero constraints, and Gui, Muller and Vohra [5] extended this to other domains. The domain constraints considered in both papers are special cases of linear inequality constraints, and it is natural to ask whether W-MON is sufficient for any domain defined by such constraints. Lavi, Mu'alem and Nisan [6] conjectured that W-MON suffices for convex domains. The main result of this paper is an affirmative answer to this conjecture:

THEOREM 1. *For any social choice function having convex domain and finite range, weak monotonicity is necessary and sufficient for truthfulness.*

Using the interpretation of weak monotonicity in terms of obstructions each having domain size 2, this provides a complete set of minimal obstructions for truthfulness within the class of social choice functions with convex domains.

The two hypotheses on the social choice function, that the domain is convex and that the range is finite, can not be omitted as is shown by the examples given in section 7.

1.1 Related Work

There is a simple and natural parametrized set of truthful social choice functions called *affine maximizers*. Roberts [10] showed that for functions with unrestricted domain, every truthful function is an affine maximizer, thus providing a parametrized representation for truthful functions with unrestricted domain. There are many known examples of truthful functions over restricted domains that are not affine maximizers (see [1], [2], [4], [6] and [7]). Each of these examples has a special structure and it seems plausible that there might be some mild restrictions on the class of all social choice functions such that all truthful functions obeying these restrictions are affine maximizers. Lavi, Mu'alem and Nisan [6] obtained a result in this direction by showing that for order-based domains, under certain technical assumptions, every truthful social choice function is "almost" an affine maximizer.

There are a number of results about truthfulness that can be viewed as providing obstruction characterizations, although the notion of obstruction is not explicitly discussed.

For a player i, a set of valuation matrices is said to be *i-local* if all of the matrices in the set are identical except for row i. Call a social choice function *i-local* if its domain is i-local and call it *local* if it is i-local for some i. The following easily proved fact is used extensively in the literature:

PROPOSITION 2. *The social choice function f is truthful if and only if every local subfunction of f is truthful.*

This implies that the set of all local non-truthful functions comprises a complete set of obstructions for truthfulness. This set is much smaller than the set of all non-truthful functions, but is still far from a minimal set of obstructions.

Rochet [11], Rozenshtrom [12] and Gui, Muller and Vohra [5] identified a necessary and sufficient condition for truthfulness (see lemma 3 below) called the *nonnegative cycle property*. This condition can be viewed as providing a minimal complete set of non-truthful functions. As is required by proposition 2, each function in the set is local. Furthermore it is one-to-one. In particular its domain has size at most the number of possible outcomes $|A|$.

As this complete set of obstructions consists of minimal non-truthful functions, this provides the optimal obstruction characterization of non-truthful functions within the class of all social choice functions. But by restricting attention to interesting subclasses of social choice functions, one may hope to get simpler sets of obstructions for truthfulness within that class.

The condition of weak monotonicity mentioned earlier can be defined by a set of obstructions, each of which is a local function of domain size exactly 2. Thus the results of Lavi, Mu'alem and Nisan [6], and of Gui, Muller and Vohra [5] give a very simple set of obstructions for truthfulness within certain subclasses of social choice functions. Theorem 1 extends these results to a much larger subclass of functions.

1.2 Weak Monotonicity and the Nonnegative Cycle Property

By proposition 2, a function is truthful if and only if each of its local subfunctions is truthful. Therefore, to get a set of obstructions for truthfulness, it suffices to obtain such a set for local functions.

The domain of an i-local function consists of matrices that are fixed on all rows but row i. Fix such a function f and let $D \subseteq \mathbb{R}^A$ be the set of allowed choices for row i. Since f depends only on row i and row i is chosen from D, we can view f as a function from D to A. Therefore, f is a social choice function having one player; we refer to such a function as a *single player function*.

Associated to any single player function f with domain D we define an edge-weighted directed graph H_f whose vertex set is the image of f. For convenience, we assume that f is surjective and so this image is A. For each $a, b \in A$, $x \in f^{-1}(a)$ there is an edge $e_x(a, b)$ from a to b with weight $x(a) - x(b)$. The weight of a set of edges is just the sum of the weights of the edges. We say that f satisfies:

- the *nonnegative cycle property* if every directed cycle has nonnegative weight.
- the *nonnegative two-cycle property* if every directed cycle between two vertices has nonnegative weight.

We say a local function g satisfies nonnegative cycle property/nonnegative two-cycle property if its associated single player function f does.

The graph H_f has a possibly infinite number of edges between any two vertices. We define G_f to be the edge-weighted directed graph with exactly one edge from a to b, whose weight δ_{ab} is the infimum (possibly $-\infty$) of all of the edge weights $e_x(a, b)$ for $x \in f^{-1}(a)$. It is easy to see that H_f has the nonnegative cycle property/nonnegative two-cycle property if and only if G_f does. G_f is called the *outcome graph* of f.

The weak monotonicity property mentioned earlier can be defined for arbitrary social choice functions by the condition that every local subfunction satisfies the nonnegative two-cycle property. The following result was obtained by Rochet [11] in a slightly different form and rediscovered by Rozenshtrom [12] and Gui, Muller and Vohra [5]:

Lemma 3. *A local social choice function is truthful if and only if it has the nonnegative cycle property. Thus a social choice function is truthful if and only if every local subfunction satisfies the nonnegative cycle property.*

In light of this, theorem 1 follows from:

Theorem 4. *For any surjective single player function $f : D \longrightarrow A$ where D is a convex subset of $\mathbb{R}^A$ and A is finite, the nonnegative two-cycle property implies the nonnegative cycle property.*

This is the result we will prove.

1.3 Overview of the Proof of Theorem 4

Let $D \subseteq \mathbb{R}^A$ be convex and let $f : D \longrightarrow A$ be a single player function such that G_f has no negative two-cycles. We want to conclude that G_f has no negative cycles. For two vertices a, b, let δ^*_{ab} denote the minimum weight of any path from a to b. Clearly $\delta^*_{ab} \leq \delta_{ab}$. Our proof shows that the δ^*-weight of every cycle is exactly 0, from which theorem 4 follows.

There seems to be no direct way to compute δ^* and so we proceed indirectly. Based on geometric considerations, we identify a subset of paths in G_f called *admissible* paths and a subset of admissible paths called *straight* paths. We prove that for any two outcomes a, b, there is a straight path from a to b (lemma 8 and corollary 10), and all straight paths from a to b have the same weight, which we denote ρ_{ab} (theorem 12). We show that $\rho_{ab} \leq \delta_{ab}$ (lemma 14) and that the ρ-weight of every cycle is 0. The key step to this proof is showing that the ρ-weight of every directed triangle is 0 (lemma 17).

It turns out that ρ is equal to δ^* (corollary 20), although this equality is not needed in the proof of theorem 4.

To expand on the above summary, we give the definitions of an admissible path and a straight path. These are somewhat technical and rely on the geometry of f. We first observe that, without loss of generality, we can assume that D is (topologically) closed (section 2). In section 3, for each $a \in A$, we enlarge the set $f^{-1}(a)$ to a closed convex set $D_a \subseteq D$ in such a way that for $a, b \in A$ with $a \neq b$, D_a and D_b have disjoint interiors. We define an admissible path to be a sequence of outcomes $(a_1, \ldots, a_k)$ such that each of the sets $I_j = D_{a_j} \cap D_{a_{j+1}}$ is nonempty (section 4). An admissible path is straight if there is a straight line that meets one point from each of the sets $I_1, \ldots, I_{k-1}$ in order (section 5).

Finally, we mention how the hypotheses of convex domain and finite range are used in the proof. Both hypotheses are needed to show: (1) the existence of a straight path from a to b for all a, b (lemma 8). (2) that the ρ-weight of a directed triangle is 0 (lemma 17). The convex domain hypothesis is also needed for the convexity of the sets D_a (section 3). The finite range hypothesis is also needed to reduce theorem 4 to the case that D is closed (section 2) and to prove that every straight path from a to b has the same δ-weight (theorem 12).

2. REDUCTION TO CLOSED DOMAIN

We first reduce the theorem to the case that D is closed. Write D^C for the closure of D. Since A is finite, $D^C = \cup_{a \in A}(f^{-1}(a))^C$. Thus for each $v \in D^C - D$, there is an $a = a(v) \in A$ such that $v \in (f^{-1}(a))^C$. Extend f to the function g on D^C by defining $g(v) = a(v)$ for $v \in D^C - D$ and $g(v) = f(v)$ for $v \in D$. It is easy to check that $\delta_{ab}(g) = \delta_{ab}(f)$ for all $a, b \in A$ and therefore it suffices to show that the nonnegative two-cycle property for g implies the nonnegative cycle property for g.

Henceforth we assume D is convex and closed.

3. A DISSECTION OF THE DOMAIN

In this section, we construct a family of closed convex sets $\{D_a : a \in A\}$ with disjoint interiors whose union is D and satisfying $f^{-1}(a) \subseteq D_a$ for each $a \in A$.

Let $R_a = \{v : \forall b \in A, v(a) - v(b) \geq \delta_{ab}\}$. R_a is a closed polyhedron containing $f^{-1}(a)$. The next proposition implies that any two of these polyhedra intersect only on their boundary.

Proposition 5. *Let $a, b \in A$. If $v \in R_a \cap R_b$ then $v(a) - v(b) = \delta_{ab} = -\delta_{ba}$.*

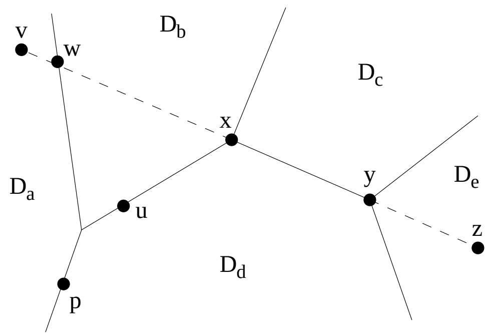

Figure 1: A 2-dimensional domain with 5 outcomes.

PROOF. $v \in R_a$ implies $v(a) - v(b) \geq \delta_{ab}$ and $v \in R_b$ implies $v(b) - v(a) \geq \delta_{ba}$ which, by the nonnegative two-cycle property, implies $v(a) - v(b) \leq \delta_{ab}$. Thus $v(a) - v(b) = \delta_{ab}$ and by symmetry $v(b) - v(a) = \delta_{ba}$. □

Finally, we restrict the collection of sets $\{R_a : a \in A\}$ to the domain D by defining $D_a = R_a \cap D$ for each $a \in A$. Clearly, D_a is closed and convex, and contains $f^{-1}(a)$. Therefore $\bigcup_{a \in A} D_a = D$. Also, by proposition 5, any point v in $D_a \cap D_b$ satisfies $v(a) - v(b) = \delta_{ab} = -\delta_{ba}$.

4. PATHS AND D-SEQUENCES

A *path of size* k is a sequence $\overrightarrow{a} = (a_1, \ldots, a_k)$ with each $a_i \in A$ (possibly with repetition). We call $\overrightarrow{a}$ an (a_1, a_k)-path. For a path $\overrightarrow{a}$, we write $|\overrightarrow{a}|$ for the size of $\overrightarrow{a}$. $\overrightarrow{a}$ is *simple* if the a_i's are distinct.

For $b, c \in A$ we write P_{bc} for the set of (b, c)-paths and SP_{bc} for the set of simple (b, c)-paths. The δ-weight of path $\overrightarrow{a}$ is defined by

$$\delta(\overrightarrow{a}) = \sum_{i=1}^{k-1} \delta_{a_i a_{i+1}}.$$

A *D-sequence of order* k is a sequence $\overrightarrow{u} = (u_0, \ldots, u_k)$ with each $u_i \in D$ (possibly with repetition). We call $\overrightarrow{u}$ a (u_0, u_k)-sequence. For a D-sequence $\overrightarrow{u}$, we write $ord(u)$ for the order of $\overrightarrow{u}$. For $v, w \in D$ we write S^{vw} for the set of (v, w)-sequences.

A compatible pair is a pair $(\overrightarrow{a}, \overrightarrow{u})$ where $\overrightarrow{a}$ is a path and $\overrightarrow{u}$ is a D-sequence satisfying $ord(\overrightarrow{u}) = |\overrightarrow{a}|$ and for each $i \in [k]$, both u_{i-1} and u_i belong to D_{a_i}.

We write $C(\overrightarrow{a})$ for the set of D-sequences $\overrightarrow{u}$ that are compatible with $\overrightarrow{a}$. We say that $\overrightarrow{a}$ is *admissible* if $C(\overrightarrow{a})$ is nonempty. For $\overrightarrow{u} \in C(\overrightarrow{a})$ we define

$$\Delta_{\overrightarrow{a}}(\overrightarrow{u}) = \sum_{i=1}^{|\overrightarrow{a}|-1} (u_i(a_i) - u_i(a_{i+1})).$$

For $v, w \in D$ and $b, c \in A$, we define C_{bc}^{vw} to be the set of compatible pairs $(\overrightarrow{a}, \overrightarrow{u})$ such that $\overrightarrow{a} \in P_{bc}$ and $\overrightarrow{u} \in S^{vw}$.

To illustrate these definitions, figure 1 gives the dissection of a domain, a 2-dimensional plane, into five regions D_a, D_b, D_c, D_d, D_e. D-sequence (v, w, x, y, z) is compatible with both path (a, b, c, e) and path (a, b, d, e); D-sequence (v, w, u, y, z) is compatible with a unique path (a, b, d, e). D-sequence (x, w, p, y, z) is compatible with a unique path (b, a, d, e). Hence (a, b, c, e), (a, b, d, e) and (b, a, d, e) are admissible paths. However, path (a, c, d) or path (b, e) is not admissible.

PROPOSITION 6. *For any compatible pair* $(\overrightarrow{a}, \overrightarrow{u})$, $\Delta_{\overrightarrow{a}}(\overrightarrow{u}) = \delta(\overrightarrow{a})$.

PROOF. Let $k = ord(\overrightarrow{u}) = |\overrightarrow{a}|$. By the definition of a compatible pair, $u_i \in D_{a_i} \cap D_{a_{i+1}}$ for $i \in [k-1]$. $u_i(a_i) - u_i(a_{i+1}) = \delta_{a_i a_{i+1}}$ from proposition 5. Therefore,

$$\Delta_{\overrightarrow{a}}(\overrightarrow{u}) = \sum_{i=1}^{k-1} (u_i(a_i) - u_i(a_{i+1})) = \sum_{i=1}^{k-1} \delta_{a_i a_{i+1}} = \delta(\overrightarrow{a}).$$

□

LEMMA 7. *Let* $b, c \in A$ *and let* $\overrightarrow{a}, \overrightarrow{a}' \in P_{bc}$. *If* $C(\overrightarrow{a}) \cap C(\overrightarrow{a}') \neq \emptyset$ *then* $\delta(\overrightarrow{a}) = \delta(\overrightarrow{a}')$.

PROOF. Let $\overrightarrow{u}$ be a D-sequence in $C(\overrightarrow{a}) \cap C(\overrightarrow{a}')$. By proposition 6, $\delta(\overrightarrow{a}) = \Delta_{\overrightarrow{a}}(\overrightarrow{u})$ and $\delta(\overrightarrow{a}') = \Delta_{\overrightarrow{a}'}(\overrightarrow{u})$, it suffices to show $\Delta_{\overrightarrow{a}}(\overrightarrow{u}) = \Delta_{\overrightarrow{a}'}(\overrightarrow{u})$.

Let $k = ord(\overrightarrow{u}) = |\overrightarrow{a}| = |\overrightarrow{a}'|$. Since

$$\begin{aligned}\Delta_{\overrightarrow{a}}(\overrightarrow{u}) &= \sum_{i=1}^{k-1} (u_i(a_i) - u_i(a_{i+1})) \\ &= u_1(a_1) + \sum_{i=2}^{k-1} (u_i(a_i) - u_{i-1}(a_i)) - u_{k-1}(a_k) \\ &= u_1(b) + \sum_{i=2}^{k-1} (u_i(a_i) - u_{i-1}(a_i)) - u_{k-1}(c),\end{aligned}$$

$$\begin{aligned}&\Delta_{\overrightarrow{a}}(\overrightarrow{u}) - \Delta_{\overrightarrow{a}'}(\overrightarrow{u}) \\ &= \sum_{i=2}^{k-1} ((u_i(a_i) - u_{i-1}(a_i)) - (u_i(a_i') - u_{i-1}(a_i'))) \\ &= \sum_{i=2}^{k-1} ((u_i(a_i) - u_i(a_i')) - (u_{i-1}(a_i) - u_{i-1}(a_i'))).\end{aligned}$$

Noticing both u_{i-1} and u_i belong to $D_{a_i} \cap D_{a_i'}$, we have by proposition 5

$$u_{i-1}(a_i) - u_{i-1}(a_i') = \delta_{a_i a_i'} = u_i(a_i) - u_i(a_i').$$

Hence $\Delta_{\overrightarrow{a}}(\overrightarrow{u}) - \Delta_{\overrightarrow{a}'}(\overrightarrow{u}) = 0$. □

5. LINEAR D-SEQUENCES AND STRAIGHT PATHS

For $v, w \in D$ we write $\overline{vw}$ for the (closed) line segment joining v and w.

A D-sequence $\overrightarrow{u}$ of order k is *linear* provided that there is a sequence of real numbers $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_k = 1$ such that $u_i = (1 - \lambda_i)u_0 + \lambda_i u_k$. In particular, each u_i belongs to $\overline{u_0 u_k}$. For $v, w \in D$ we write L^{vw} for the set of linear (v, w)-sequences.

For $b, c \in A$ and $v, w \in D$ we write LC_{bc}^{vw} for the set of compatible pairs $(\overrightarrow{a}, \overrightarrow{u})$ such that $\overrightarrow{a} \in P_{bc}$ and $\overrightarrow{u} \in L^{vw}$.

For a path $\overrightarrow{a}$, we write $L(\overrightarrow{a})$ for the set of linear sequences compatible with $\overrightarrow{a}$. We say that $\overrightarrow{a}$ is *straight* if $L(\overrightarrow{a}) \neq \emptyset$.

For example, in figure 1, D-sequence (v, w, x, y, z) is linear while (v, w, u, y, z), (x, w, p, y, z), and (x, v, w, y, z) are not. Hence path (a, b, c, e) and (a, b, d, e) are both straight. However, path (b, a, d, e) is not straight.

Lemma 8. *Let $b, c \in A$ and $v \in D_b, w \in D_c$. There is a simple path $\overrightarrow{a}$ and D-sequence $\overrightarrow{u}$ such that $(\overrightarrow{a}, \overrightarrow{u}) \in LC^{vw}_{bc}$. Furthermore, for any such path $\overrightarrow{a}$, $\delta(\overrightarrow{a}) \leq v(b) - v(c)$.*

Proof. By the convexity of D, any sequence of points on $\overline{vw}$ is a D-sequence.

If $b = c$, singleton path $\overrightarrow{a} = (b)$ and D-sequence $\overrightarrow{u} = (v, w)$ are obviously compatible. $\delta(\overrightarrow{a}) = 0 = v(b) - v(c)$.

So assume $b \neq c$. If $D_b \cap D_c \cap \overline{vw} \neq \emptyset$, we pick an arbitrary x from this set and let $\overrightarrow{a} = (b, c) \in SP_{bc}$, $\overrightarrow{u} = (v, x, w) \in L^{vw}$. Again it is easy to check the compatibility of $(\overrightarrow{a}, \overrightarrow{u})$. Since $v \in D_b$, $v(b) - v(c) \geq \delta_{bc} = \delta(\overrightarrow{a})$.

For the remaining case $b \neq c$ and $D_b \cap D_c \cap \overline{vw} = \emptyset$, notice $v \neq w$ otherwise $v = w \in D_b \cap D_c \cap \overline{vw}$. So we can define λ_x for every point x on $\overline{vw}$ to be the unique number in $[0, 1]$ such that $x = (1 - \lambda_x)v + \lambda_x w$. For convenience, we write $x \leq y$ for $\lambda_x \leq \lambda_y$.

Let $I_a = D_a \cap \overline{vw}$ for each $a \in A$. Since $D = \cup_{a \in A} D_a$, we have $\overline{vw} = \cup_{a \in A} I_a$. Moreover, by the convexity of D_a and $\overline{vw}$, I_a is a (possibly trivial) closed interval.

We begin by considering the case that I_b and I_c are each a single point, that is, $I_b = \{v\}$ and $I_c = \{w\}$.

Let S be a minimal subset of A satisfying $\cup_{s \in S} I_s = \overline{vw}$. For each $s \in S$, I_s is maximal, i.e., not contained in any other I_t, for $t \in S$. In particular, the intervals $\{I_s : s \in S\}$ have all left endpoints distinct and all right endpoints distinct and the order of the left endpoints is the same as that of the right endpoints. Let $k = |S| + 2$ and index S as $a_2, \ldots, a_{k-1}$ in the order defined by the right endpoints. Denote the interval I_{a_i} by $[l_i, r_i]$. Thus $l_2 < l_3 < \ldots < l_{k-1}$, $r_2 < r_3 < \ldots < r_{k-1}$ and the fact that these intervals cover $\overline{vw}$ implies $l_2 = v$, $r_{k-1} = w$ and for all $2 \leq i \leq k-2$, $l_{i+1} \leq r_i$ which further implies $l_i < r_i$. Now we define the path $\overrightarrow{a} = (a_1, a_2, \ldots, a_{k-1}, a_k)$ with $a_1 = b$, $a_k = c$ and $a_2, a_3, \ldots, a_{k-1}$ as above. Define the linear D-sequence $\overrightarrow{u} = (u_0, u_1, \ldots, u_k)$ by $u_0 = u_1 = v$, $u_k = w$ and for $2 \leq i \leq k-1$, $u_i = r_i$. It follows immediately that $(\overrightarrow{a}, \overrightarrow{u}) \in LC^{vw}_{bc}$. Neither b nor c is in S since $l_b = r_b$ and $l_c = r_c$. Thus $\overrightarrow{a}$ is simple.

Finally to show $\delta(\overrightarrow{a}) \leq v(b) - v(c)$, we note

$$v(b) - v(c) = v(a_1) - v(a_k) = \sum_{i=1}^{k-1} (v(a_i) - v(a_{i+1}))$$

and

$$\delta(\overrightarrow{a}) = \Delta_{\overrightarrow{a}}(\overrightarrow{u}) = \sum_{i=1}^{k-1} (u_i(a_i) - u_i(a_{i+1}))$$

$$= v(a_1) - v(a_2) + \sum_{i=2}^{k-1} (r_i(a_i) - r_i(a_{i+1})).$$

For two outcomes $d, e \in A$, let us define $f_{de}(z) = z(d) - z(e)$ for all $z \in D$. It suffices to show $f_{a_i a_{i+1}}(r_i) \leq f_{a_i a_{i+1}}(v)$ for $2 \leq i \leq k-1$.

Fact 9. *For $d, e \in A$, $f_{de}(z)$ is a linear function of z. Furthermore, if $x \in D_d$ and $y \in D_e$ with $x \neq y$, then $f_{de}(x) = x(d) - x(e) \geq \delta_{de} \geq -\delta_{ed} \geq -(y(e) - y(d)) = f_{de}(y)$. Therefore $f_{de}(z)$ is monotonically nonincreasing along the line $\overleftrightarrow{xy}$ as z moves in the direction from x to y.*

Applying this fact with $d = a_i$, $e = a_{i+1}$, $x = l_i$ and $y = r_i$ gives the desired conclusion. This completes the proof for the case that $I_b = \{v\}$ and $I_c = \{w\}$.

For general I_b, I_c, $r_b < l_c$ otherwise $D_b \cap D_c \cap \overline{vw} = I_b \cap I_c \neq \emptyset$. Let $v' = r_b$ and $w' = l_c$. Clearly we can apply the above conclusion to $v' \in D_b$, $w' \in D_c$ and get a compatible pair $(\overrightarrow{a}, \overrightarrow{u}') \in LC^{v'w'}_{bc}$ with $\overrightarrow{a}$ simple and $\delta(\overrightarrow{a}) \leq v'(b) - v'(c)$. Define the linear D-sequence $\overrightarrow{u}$ by $u_0 = v, u_k = w$ and $u_i = u'_i$ for $i \in [k-1]$. $(\overrightarrow{a}, \overrightarrow{u}) \in LC^{vw}_{bc}$ is evident. Moreover, applying the above fact with $d = b$, $e = c$, $x = v$ and $y = w$, we get $v(b) - v(c) \geq v'(b) - v'(c) \geq \delta(\overrightarrow{a})$. □

Corollary 10. *For any $b, c \in A$ there is a straight (b, c)-path.*

The main result of this section (theorem 12) says that for any $b, c \in A$, every straight (b, c)-path has the same δ-weight. To prove this, we first fix $v \in D_b$ and $w \in D_c$ and show (lemma 11) that every straight (b, c)-path compatible with some linear (v, w)-sequence has the same δ-weight $\rho_{bc}(v, w)$. We then show in theorem 12 that $\rho_{bc}(v, w)$ is the same for all choices of $v \in D_b$ and $w \in D_c$.

Lemma 11. *For $b, c \in A$, there is a function $\rho_{bc} : D_b \times D_c \longrightarrow \mathbb{R}$ satisfying that for any $(\overrightarrow{a}, \overrightarrow{u}) \in LC^{vw}_{bc}$, $\delta(\overrightarrow{a}) = \rho_{bc}(v, w)$.*

Proof. Let $(\overrightarrow{a}', \overrightarrow{u}'), (\overrightarrow{a}'', \overrightarrow{u}'') \in LC^{vw}_{bc}$. It suffices to show $\delta(\overrightarrow{a}') = \delta(\overrightarrow{a}'')$. To do this we construct a linear (v, w)-sequence $\overrightarrow{u}$ and paths $\overrightarrow{a}^*, \overrightarrow{a}^{**} \in P_{bc}$, both compatible with $\overrightarrow{u}$, satisfying $\delta(\overrightarrow{a}^*) = \delta(\overrightarrow{a}')$ and $\delta(\overrightarrow{a}^{**}) = \delta(\overrightarrow{a}'')$. Lemma 7 implies $\delta(\overrightarrow{a}^*) = \delta(\overrightarrow{a}^{**})$, which will complete the proof.

Let $|\overrightarrow{a}'| = ord(\overrightarrow{u}') = k$ and $|\overrightarrow{a}''| = ord(\overrightarrow{u}'') = l$. We select $\overrightarrow{u}$ to be any linear (v, w)-sequence $(u_0, u_1, \ldots, u_t)$ such that $\overrightarrow{u}'$ and $\overrightarrow{u}''$ are both subsequences of $\overrightarrow{u}$, i.e., there are indices $0 = i_0 < i_1 < \cdots < i_k = t$ and $0 = j_0 < j_1 < \cdots < j_l = t$ such that $\overrightarrow{u}' = (u_{i_0}, u_{i_1}, \ldots, u_{i_k})$ and $\overrightarrow{u}'' = (u_{j_0}, u_{j_1}, \ldots, u_{j_l})$. We now construct a (b, c)-path $\overrightarrow{a}^*$ compatible with $\overrightarrow{u}$ such that $\delta(\overrightarrow{a}^*) = \delta(\overrightarrow{a}')$. (An analogous construction gives $\overrightarrow{a}^{**}$ compatible with $\overrightarrow{u}$ such that $\delta(\overrightarrow{a}^{**}) = \delta(\overrightarrow{a}'')$.) This will complete the proof.

$\overrightarrow{a}^*$ is defined as follows: for $1 \leq j \leq t$, $a^*_j = a'_r$ where r is the unique index satisfying $i_{r-1} < j \leq i_r$. Since both $u_{i_{r-1}} = u'_{r-1}$ and $u_{i_r} = u'_r$ belong to $D_{a'_r}$, $u_j \in D_{a'_r}$ for $i_{r-1} \leq j \leq i_r$ by the convexity of $D_{a'_r}$. The compatibility of $(\overrightarrow{a}^*, \overrightarrow{u})$ follows immediately. Clearly, $a^*_1 = a'_1 = b$ and $a^*_t = a'_k = c$, so $\overrightarrow{a}^* \in P_{bc}$. Furthermore, as $\delta_{a^*_j a^*_{j+1}} = \delta_{a'_r a'_r} = 0$ for each $r \in [k], i_{r-1} < j < i_r$,

$$\delta(\overrightarrow{a}^*) = \sum_{r=1}^{k-1} \delta_{a^*_{i_r} a^*_{i_r+1}} = \sum_{r=1}^{k-1} \delta_{a'_r a'_{r+1}} = \delta(\overrightarrow{a}').$$

□

We are now ready for the main theorem of the section:

Theorem 12. *ρ_{bc} is a constant map on $D_b \times D_c$. Thus for any $b, c \in A$, every straight (b, c)-path has the same δ-weight.*

Proof. For a path $\overrightarrow{a}$, (v, w) is compatible with $\overrightarrow{a}$ if there is a linear (v, w)-sequence compatible with $\overrightarrow{a}$. We write $CP(\overrightarrow{a})$ for the set of pairs (v, w) compatible with $\overrightarrow{a}$. ρ_{bc} is constant on $CP(\overrightarrow{a})$ because for each $(v, w) \in CP(\overrightarrow{a})$, $\rho_{bc}(v, w) = \delta(\overrightarrow{a})$. By lemma 8, we also have $\bigcup_{\overrightarrow{a} \in SP_{bc}} CP(\overrightarrow{a}) = D_b \times D_c$. Since A is finite, SP_{bc}, the set of simple paths from b to c, is finite as well.

Next we prove that for any path $\overrightarrow{a}$, $CP(\overrightarrow{a})$ is closed.

Let $((v^n, w^n) : n \in \mathbb{N})$ be a convergent sequence in $CP(\overrightarrow{a})$ and let (v, w) be the limit. We want to show that $(v, w) \in CP(\overrightarrow{a})$. For each $n \in \mathbb{N}$, since $(v^n, w^n) \in CP(\overrightarrow{a})$, there is a linear (v^n, w^n)-sequence u^n compatible with $\overrightarrow{a}$, i.e., there are $0 = \lambda_0^n \le \lambda_1^n \le \ldots \le \lambda_k^n = 1$ ($k = |\overrightarrow{a}|$) such that $u_j^n = (1 - \lambda_j^n)v^n + \lambda_j^n w^n$ ($j = 0, 1, \ldots, k$). Since for each n, $\lambda^n = (\lambda_0^n, \lambda_1^n, \ldots, \lambda_k^n)$ belongs to the closed bounded set $[0, 1]^{k+1}$ we can choose an infinite subset $I \subseteq \mathbb{N}$ such that the sequence $(\lambda^n : n \in I)$ converges. Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k)$ be the limit. Clearly $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_k = 1$.

Define the linear (v, w)-sequence $\overrightarrow{u}$ by $u_j = (1 - \lambda_j)v + \lambda_j w$ ($j = 0, 1, \ldots, k$). Then for each $j \in \{0, \ldots, k\}$, u_j is the limit of the sequence $(u_j^n : n \in I)$. For $j > 0$, each u_j^n belongs to the closed set D_{a_j}, so $u_j \in D_{a_j}$. Similarly, for $j < k$ each u_j^n belongs to the closed set $D_{a_{j+1}}$, so $u_j \in D_{a_{j+1}}$. Hence $(\overrightarrow{a}, \overrightarrow{u})$ is compatible, implying $(v, w) \in CP(\overrightarrow{a})$.

Now we have $D_b \times D_c$ covered by finitely many closed subsets on each of them ρ_{bc} is a constant.

Suppose for contradiction that there are $(v, w), (v', w') \in D_b \times D_c$ such that $\rho_{bc}(v, w) \neq \rho_{bc}(v', w')$.

$$L = \{((1 - \lambda)v + \lambda v', (1 - \lambda)w + \lambda w') : \lambda \in [0, 1]\}$$

is a line segment in $D_b \times D_c$ by the convexity of D_b, D_c. Let

$$L_1 = \{(x, y) \in L : \rho_{bc}(x, y) = \rho_{bc}(v, w)\}$$

and $L_2 = L - L_1$. Clearly $(v, w) \in L_1$, $(v', w') \in L_2$. Let

$$P = \{\overrightarrow{a} \in SP_{bc} : \delta(\overrightarrow{a}) = \rho_{bc}(v, w)\}.$$

$L_1 = \left(\bigcup_{\overrightarrow{a} \in P} CP(\overrightarrow{a})\right) \cap L$, $L_2 = \left(\bigcup_{\overrightarrow{a} \in SP_{bc} - P} CP(\overrightarrow{a})\right) \cap L$ are closed by the finiteness of P. This is a contradiction, since it is well known (and easy to prove) that a line segment can not be expressed as the disjoint union of two nonempty closed sets. □

Summarizing corollary 10, lemma 11 and theorem 12, we have

Corollary 13. *For any $b, c \in A$, there is a real number ρ_{bc} with the property that (1) There is at least one straight (b, c)-path of δ-weight ρ_{bc} and (2) Every straight (b, c)-path has δ-weight ρ_{bc}.*

6. PROOF OF THEOREM 4

Lemma 14. *$\rho_{bc} \le \delta_{bc}$ for all $b, c \in A$.*

Proof. For contradiction, suppose $\rho_{bc} - \delta_{bc} = \epsilon > 0$. By the definition of δ_{bc}, there exists $v \in f^{-1}(b) \subseteq D_b$ with $v(b) - v(c) < \delta_{bc} + \epsilon = \rho_{bc}$. Pick an arbitrary $w \in D_c$. By lemma 8, there is a compatible pair $(\overrightarrow{a}, \overrightarrow{u}) \in LC_{bc}^{vw}$ with $\delta(\overrightarrow{a}) \le v(b) - v(c)$. Since $\overrightarrow{a}$ is a straight (b, c)-path, $\rho_{bc} = \delta(\overrightarrow{a}) \le v(b) - v(c)$, leading to a contradiction. □

Define another edge-weighted complete directed graph G'_f on vertex set A where the weight of arc (a, b) is ρ_{ab}. Immediately from lemma 14, the weight of every directed cycle in G_f is bounded below by its weight in G'_f. To prove theorem 4, it suffices to show the zero cycle property of G'_f, i.e., every directed cycle has weight zero. We begin by considering two-cycles.

Lemma 15. *$\rho_{bc} + \rho_{cb} = 0$ for all $b, c \in A$.*

Proof. Let $\overrightarrow{a}$ be a straight (b, c)-path compatible with linear sequence $\overrightarrow{u}$. let $\overrightarrow{a}'$ be the reverse of $\overrightarrow{a}$ and $\overrightarrow{u}'$ the reverse of $\overrightarrow{u}$. Obviously, $(\overrightarrow{a}', \overrightarrow{u}')$ is compatible as well and thus $\overrightarrow{a}'$ is a straight (c, b)-path. Therefore,

$$\rho_{bc} + \rho_{cb} = \delta(\overrightarrow{a}) + \delta(\overrightarrow{a}') = \sum_{i=1}^{k-1} \delta_{a_i a_{i+1}} + \sum_{i=1}^{k-1} \delta_{a_{i+1} a_i}$$

$$= \sum_{i=1}^{k-1} (\delta_{a_i a_{i+1}} + \delta_{a_{i+1} a_i}) = 0,$$

where the final equality uses proposition 5. □

Next, for three cycles, we first consider those compatible with linear triples.

Lemma 16. *If there are collinear points $u \in D_a$, $v \in D_b$, $w \in D_c$ ($a, b, c \in A$), $\rho_{ab} + \rho_{bc} + \rho_{ca} = 0$.*

Proof. First, we prove for the case where v is between u and w. From lemma 8, there are compatible pairs $(\overrightarrow{a}', \overrightarrow{u}') \in LC_{ab}^{uv}$, $(\overrightarrow{a}'', \overrightarrow{u}'') \in LC_{bc}^{vw}$. Let $|\overrightarrow{a}'| = ord(\overrightarrow{u}') = k$ and $|\overrightarrow{a}''| = ord(\overrightarrow{u}'') = l$. We paste $\overrightarrow{a}'$ and $\overrightarrow{a}''$ together as

$$\overrightarrow{a}''' = (a = a'_1, a'_2, \ldots, a'_{k-1}, a'_k, a''_1, \ldots, a''_l = c),$$

$\overrightarrow{u}'$ and $\overrightarrow{u}''$ as

$$\overrightarrow{u}''' = (u = u'_0, u'_1, \ldots, u'_k = v = u''_0, u''_1, \ldots, u''_l = w).$$

Clearly $(\overrightarrow{a}''', \overrightarrow{u}''') \in LC_{ac}^{uw}$ and

$$\begin{aligned}\delta(\overrightarrow{a}''') &= \sum_{i=1}^{k-1} \delta_{a'_i a'_{i+1}} + \delta_{a'_k a''_1} + \sum_{i=1}^{l-1} \delta_{a''_i a''_{i+1}} \\ &= \delta(\overrightarrow{a}') + \delta_{bb} + \delta(\overrightarrow{a}'') \\ &= \delta(\overrightarrow{a}') + \delta(\overrightarrow{a}'').\end{aligned}$$

Therefore, $\rho_{ac} = \delta(\overrightarrow{a}''') = \delta(\overrightarrow{a}') + \delta(\overrightarrow{a}'') = \rho_{ab} + \rho_{bc}$. Moreover, $\rho_{ac} = -\rho_{ca}$ by lemma 15, so we get $\rho_{ab} + \rho_{bc} + \rho_{ca} = 0$.

Now suppose w is between u and v. By the above argument, we have $\rho_{ac} + \rho_{cb} + \rho_{ba} = 0$ and by lemma 15, $\rho_{ab} + \rho_{bc} + \rho_{ca} = -\rho_{ba} - \rho_{cb} - \rho_{ac} = 0$.

The case that u is between v and w is similar. □

Now we are ready for the zero three-cycle property:

Lemma 17. *$\rho_{ab} + \rho_{bc} + \rho_{ca} = 0$ for all $a, b, c \in A$.*

Proof. Let $S = \{(a, b, c) : \rho_{ab} + \rho_{bc} + \rho_{ca} \neq 0\}$ and for contradiction, suppose $S \neq \emptyset$. S is finite. For each $a \in A$, choose $v_a \in D_a$ arbitrarily and let T be the convex hull of $\{v_a : a \in A\}$. For each $(a, b, c) \in S$, let $R_{abc} = D_a \times D_b \times D_c \cap T^3$. Clearly, each R_{abc} is nonempty and compact. Moreover, by lemma 16, no $(u, v, w) \in R_{abc}$ is collinear.

Define $f : D^3 \to \mathbb{R}$ by $f(u, v, w) = |v - u| + |w - v| + |u - w|$. For $(a, b, c) \in S$, the restriction of f to the compact set R_{abc} attains a minimum $m(a, b, c)$ at some point $(u, v, w) \in R_{abc}$ by the continuity of f, i.e., there exists a triangle Δuvw of minimum perimeter within T with $u \in D_a, v \in D_b, w \in D_c$.

Choose $(a^*, b^*, c^*) \in S$ so that $m(a^*, b^*, c^*)$ is minimum and let $(u^*, v^*, w^*) \in R_{a^* b^* c^*}$ be a triple achieving it. Pick an arbitrary point p in the interior of $\Delta u^* v^* w^*$. By the convexity of domain D, there is $d \in A$ such that $p \in D_d$.

Consider triangles Δu^*pw^*, Δw^*pv^* and Δv^*pu^*. Since each of them has perimeter less than that of $\Delta u^*v^*w^*$ and all three triangles are contained in T, by the minimality of $\Delta u^*v^*w^*$, (a^*, d, c^*), (c^*, d, b^*), $(b^*, d, a^*) \notin S$. Thus

$$\rho_{a^*d} + \rho_{dc^*} + \rho_{c^*a^*} = 0,$$

$$\rho_{c^*d} + \rho_{db^*} + \rho_{b^*c^*} = 0,$$

$$\rho_{b^*d} + \rho_{da^*} + \rho_{a^*b^*} = 0.$$

Summing up the three equalities,

$$(\rho_{a^*d} + \rho_{dc^*} + \rho_{c^*d} + \rho_{db^*} + \rho_{b^*d} + \rho_{da^*})$$

$$+(\rho_{c^*a^*} + \rho_{b^*c^*} + \rho_{a^*b^*}) = 0,$$

which yields a contradiction

$$\rho_{a^*b^*} + \rho_{b^*c^*} + \rho_{c^*a^*} = 0.$$

□

With the zero two-cycle and three-cycle properties, the zero cycle property of G'_f is immediate. As noted earlier, this completes the proof of theorem 4.

THEOREM 18. *Every directed cycle of G'_f has weight zero.*

PROOF. Clearly, zero two-cycle and three-cycle properties imply triangle equality $\rho_{ab} + \rho_{bc} = \rho_{ac}$ for all $a, b, c \in A$. For a directed cycle $C = a_1a_2 \dots a_ka_1$, by inductively applying triangle equality, we have $\sum_{i=1}^{k-1} \rho_{a_ia_{i+1}} = \rho_{a_1a_k}$. Therefore, the weight of C is

$$\sum_{i=1}^{k-1} \rho_{a_ia_{i+1}} + \rho_{a_ka_1} = \rho_{a_1a_k} + \rho_{a_ka_1} = 0.$$

□

As final remarks, we note that our result implies the following strengthenings of theorem 12:

COROLLARY 19. *For any $b, c \in A$, every admissible (b, c)-path has the same δ-weight ρ_{bc}.*

PROOF. First notice that for any $b, c \in A$, if $D_b \cap D_c \neq \emptyset$, $\delta_{bc} = \rho_{bc}$. To see this, pick $v \in D_b \cap D_c$ arbitrarily. Obviously, path $\overrightarrow{a} = (b, c)$ is compatible with linear sequence $\overrightarrow{u} = (v, v, v)$ and is thus a straight (b, c)-path. Hence $\rho_{bc} = \delta(\overrightarrow{a}) = \delta_{bc}$.

Now for any $b, c \in A$ and any (b, c)-path $\overrightarrow{a}$ with $C(\overrightarrow{a}) \neq \emptyset$, let $\overrightarrow{u} \in C(\overrightarrow{a})$. Since $u_i \in D_{a_i} \cap D_{a_{i+1}}$ for $i \in [|\overrightarrow{a}| - 1]$,

$$\delta(\overrightarrow{a}) = \sum_{i=1}^{|\overrightarrow{a}|-1} \delta_{a_ia_{i+1}} = \sum_{i=1}^{|\overrightarrow{a}|-1} \rho_{a_ia_{i+1}},$$

which by theorem 18, $= -\rho_{a_{|\overrightarrow{a}|}a_1} = \rho_{a_1a_{|\overrightarrow{a}|}} = \rho_{bc}$. □

COROLLARY 20. *For any $b, c \in A$, ρ_{bc} is equal to δ^*_{bc}, the minimum δ-weight over all (b, c)-paths.*

PROOF. Clearly $\rho_{bc} \geq \delta^*_{bc}$ by corollary 13. On the other hand, for every (b, c)-path $\overrightarrow{a} = (b = a_1, a_2, \dots, a_k = c)$, by lemma 14,

$$\delta(\overrightarrow{a}) = \sum_{i=1}^{k-1} \delta_{a_ia_{i+1}} \geq \sum_{i=1}^{k-1} \rho_{a_ia_{i+1}},$$

which by theorem 18, $= -\rho_{a_ka_1} = \rho_{a_1a_k} = \rho_{bc}$. Hence $\rho_{bc} \leq \delta^*_{bc}$, which completes the proof. □

7. COUNTEREXAMPLES TO STRONGER FORMS OF THEOREM 4

Theorem 4 applies to social choice functions with convex domain and finite range. We now show that neither of these hypotheses can be omitted. Our examples are single player functions.

The first example illustrates that convexity can not be omitted. We present an untruthful single player social choice function with three outcomes a, b, c satisfying W-MON on a path-connected but non-convex domain. The domain is the boundary of a triangle whose vertices are $x = (0, 1, -1)$, $y = (-1, 0, 1)$ and $z = (1, -1, 0)$. x and the open line segment $\overline{zx}$ is assigned outcome a, y and the open line segment $\overline{xy}$ is assigned outcome b, and z and the open line segment $\overline{yz}$ is assigned outcome c. Clearly, $\delta_{ab} = -\delta_{ba} = \delta_{bc} = -\delta_{cb} = \delta_{ca} = -\delta_{ac} = -1$, W-MON (the nonnegative two-cycle property) holds. Since there is a negative cycle $\delta_{ab} + \delta_{bc} + \delta_{ca} = -3$, by lemma 3, this is not a truthful choice function.

We now show that the hypothesis of finite range can not be omitted. We construct a family of single player social choice functions each having a convex domain and an infinite number of outcomes, and satisfying weak monotonicity but not truthfulness.

Our examples will be specified by a positive integer n and an $n \times n$ matrix M satisfying the following properties: (1) M is non-singular. (2) M is positive semidefinite. (3) There are distinct $i_1, i_2, \dots, i_k \in [n]$ satisfying

$$\sum_{j=1}^{k-1} (M(i_j, i_j) - M(i_j, i_{j+1})) + (M(i_k, i_k) - M(i_k, i_1)) < 0.$$

Here is an example matrix with $n = 3$ and $(i_1, i_2, i_3) = (1, 2, 3)$:

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Let $e_1, e_2, \dots, e_n$ denote the standard basis of $\mathbb{R}^n$. Let S_n denote the convex hull of $\{e_1, e_2 \dots, e_n\}$, which is the set of vectors in $\mathbb{R}^n$ with nonnegative coordinates that sum to 1. The range of our social choice function will be the set S_n and the domain D will be indexed by S_n, that is $D = \{y_\lambda : \lambda \in S_n\}$, where y_λ is defined below. The function f maps y_λ to λ.

Next we specify y_λ. By definition, D must be a set of functions from S_n to $\mathbb{R}$. For $\lambda \in S_n$, the domain element $y_\lambda : S_n \longrightarrow \mathbb{R}$ is defined by $y_\lambda(\alpha) = \lambda^T M \alpha$. The non-singularity of M guarantees that $y_\lambda \neq y_\mu$ for $\lambda \neq \mu \in S_n$. It is easy to see that D is a convex subset of the set of all functions from S_n to $\mathbb{R}$.

The outcome graph G_f is an infinite graph whose vertex set is the outcome set $A = S_n$. For outcomes $\lambda, \mu \in A$, the edge weight $\delta_{\lambda\mu}$ is equal to

$$\delta_{\lambda\mu} = inf\{v(\lambda) - v(\mu) : f(v) = \lambda\}$$

$$= y_\lambda(\lambda) - y_\lambda(\mu) = \lambda^T M \lambda - \lambda^T M \mu = \lambda^T M (\lambda - \mu).$$

We claim that G_f satisfies the nonnegative two-cycle property (W-MON) but has a negative cycle (and hence is not truthful).

For outcomes $\lambda, \mu \in A$,

$$\delta_{\lambda\mu} + \delta_{\mu\lambda} = \lambda^T M(\lambda - \mu) + \mu^T M(\mu - \lambda) = (\lambda - \mu)^T M(\lambda - \mu),$$

which is nonnegative since M is positive semidefinite. Hence the nonnegative two-cycle property holds. Next we show that G_f has a negative cycle. Let $i_1, i_2, \ldots, i_k$ be a sequence of indices satisfying property 3 of M. We claim $e_{i_1} e_{i_2} \ldots e_{i_k} e_{i_1}$ is a negative cycle. Since

$$\delta_{e_i e_j} = e_i^T M(e_i - e_j) = M(i,i) - M(i,j)$$

for any $i, j \in [k]$, the weight of the cycle

$$\sum_{j=1}^{k-1} \delta_{e_{i_j} e_{i_{j+1}}} + \delta_{e_{i_k} e_{i_1}}$$
$$= \sum_{j=1}^{k-1} (M(i_j, i_j) - M(i_j, i_{j+1})) + (M(i_k, i_k) - M(i_k, i_1)) < 0,$$

which completes the proof.

Finally, we point out that the third property imposed on the matrix M has the following interpretation. Let $R(M) = \{r_1, r_2, \ldots, r_n\}$ be the set of row vectors of M and let h_M be the single player social choice function with domain $R(M)$ and range $\{1, 2, \ldots, n\}$ mapping r_i to i. Property 3 is equivalent to the condition that the outcome graph G_{h_M} has a negative cycle. By lemma 3, this is equivalent to the condition that h_M is untruthful.

8. FUTURE WORK

As stated in the introduction, the goal underlying the work in this paper is to obtain useful and general characterizations of truthfulness.

Let us say that a set D of $P \times A$ real valuation matrices is a *WM-domain* if any social choice function on D satisfying weak monotonicity is truthful. In this paper, we showed that for finite A, any convex D is a WM-domain. Typically, the domains of social choice functions considered in mechanism design are convex, but there are interesting examples with non-convex domains, e.g., combinatorial auctions with unknown single-minded bidders. It is intriguing to find the most general conditions under which a set D of real matrices is a WM-domain. We believe that convexity is the main part of the story, i.e., a WM-domain is, after excluding some exceptional cases, "essentially" a convex set.

Turning to parametric representations, let us say a set D of $P \times A$ matrices is an *AM-domain* if any truthful social choice function with domain D is an affine maximizer. Roberts' theorem says that the unrestricted domain is an AM-domain. What are the most general conditions under which a set D of real matrices is an AM-domain?

Acknowledgments

We thank Ron Lavi for helpful discussions and the two anonymous referees for helpful comments.

9. REFERENCES

[1] A. Archer and E. Tardos. Truthful mechanisms for one-parameter agents. In *IEEE Symposium on Foundations of Computer Science*, pages 482–491, 2001.

[2] Y. Bartal, R. Gonen, and N. Nisan. Incentive compatible multi unit combinatorial auctions. In *TARK '03: Proceedings of the 9th conference on Theoretical aspects of rationality and knowledge*, pages 72–87. ACM Press, 2003.

[3] S. Bikhchandani, S. Chatterjee, and A. Sen. Incentive compatibility in multi-unit auctions. Technical report, UCLA Department of Economics, Dec. 2004.

[4] A. Goldberg, J. Hartline, A. Karlin, M. Saks and A. Wright. Competitive Auctions, 2004.

[5] H. Gui, R. Muller, and R. Vohra. Dominant strategy mechanisms with multidimensional types. Technical Report 047, Maastricht: METEOR, Maastricht Research School of Economics of Technology and Organization, 2004. available at http://ideas.repec.org/p/dgr/umamet/2004047.html.

[6] R. Lavi, A. Mu'alem, and N. Nisan. Towards a characterization of truthful combinatorial auctions. In *FOCS '03: Proceedings of the 44th Annual IEEE Symposium on Foundations of Computer Science*, page 574. IEEE Computer Society, 2003.

[7] D. Lehmann, L. O'Callaghan, and Y. Shoham. Truth revelation in approximately efficient combinatorial auctions. *J. ACM*, 49(5):577–602, 2002.

[8] A. Mas-Colell, M. Whinston, and J. Green. *Microeconomic Theory*. Oxford University Press, 1995.

[9] N. Nisan. Algorithms for selfish agents. *Lecture Notes in Computer Science*, 1563:1–15, 1999.

[10] K. Roberts. The characterization of implementable choice rules. *Aggregation and Revelation of Preferences*, J-J. Laffont (ed.), North Holland Publishing Company.

[11] J.-C. Rochet. A necessary and sufficient condition for rationalizability in a quasi-linear context. *Journal of Mathematical Economics*, 16:191–200, 1987.

[12] I. Rozenshtrom. Dominant strategy implementation with quasi-linear preferences. Master's thesis, Dept. of Economics, The Hebrew University, Jerusalem, Israel, 1999.