Graduate Artificial Intelligence 15-780

Homework 6: *Automated Mechanism Design*

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An optional homework, for extra homework credit only.
Out on April 26
Due on May 5

First, we're going to need to review some game theory concepts. In the simple auction setting, an agent's *type* simply corresponds to their *value for the item.* We denote the type of agent i as $\theta_i \in \Theta_i$.

A (single-item) auction works by receiving a vector of type reports $(\theta_1, \ldots, \theta_n) \in \Theta_1 \times \cdots \times \Theta_n$ from the n agents, and producing an allocation a_i and payment π_i for each agent, where a_i is binary and π_i is non-negative. We can add an extra agent 0 to denote the seller keeping the item; if $a_0 = 1$ the item is unallocated. In a single-item auction, the a_i must be allocated exactly once:

$$\sum_i a_i = 1$$

The *expected revenue* of an auction is just the expectation taken over the payments π.

Agent i's *utility* is given by the difference between their value for their allocation and the price they pay

$$u_i = a_i\theta_i - \pi_i$$

Since the auction mechanism determines a_i and π_i from the type reports, we can abuse notation and define

$$u_i(\theta, \vec{\theta}_{-i})$$

as agent i's utility from reporting θ when the other agents report $\vec{\theta}_{-i}$.

Throughout this problem, we will assume that the probability distribution over true types is common knowledge, known to all agents and the auctioneer.

Now we can define various properties of an auction.

Definition. *An auction is* efficient *if it always allocates the good to the agent that reports the highest type. Formally*

$$a_i = 1 \longrightarrow \theta_i = \max_j \theta_j$$

Definition. *An auction is* individually rational *if an agent is never charged more than their value for the allocation they receive. Formally, for all agents i, all true types θ_i, and all set of other type reports $\vec{\theta}_{-i}$*

$$u_i(\theta_i, \vec{\theta}_{-i}) \geq 0$$

Definition. *An auction is* truth promoting in dominant strategies *for agent i if that agent is never incentivized to misreport their true type. Formally, for all true types θ_i, all other reports of agents $\vec{\theta}_{-i}$, and all misreported types $\hat{\theta}$*

$$u_i(\theta_i, \vec{\theta}_{-i}) \geq u_i(\hat{\theta}, \vec{\theta}_{-i})$$

if this holds for all agents, it is a dominant strategy equilibrium (DSE).

We can also relax DSE, so that agents are only incentivized to report truthfully *in expectation* over the reports of the other agents.

Definition. *An auction is* truth promoting in expectation *for agent i if that agent is never incentivized* in expectation *to misreport their true type. Formally, for all true types θ_i and all misreported types $\hat{\theta}$*

$$\mathbb{E}_{\vec{\theta}_{-i}} u_i(\theta_i, \vec{\theta}_{-i}) \geq \mathbb{E}_{\vec{\theta}_{-i}} u_i(\hat{\theta}, \vec{\theta}_{-i})$$

where the expectation is taken over the type reports of the other agents. If this holds for all agents, it is a Bayes-Nash equilibrium (BNE).

Observe that DSE is strictly stronger than BNE.

The key insight of *automated mechanism design* is that all of these properties can be embedded into a MIP in a straightforward fashion. Consider, for instance, how to embed the individual rationality constraints into a MIP. Let $a_i^{\vec{\theta}}$ be a binary representing whether agent i is allocated the item when the agents report types $\vec{\theta}$, and let $\pi_i^{\vec{\theta}}$ similarly represent the payment of agent i for that report. Then for every agent i, every true type $\theta \in \Theta_i$, and every agent type report $\vec{\theta}$ the following relation must hold:

$$\theta a_i^{\vec{\theta}} - \pi_i^{\vec{\theta}} \geq 0$$

As another example, it is possible to mandate efficiency just by hardcoding in the proper values for the $a_i^{\vec{\theta}}$ variables (to select the agent with the highest valuation).

Problem 1: Auction Design (60 points)

A plutocrat football-team owner has installed a massive luxury suite in his new stadium. In this problem, you will use your knowledge of game theory and mixed-integer programming to get him the highest dollar for season tickets to his luxury suite by designing an optimal auction.

Here, there are three possible agents interested in the tickets, each of whom has one of two possible valuations:

a) The *A*dvertising Executive. He's got a reasonable shot at needing to entertain a big account. He values the tickets at 400 thousand dollars with probability 1/4, and values the tickets at 100 thousand with probability 3/4.

b) The Lumber *B*aron. Being a comfortable and secure captain of industry, he is relatively comfortable and secure in his valuations. He values the tickets at 300 thousand or at 250 thousand, each with probability 1/2.

c) The Disgraced *C*EO. He's currently facing federal charges, but there's a small chance he could wiggle his way out of liability. He values the tickets at 500 thousand dollars with probability 1/6, and values the tickets at 75 thousand with probability 5/6.

All values for the agents are drawn independently. Agents know their own valuations, and the probabilities of agents having their valuations are common knowledge, including common knowledge to the auctioneer. Therefore, the revenue-maximizing auction maximizes revenue *in expectation*.

- Write out and solve the MIP corresponding to the *revenue-maximizing, individually-rational, efficient, truth promoting in DSE* auction. That is, describe the allocations and payments for all 8 possible inputs.
- Solve for the *revenue-maximizing, individually-rational auction that truth promotes in DSE*. What is the expected revenue? How often are the tickets not sold to the highest bidder? How often are the tickets not sold at all? (Recall that you can model the tickets not being sold by introducing a dummy agent.)
- Solve for the *revenue-maximizing, individually-rational auction that truth promotes in BNE*. What is the expected revenue? How often are the tickets not sold to the highest bidder?
- Write out a MIP for the expected revenue of the revenue-maximizing, individually-rational auction that is truth-promoting in BNE, subject to the tickets being sold to the highest bidder (i.e., the auction being efficient) with probability at least p. Plot expected revenue against $p \in [0, 1]$.