15-780: Grad AI
Lecture 13: Duality; Planning

Geoff Gordon (this lecture)
Tuomas Sandholm
TAs Sam Ganzfried, Byron Boots

Review

Branch & bound (& cut)

- *Worked examples*
- *Demonstrated how to simulate resolution*
  - *and therefore DPLL+CL*

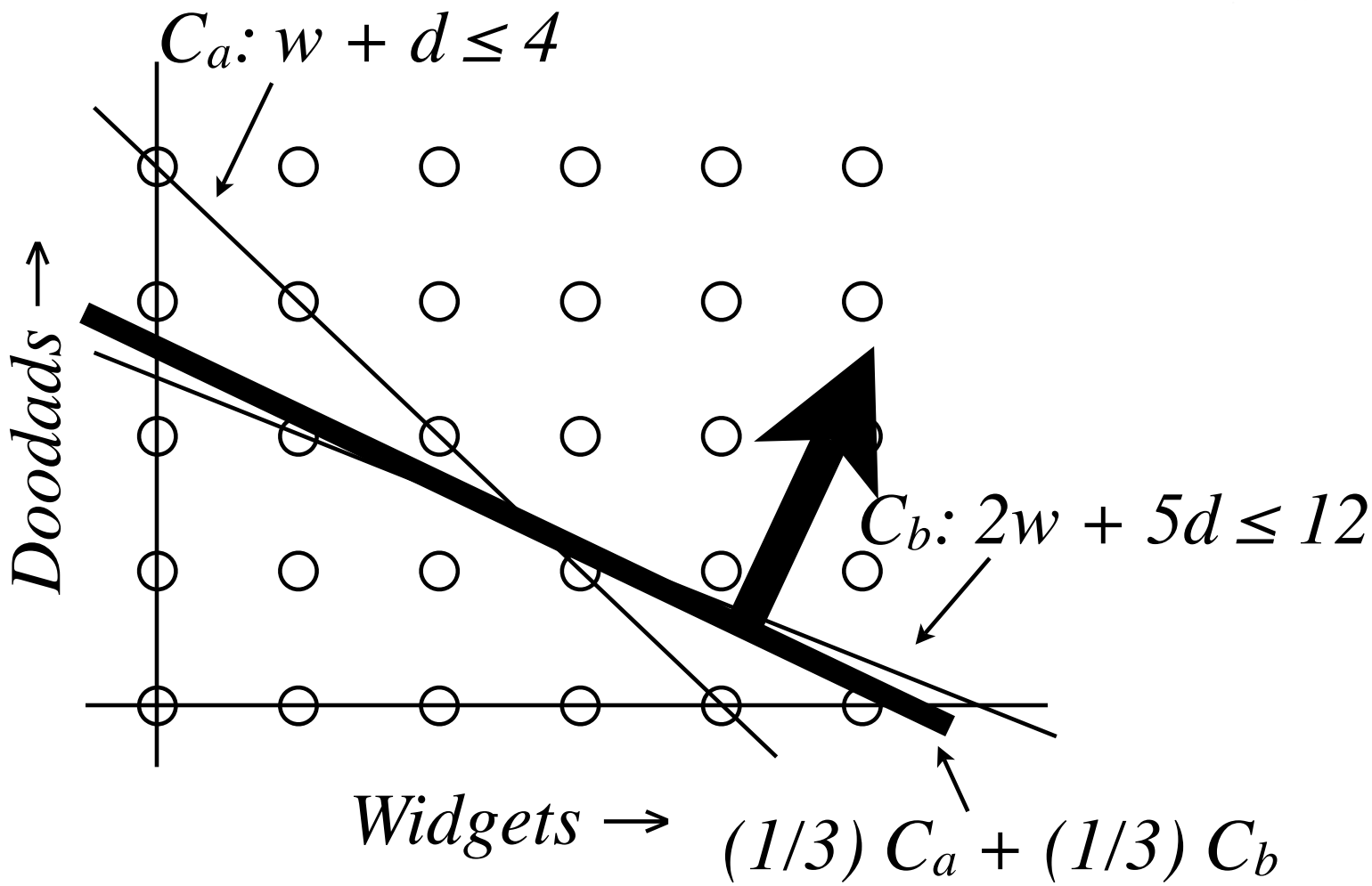
MILP examples

- *Path planning*
  - *not NP-complete; 0 integrality gap*
- *Planetary exploration planning*

Duality w/ inequalities

- *Take a linear combination of constraints to bound objective*
- $(\boxed{a + 2b})\, w + (\boxed{a + 5b})\, d \leq 4a + 12b$
- $profit = \boxed{1} w + \boxed{2} d$
- *So, if* $1 \leq (a + 2b)$ *and* $2 \leq (a + 5b)$*, we know that* $profit \leq 4a + 12b$

Duality picture



Use of duality

- *Any feasible solution to dual yields upper bound (compared with only optimal solution to primal)*
- *Dual sometimes easier to work with*

Dual dual

- *Take the dual of an LP twice, get the original LP back (called* ***primal****)*
- *Many LP solvers will give you both primal and dual solutions at the same time for no extra cost*

Duality w/ equality

Equality example

- *minimize y subject to*
- $x + y = 1$
- $2y - z = 1$
- $x, y, z \geq 0$

Equality example

- *Want to prove bound* $y \geq \ldots$
- *Look at 2nd constraint:*

  $$2y - z = 1 \quad \Rightarrow$$

  $$y - z/2 = 1/2$$

- *Since* $z \geq 0$*, dropping* $-z/2$ *can only increase LHS* $\Rightarrow$
  - $y \geq 1/2$

Duality w/ equalities

- *In general, could start from any linear combination of equality constraints*
  - *no need to restrict to +ve combination*
- $a\,(x + y - 1) + b\,(2y - z - 1) = 0$
- $a\,x + (a + 2b)\,y - b\,z = a + b$

Duality w/ equalities

- $a\,x + (a + 2b)\,y - b\,z = a + b$
- *As long as coefficients on LHS* $\leq (0, 1, 0)$,
  - *objective* $= 0\,x + 1\,y + 0\,z \geq a + b$
- *So, maximize* $a + b$ *subject to*
  - $a \leq 0$
  - $a + 2b \leq 1$
  - $-b \leq 0$

Duality recipes

Recipe for inequalities

- *If we have an LP in matrix form,*

  maximize c'x subject to

  $Ax \leq b$

  $x \geq 0$

- *Its dual is a similar-looking LP:*

  minimize b'y subject to

  $A'y \geq c$

  $y \geq 0$

$Ax \leq b$ means every component of Ax is $\leq$ corresponding component of b

Recipe with ≤ and =

- *If we have an LP with equalities,*

  maximize c'x s.t.

  $Ax \leq b$

  $Ex = f$

  $x \geq 0$

- *Its dual has some unrestricted variables:*

  minimize b'y + f'z s.t.

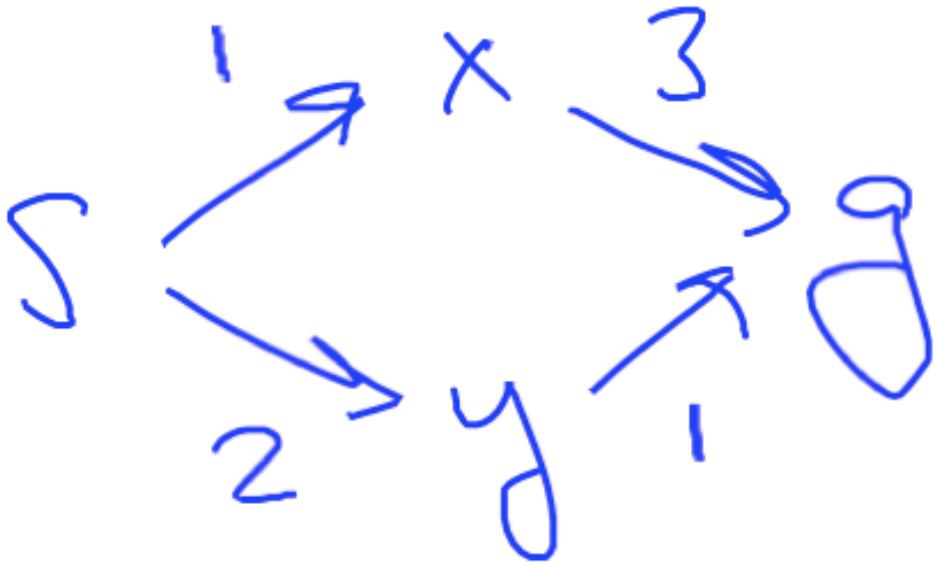
  $A'y + E'z \geq c$

  $y \geq 0$

  z unrestricted

Duality example

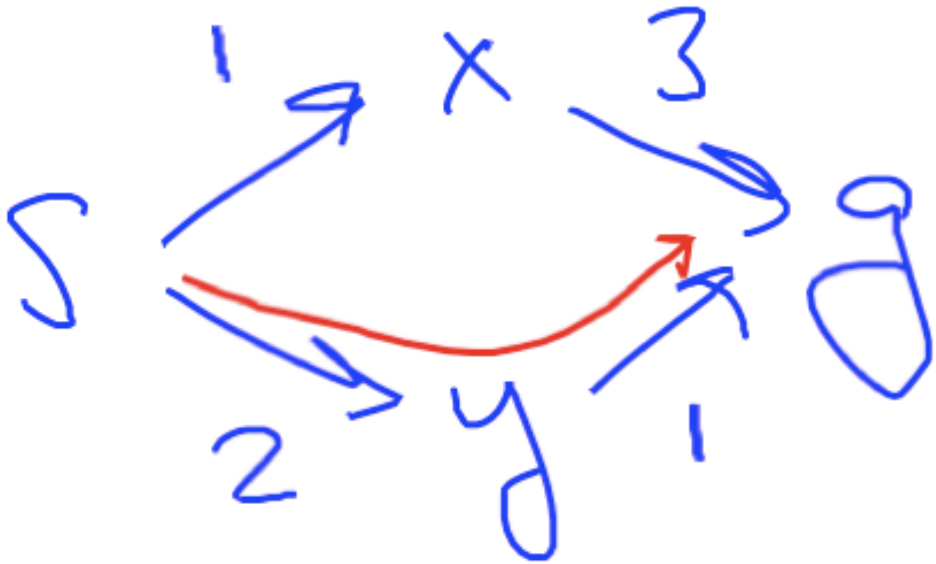
Path planning LP



- *Find the min-cost path: variables*

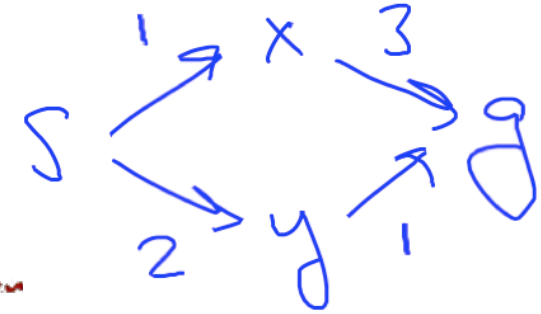
$$P_{sx},\ P_{sy},\ P_{xg},\ P_{yg} \geq 0$$

Optimal solution



$p_{sy} = p_{yg} = 1, \quad p_{sx} = p_{xg} = 0, \quad cost\ 3$

Path planning LP



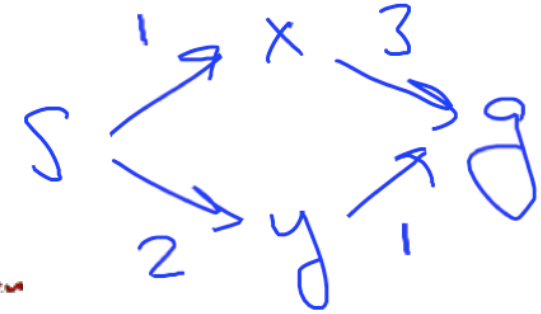
$$\min \ (1\ 3\ 2\ 1)\, p$$

st

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix} p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$p \geq 0$$

Path planning LP



$$\min \ (1\ 3\ 2\ 1)\, p$$

st

$$\begin{array}{c} \lambda_s \\ \lambda_x \\ \lambda_y \\ \lambda_g \end{array} \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix} p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$p \geq 0$$

Deriving dual

$$\min c^T p \quad \text{s.t.}$$

$$Ap = b \qquad p \geq 0$$

$$\lambda^T A p = \lambda^T b$$

linear combo of constrs

$p \geq 0$

$$\lambda^T A \leq c^T \Rightarrow \lambda^T A p \leq c^T p$$

bound

$$\Rightarrow \lambda^T b \leq c^T p$$

$$\max \lambda^T b \quad \text{s.t.}$$

$$\lambda^T A \leq c^T \iff A^T \lambda \leq c$$

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Dual

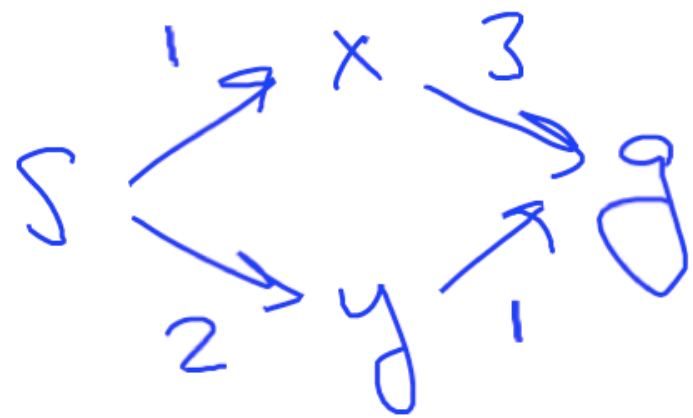
$$\max \quad \lambda_s - \lambda_g$$

$$\text{st} \quad \lambda_s - \lambda_x \le 1$$

$$\lambda_x - \lambda_g \le 3$$

$$\lambda_s - \lambda_y \le 2$$

$$\lambda_y - \lambda_g \le 1$$



Optimal dual solution

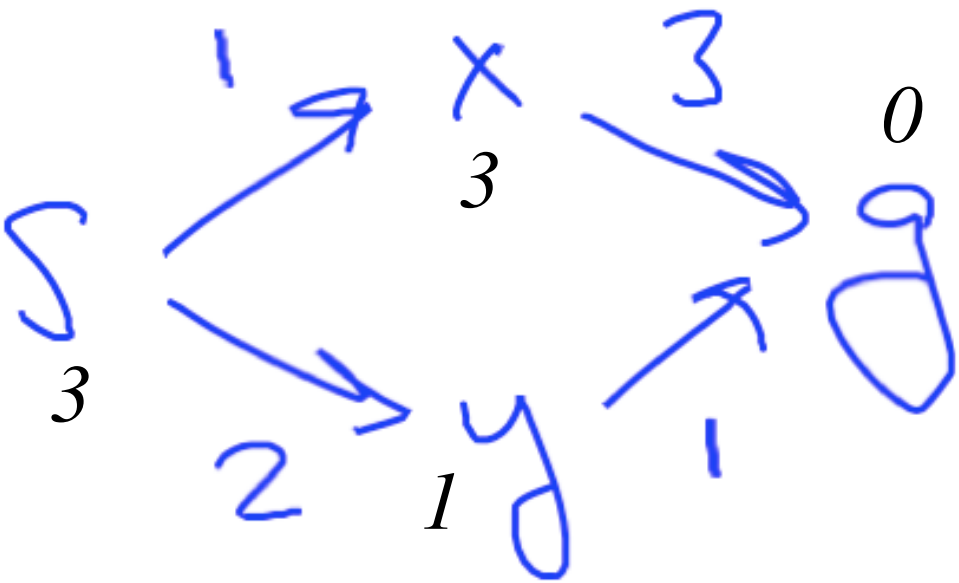
$$\max \quad \lambda_s - \lambda_g$$

$$\text{st} \quad \lambda_s - \lambda_x \le 1$$

$$\lambda_x - \lambda_g \le 3$$

$$\lambda_s - \lambda_y \le 2$$

$$\lambda_y - \lambda_g \le 1$$



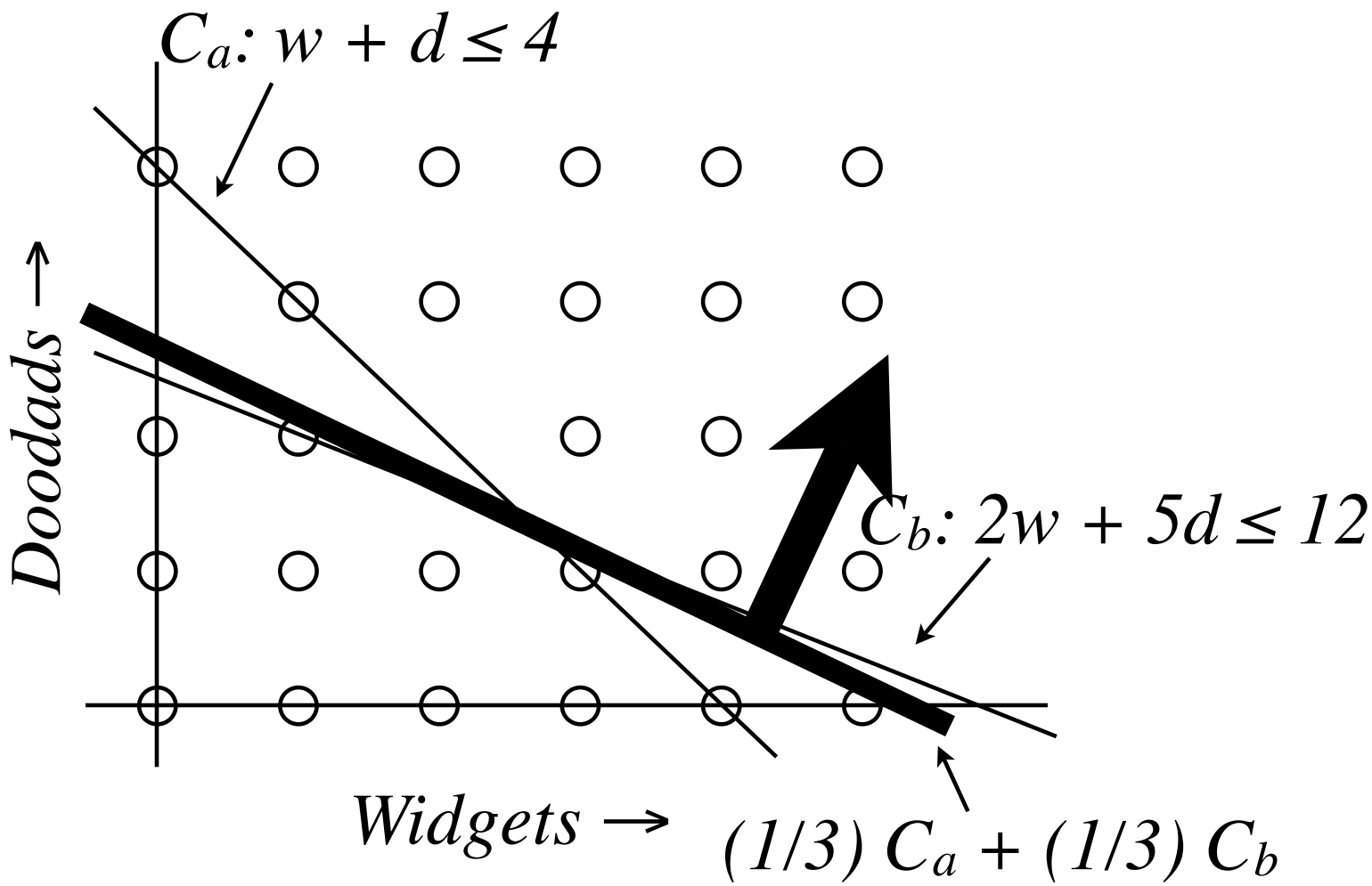
Any solution which adds a constant to all λs also works; $\lambda_x = 2$ also works

Interpreting the dual

Interpreting the dual variables

- *The primal variable variables in the factory LP were how many widgets and doodads to produce*
- *We interpreted dual variables as multipliers for primal constraints*

Dual variables as multipliers



Dual variables as prices

- *"Multiplier" interpretation doesn't give much intuition*
- *It is often possible to interpret dual variables as* ***prices*** *for primal constraints*

Dual variables as prices

- *Suppose someone offered us a quantity* ε *of wood, loosening constraint to*

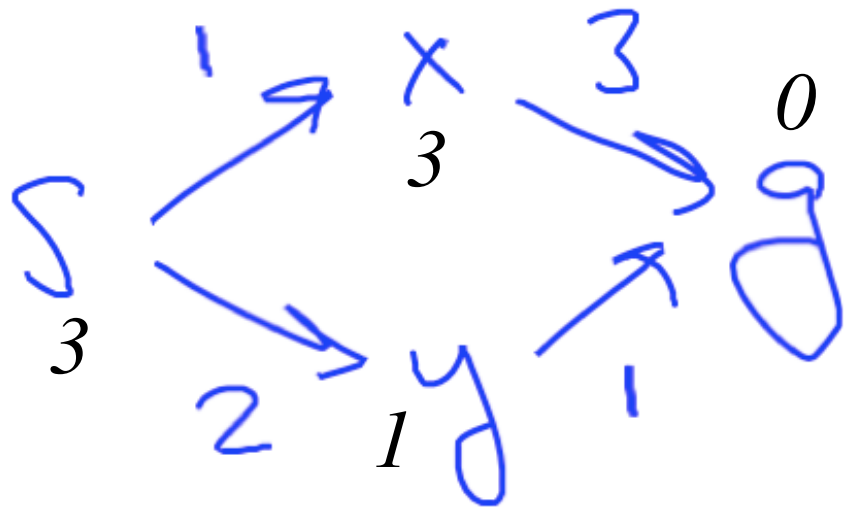
$$w + d \leq 4 + \varepsilon$$

- *How much should we be willing to pay for this wood?*

Dual variables as prices

- *RHS in primal is objective in dual*
- *So, dual constraints stay same, previous solution $a = b = 1/3$ still dual feasible*
  - *still optimal if ε small enough*
- *Bound changes to $(4 + \varepsilon)\, a + 12\, b$, difference of $\varepsilon * 1/3$*
- *So we should pay up to \$1/3 per unit of wood (in small quantities)*

Price example: path planning



- *Dual variables are prices on nodes: how much does it cost to start there?*
- *Dual constraints are local price constraints: edge* xg *(cost 3) means that node x can't cost more than 3 + price of node g*

Planning

Time

- *Recall **fluents***
- *For KBs that evolve, add extra argument to each predicate saying when it was true*
  - *at(Robot, Wean5409)*
  - *at(Robot, Wean5409, 17)*

Operators

- *Given a representation like this, can define* ***operators*** *that change state*
- *E.g., given*
  - *at(Robot, Wean5409, 17)*
  - *moves(Robot, Wean5409, corridor, 17)*
- *results might be*
  - *at(Robot, corridor, 18)*
  - *¬at(Robot, Wean5409, 18)*

Goals

- *Want our robot to, e.g., get sandwich*
- *Search for proof of has(Geoff, Sandwich, t)*
- *Try to analyze proof tree to find sequence of operators that make goal true*

Complications

- *This strategy yields lots of complications*
  - *need axioms describing natural numbers (for time)*
  - *frame or successor-state axioms (facts don't change unless operator does it)*
  - *unique names, exactly one action per step, generalization of answer literal…*
- *Result can be slow inference*

Planning

- *Alternate solution: define a subset of FOL especially for planning*
- *E.g., STRIPS language*
  - *no functions, limited quantification, …*
- *STanford Research Institute Problem Solver*

STRIPS

- *State of world at each time = { propositions }*
- *Each proposition is ground literal*
- *For brevity, list only true literals*
- *Time is implicit*

STRIPS state example

STRIPS state example

- *food(N)*
- *hungry(M)*
- *at(N, W)*
- *at(M, X)*
- *at(B1, Y)*
- *at(B2, Y)*
- *at(B3, Z)*
- *on(B2, B1)*
- *clear(B2)*
- *clear(B3)*
- *height(M, Low)*
- *height(N, High)*

STRIPS operators

- *Operator = { preconditions }, { effects }*
- *If preconditions are true at time t,*
  - *can apply operator at time t*
  - *effects will be true at time t+1*
  - *rest of state unaffected*
- *Basic STRIPS: one operator per step*

Quantification in operators

- *Preconditions of operator may contain variables (implicit ∀)*
- *Operator can apply if preconditions unify with state t (using binding X)*
- *state t+1 has e / X for each e in effects*

Operator example

- $Eat(target, p, l)$
  - ***pre***: $hungry(M), food(target), at(M, p), at(target, p), level(M, l), level(target, l)$
  - ***eff***: $\neg hungry(M), full(M), \neg at(target, p), \neg level(target, l)$

Operator example

- *Move(from, to)*
  - ***pre***: *at(M, from), level(M, Low)*
  - ***eff***: *at(M, to), ¬at(M, from)*
- *Push(object, from, to)*
  - ***pre***: *at(object, from), at(M, from), clear(object)*
  - ***eff***: *at(M, to), at(object, to), ¬at(object, from), ¬at(M, from)*

Operator example

- *Climb(object, p)*
  - ***pre***: *at(M, p), at(object, p), level(M, Low), clear(object)*
  - ***eff***: *level(M, High), ¬level(M, Low)*
- *ClimbDown()*
  - ***pre***: *level(M, High)*
  - ***eff***: *¬level(M, High), level(M, Low)*

Plan search

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Plan search

- *Given a planning problem (start state, operator descriptions, goal)*
- *Run standard search algorithms to find plan*
- *Decisions: search state representation, neighborhood, search algorithm*

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Linear planner

- *Simplest choice:* ***linear planner***
- *Search state = sequence of operators*
- *Neighbor: add an operator to end of sequence*
- *Bind variables as necessary*
  - *both operator and binding are choice points*

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Linear planner

- *Can search forward from start or backward from goal*
- *Or mix the two*
- *Goal is often incompletely specified*
- *Example heuristic: number of open literals*

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Goal: full(M)

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STRIPS state example

- *food(N)*
- *hungry(M)*
- *at(N, W)*
- *at(M, X)*
- *at(B1, Y)*
- *at(B2, Y)*
- *at(B3, Z)*
- *on(B2, B1)*
- *clear(B2)*
- *clear(B3)*
- *level(M, Low)*
- *level(N, High)*

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Linear planner example

- *Start w/ empty plan [ ], initial world state*
- *Pick an operator, e.g.,*
  - *Move(from, to)*
    - *at(M, from), level(M, Low)*
    - *at(M, to), ¬at(M, from)*

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Linear planner example

- *Bind variables so that preconditions match world state*
  - *e.g., from: X, to: Y*
  - *pre: at(M, X), level(M, Low)*
  - *post: at(M, Y), ¬at(M, X)*

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Apply operator

- *food(N)*
- *hungry(M)*
- *at(N, W)*
- *at(M, X)*
- *at(B1, Y)*
- *at(B2, Y)*
- *at(B3, Z)*
- *on(B2, B1)*
- *clear(B2)*
- *clear(B3)*
- *level(M, Low)*
- *level(N, High)*

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Apply operator

- *food(N)*
- *hungry(M)*
- *at(N, W)*
- *at(B1, Y)*
- *at(B2, Y)*
- *at(B3, Z)*
- *on(B2, B1)*
- *clear(B2)*
- *clear(B3)*
- *level(M, Low)*
- *level(N, High)*

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Apply operator

- *food(N)*
- *hungry(M)*
- *at(N, W)*
- *at(M, Y)*
- *at(B1, Y)*
- *at(B2, Y)*
- *at(B3, Z)*
- *on(B2, B1)*
- *clear(B2)*
- *clear(B3)*
- *level(M, Low)*
- *level(N, High)*

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Repeat…

- *Plan is now [move(X, Y)]*
- *World state is as in previous slide*
- *Pick another operator and binding*
  - *Climb(object, p), p: Y*
    - *at(M, p), at(object, p), level(M, Low), clear(object)*
    - *level(M, High), ¬level(M, Low)*

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Apply operator

- *food(N)*
- *hungry(M)*
- *at(N, W)*
- *at(M, Y)*
- *at(B1, Y)*
- *at(B2, Y)*
- *at(B3, Z)*
- *on(B2, B1)*
- *clear(B2)*
- *clear(B3)*
- *level(M, Low)*
- *level(N, High)*

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Apply operator

- *food(N)*
- *hungry(M)*
- *at(N, W)*
- *at(M, Y)*
- *at(B1, Y)*
- *at(B2, Y)*
- *at(B3, Z)*
- *on(B2, B1)*
- *clear(B2)*
- *clear(B3)*
- *level(N, High)*

55

Apply operator

- *food(N)*
- *hungry(M)*
- *at(N, W)*
- *at(M, Y)*
- *at(B1, Y)*
- *at(B2, Y)*
- *at(B3, Z)*
- *on(B2, B1)*
- *clear(B2)*
- *clear(B3)*
- *level(M, High)*
- *level(N, High)*

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And so forth

- *Goal: full(M)*
- *A possible plan:*
  - *move(X, Y), move(Y, Z), push(B3, Z, Y), push(B3, Y, X), push(B3, X, W), climb(B3, W), eat(N, W, High)*
- *DFS will try moving XYX, climbing on boxes unnecessarily, etc.*

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Partial-order planner

- *Linear planner can be wasteful: backtrack undoes most recent action, rather than one that might have caused failure*
- ***Partial order planner** tries to fix this*
- *Avoids committing to details of plan until it has to (**principle of least commitment**)*

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Partial-order planner

- *Search state:*
  - *set of operators (partially bound)*
  - *ordering constraints*
  - *causal links (also called* ***guards****)*
  - *open preconditions*

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Set of operators

- *Might include move(X, p) "I will move somewhere from X", eat(target) "I will eat something"*
- *Also includes extra operators START, FINISH*
  - *effects of START are initial state*
  - *preconditions of FINISH are goals*

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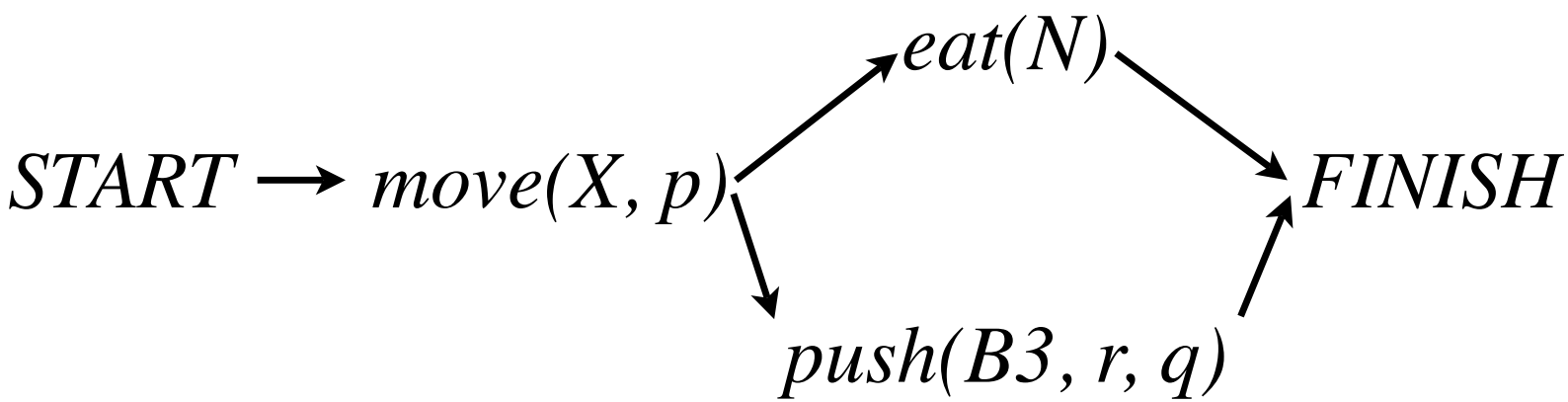
Partial ordering



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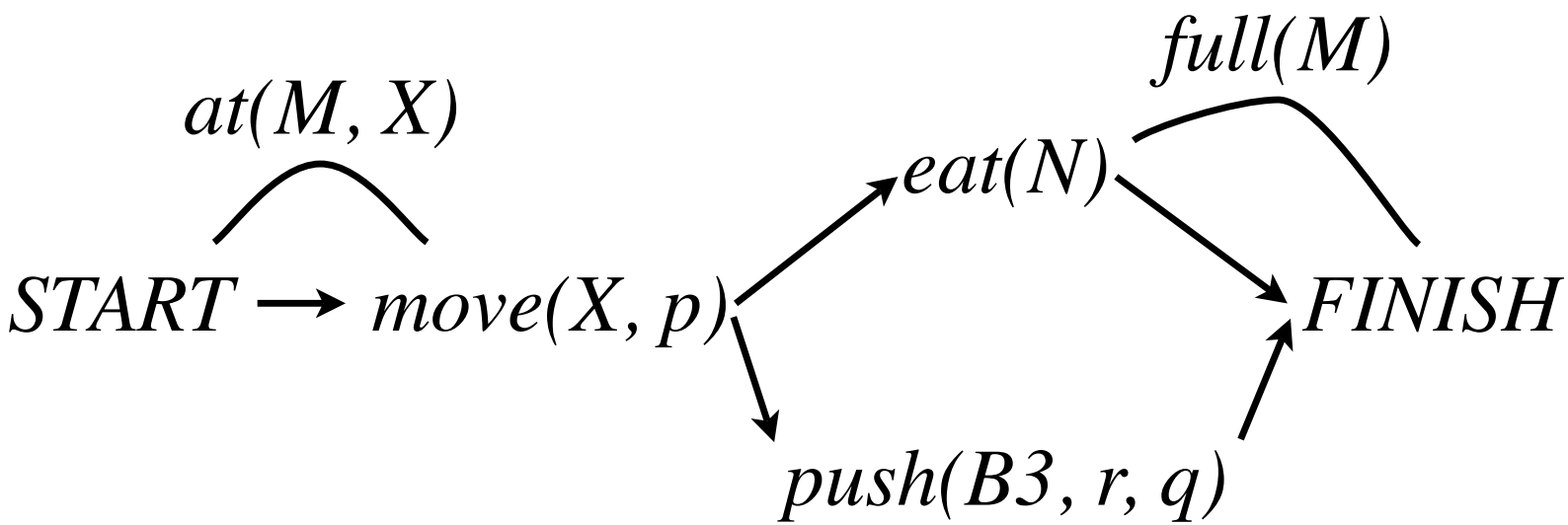
Guards



- *Describe where preconditions are satisfied*

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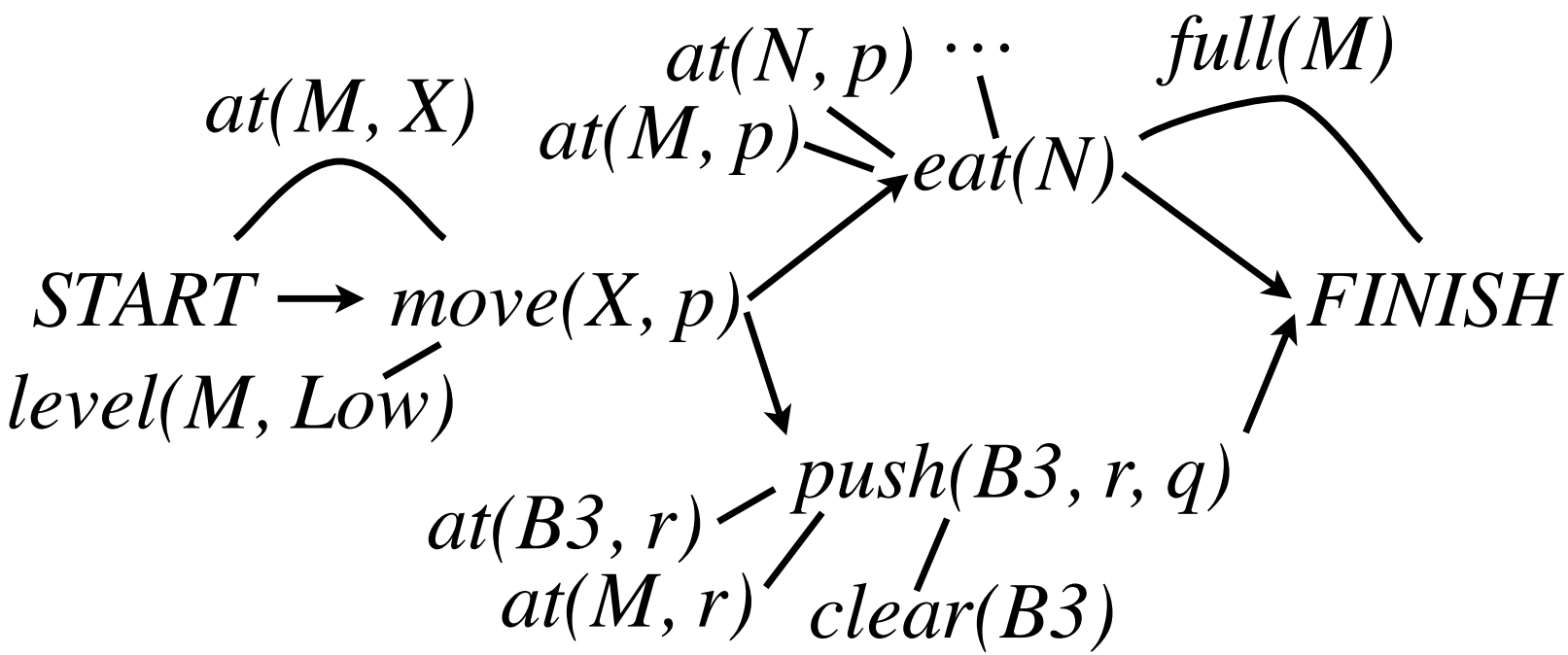
Open preconditions



- *All unsatisfied preconditions of any action*
- *Unsatisfied = doesn't have a guard*

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Partial-order planner

- *Neighborhood: plan refinement*
- *Add an operator, guard, or ordering constraint*

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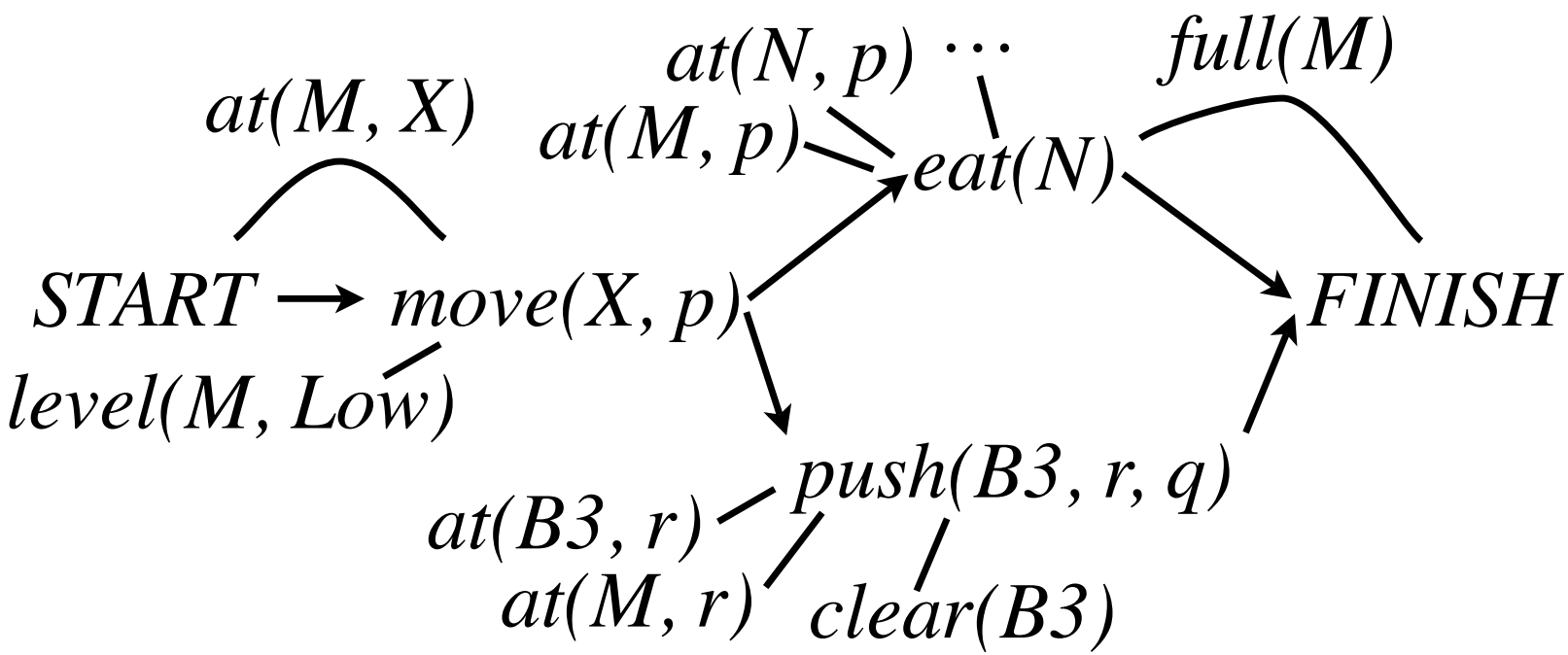
Adding an ordering constraint



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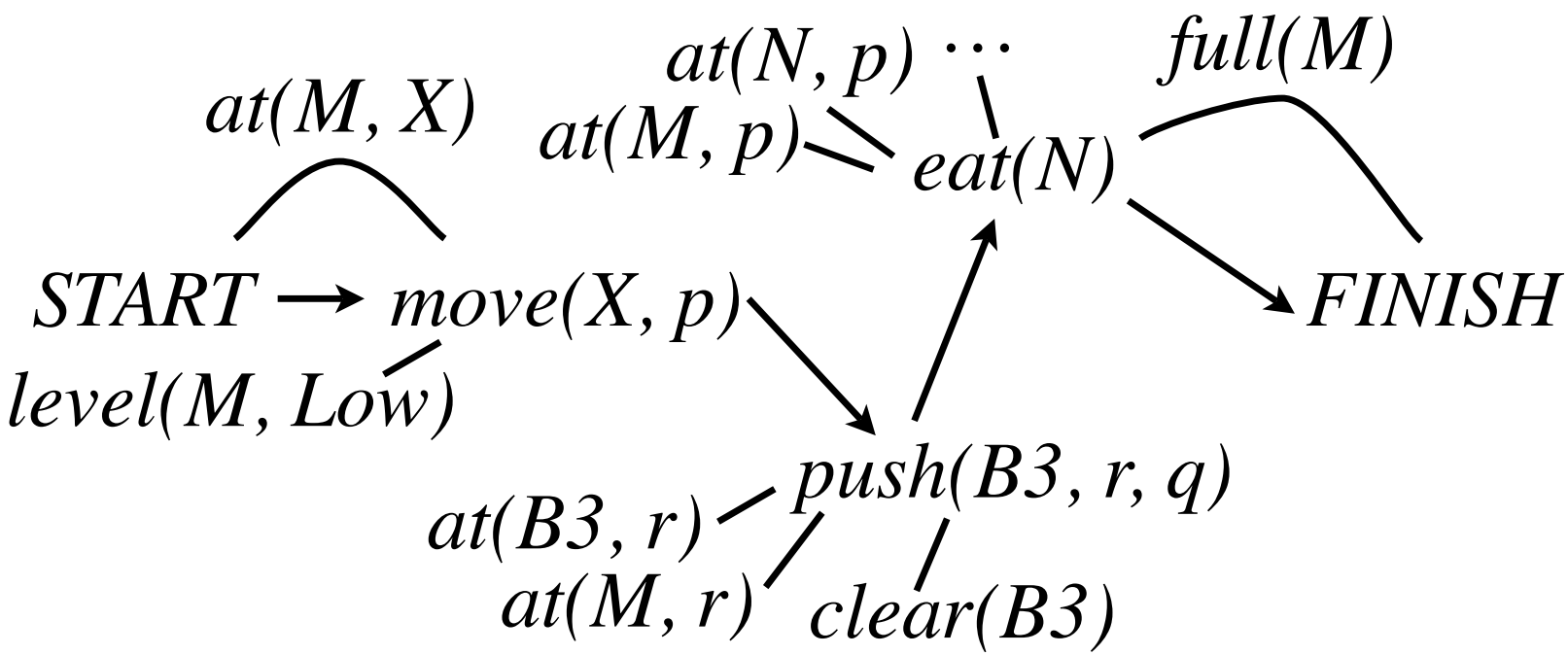
Adding an ordering constraint



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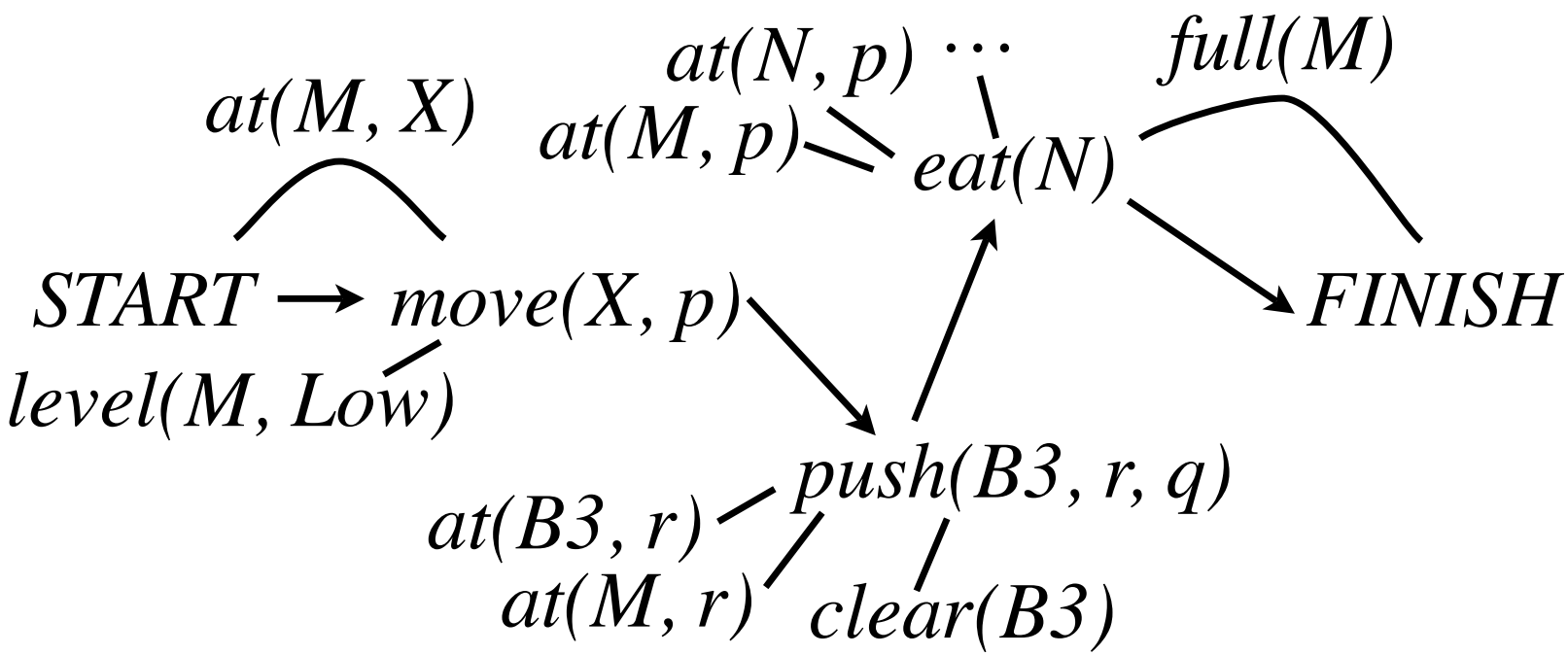
Adding an ordering constraint



- *Wouldn't ever add ordering on its own—but may need to when adding operator or guard*

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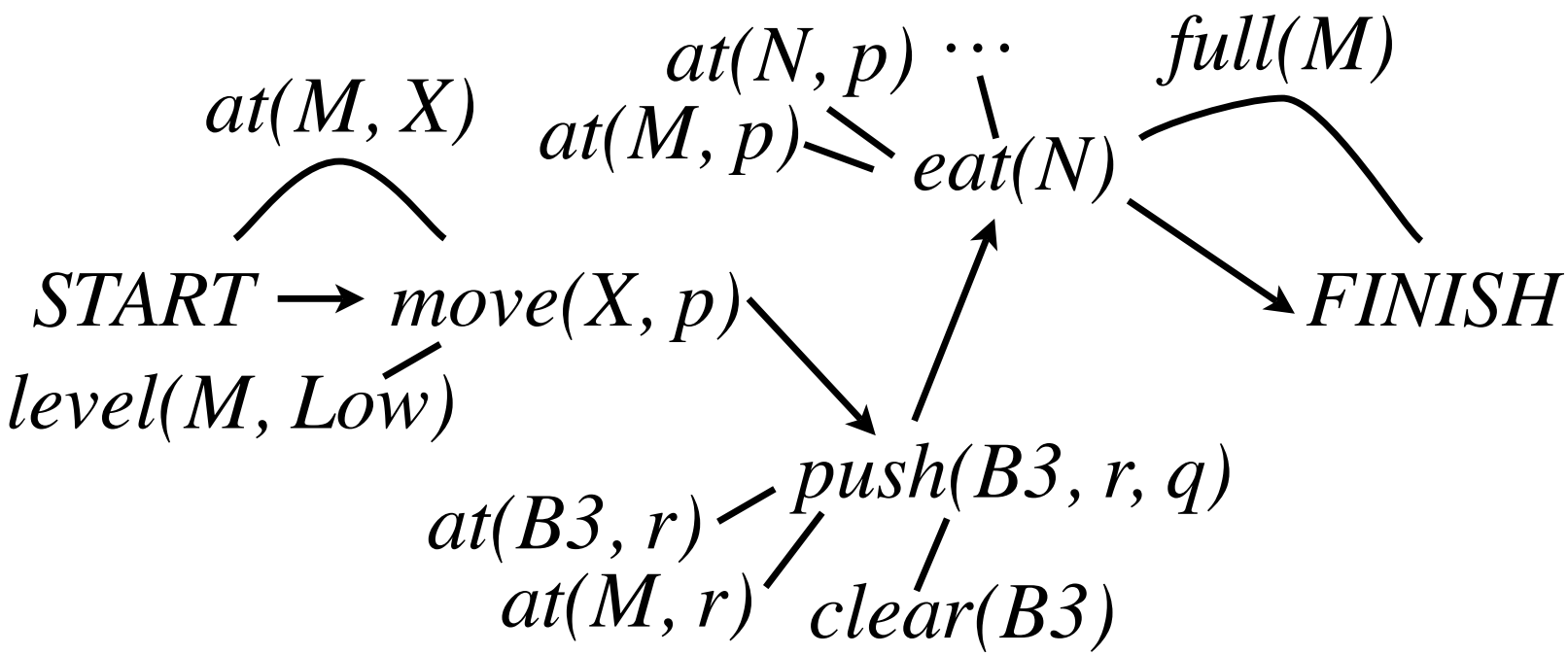
Adding a guard



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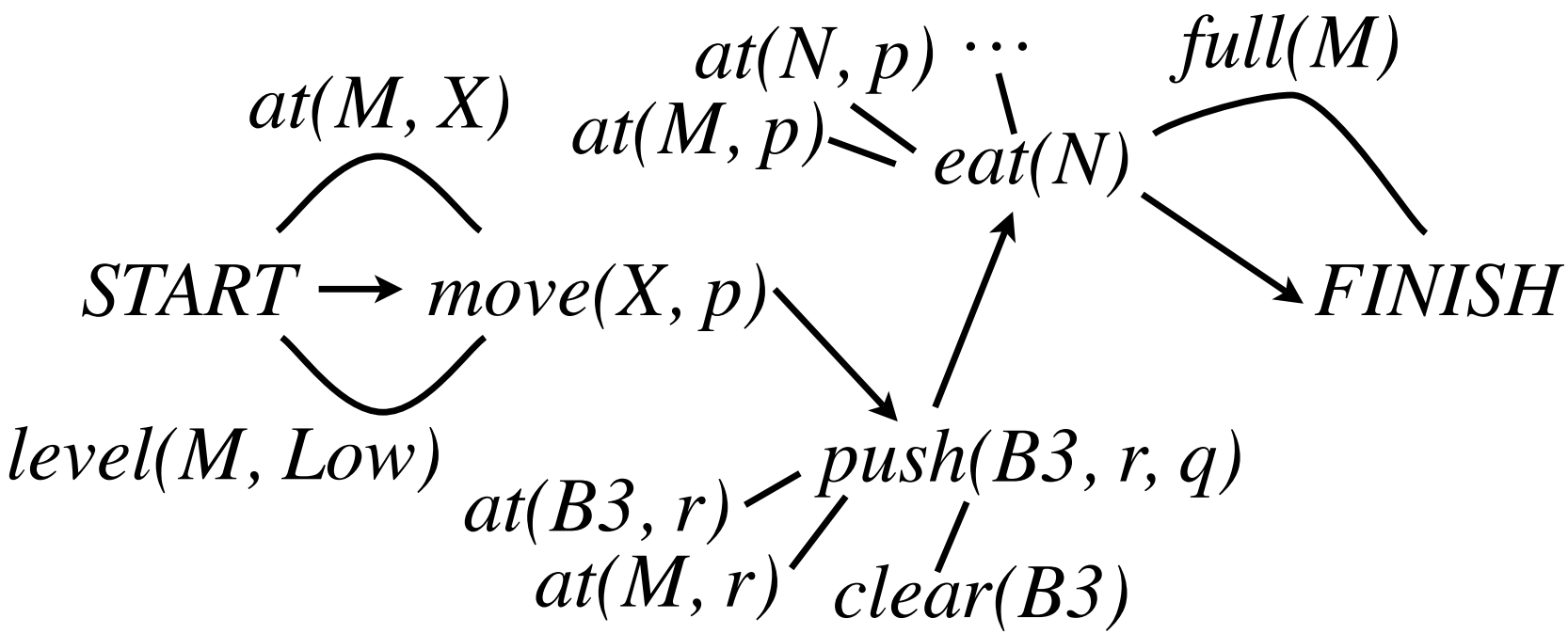
Adding a guard



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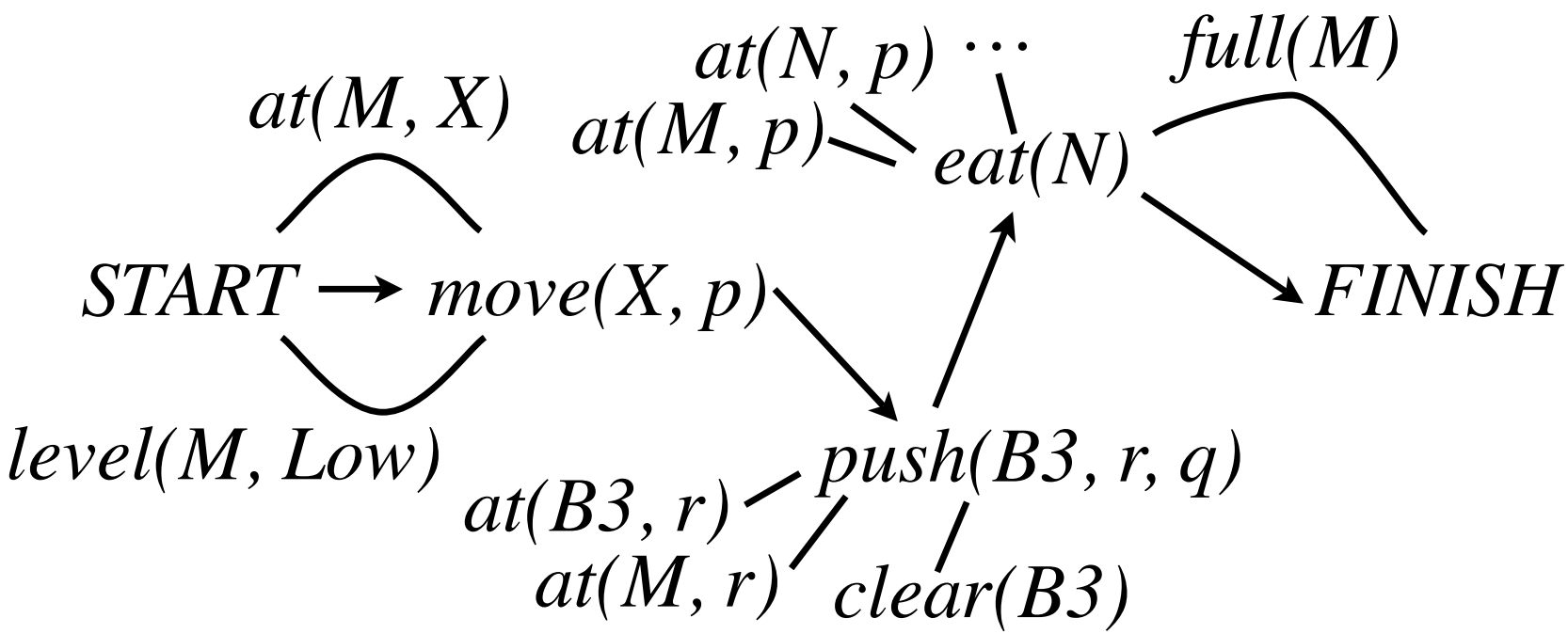
Adding a guard



- *Must go forward (may need to add ordering)*
- *Can't cross operator that affects condition*

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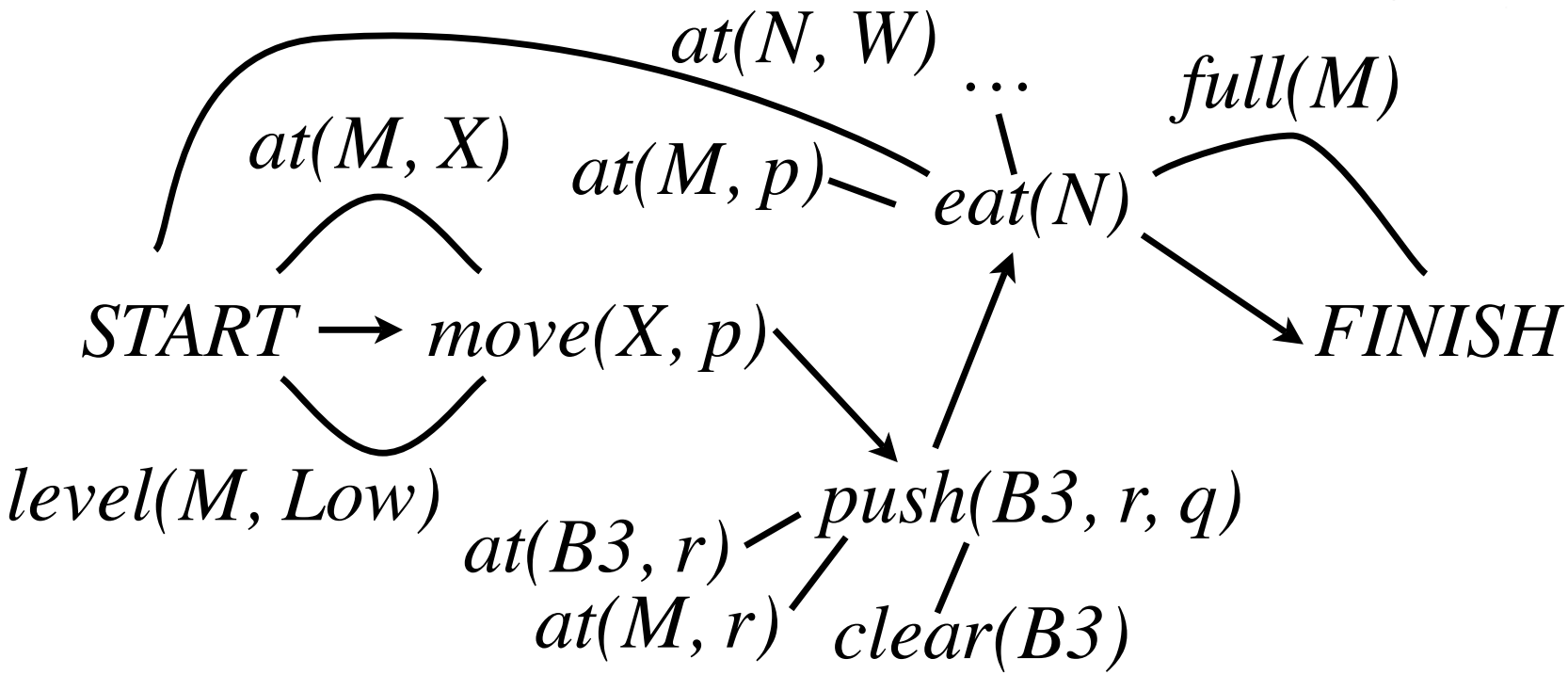
Adding a guard



- *Might involve binding a variable (may be more than one way to do so)*

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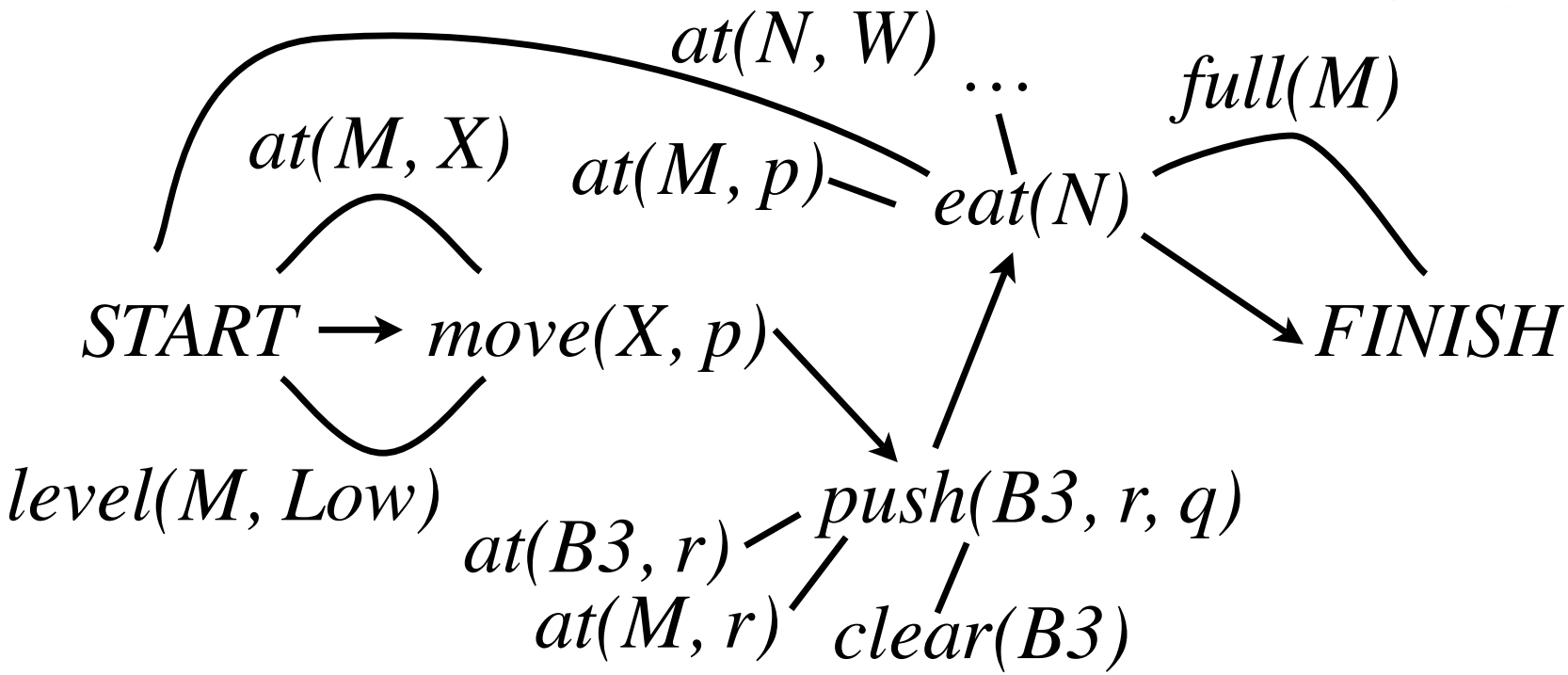
Adding an operator



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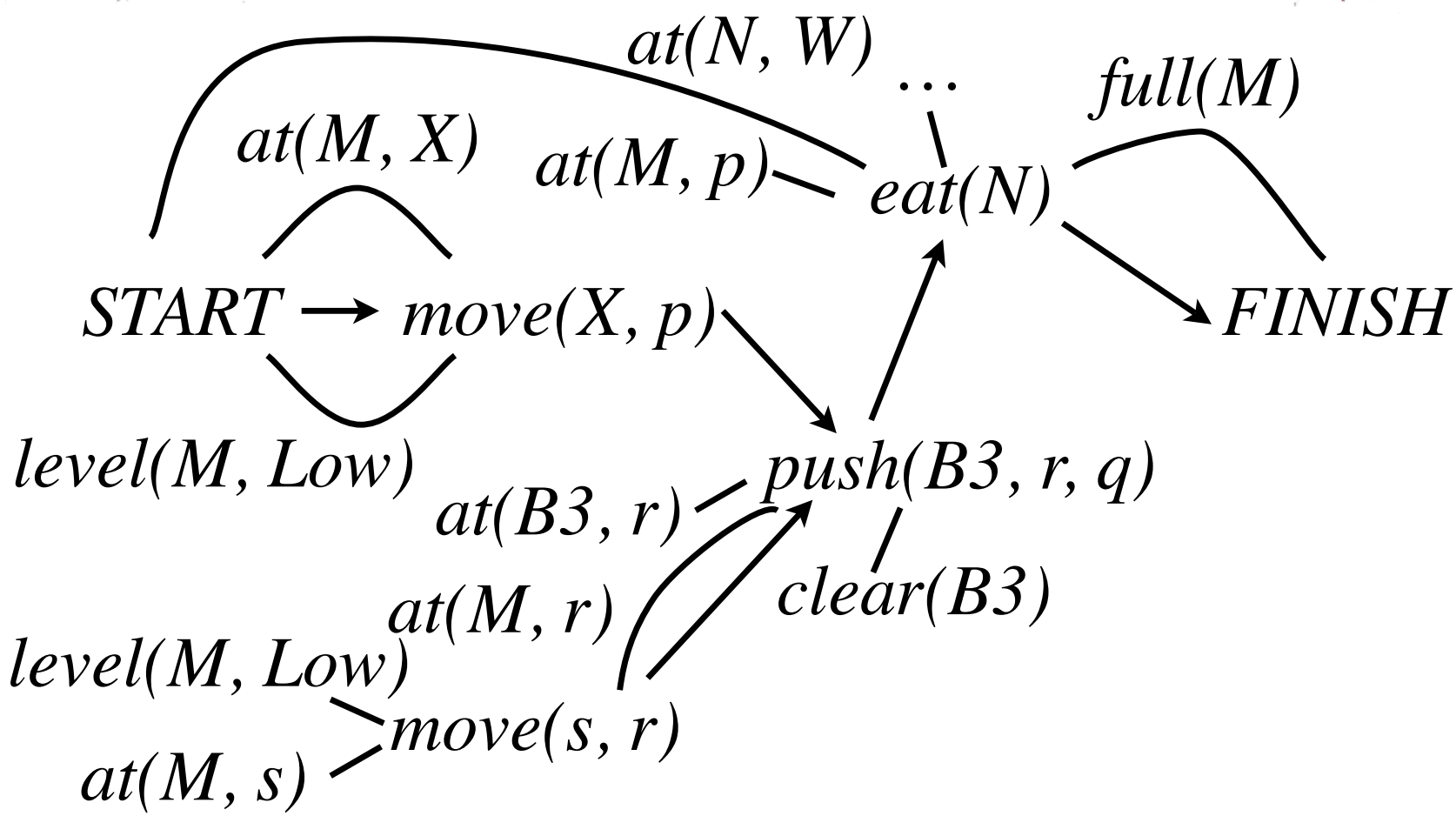
Adding an operator



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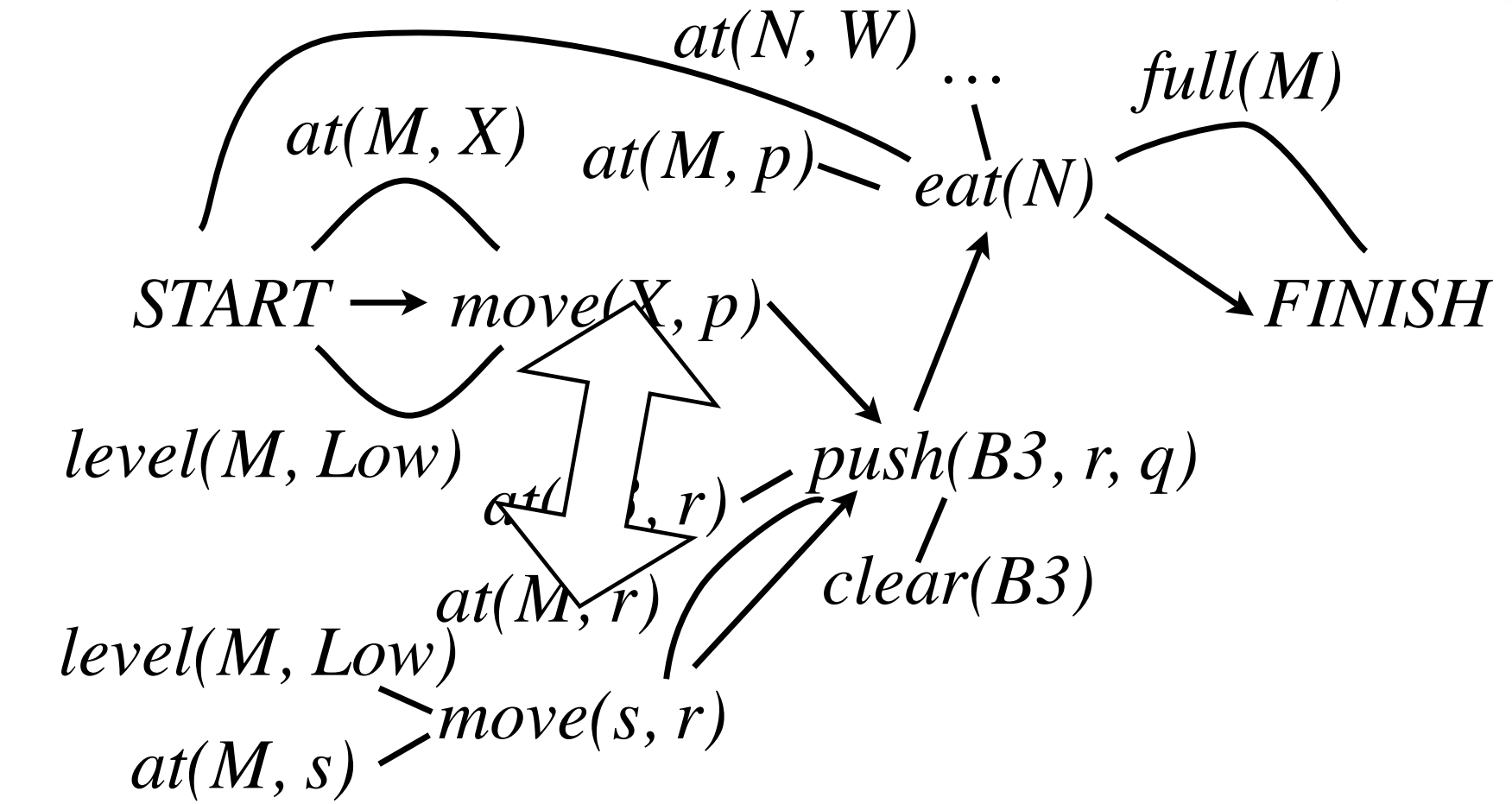
Adding an operator



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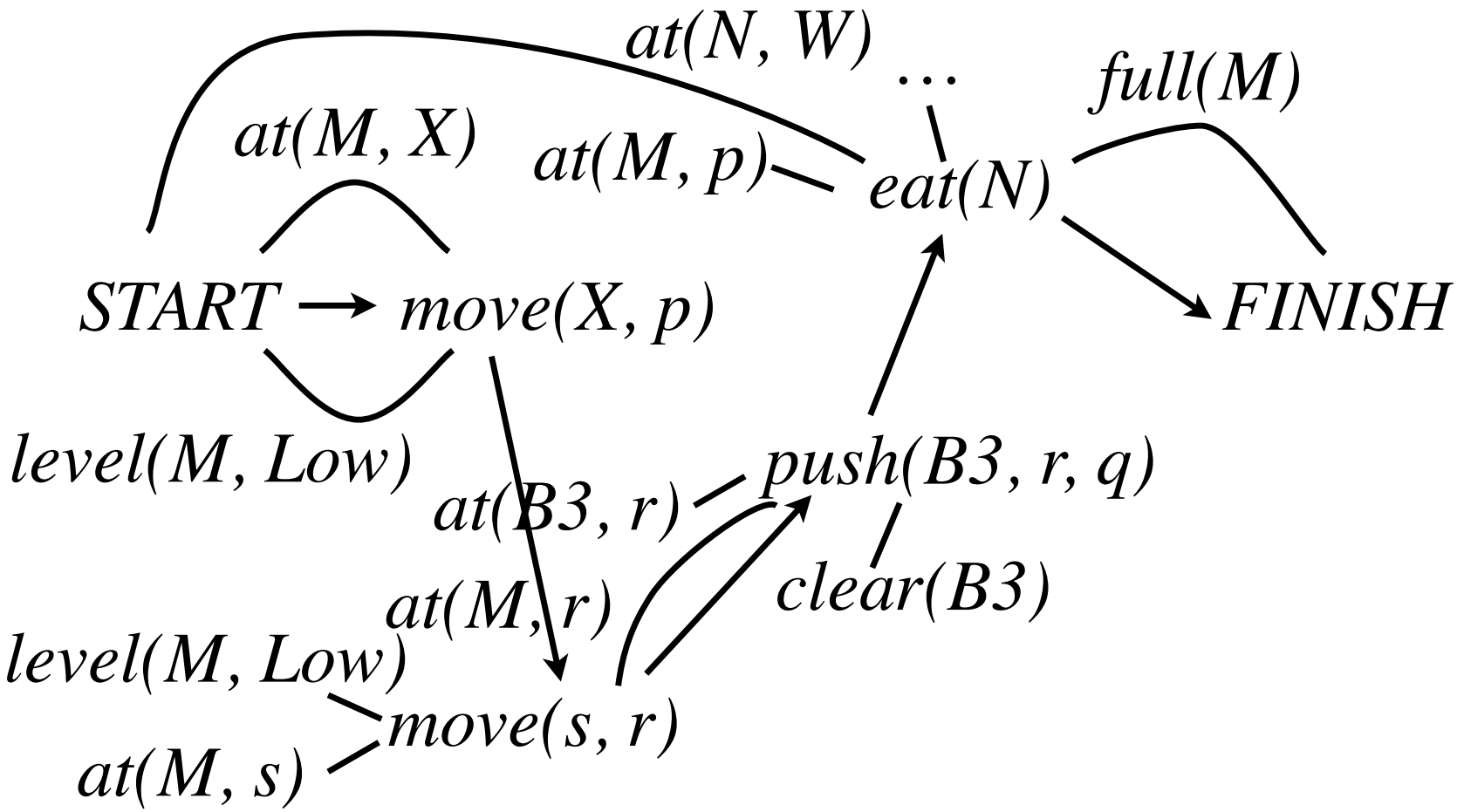
Resolving conflict



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Recap of neighborhood

- *Pick an open precondition*
- *Pick an operator and binding that can satisfy it*
  - *may need to add a new op*
  - *or can use existing op*
- *Add an ordering constraint and guard*
- *Resolve conflicts by adding more ordering constraints or bindings*

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Consistency & completeness

- *Consistency: no cycles in ordering, preconditions guaranteed true throughout guard intervals*
- *Completeness: no open preconditions*
- *Search maintains consistency, terminates when complete*

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Execution

- *A consistent, complete plan can be executed by* ***linearizing*** *it*
- *Execute actions in any order that matches the ordering constraints*
- *Fill in unbound variables in any consistent way*

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Plan Graphs

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Planning & model search

- *For a long time, it was thought that SAT-style model search was a non-starter as a planning algorithm*
- *More recently, people have written fast planners that*
  - *propositionalize the domain*
  - *turn it into a CSP or SAT problem*
  - *search for a model*

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Plan graph

- *Tool for making good CSPs: plan graph*
- *Encodes a subset of the constraints that plans must satisfy*
- *Remaining constraints are handled during search (by rejecting solutions that violate them)*

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Example

- *Start state: have(Cake)*
- *Goal: have(Cake) ∧ eaten(Cake)*
- *Operators: bake, eat*

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Operators

- *Bake*
  - *pre: ¬have(Cake)*
  - *post: have(Cake)*
- *Eat*
  - *pre: have(Cake)*
  - *post: ¬have(Cake), eaten(Cake)*

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Propositionalizing

- *Note: this domain is fully propositional*
- *If we had a general STRIPS domain, would have to pick a universe and propositionalize*
- *E.g., eat(x) would become eat(Banana), eat(Cake), eat(Fred), …*

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Plan graph

have

¬*eaten*

- *Alternating levels: states and actions*
- *First level: initial state*

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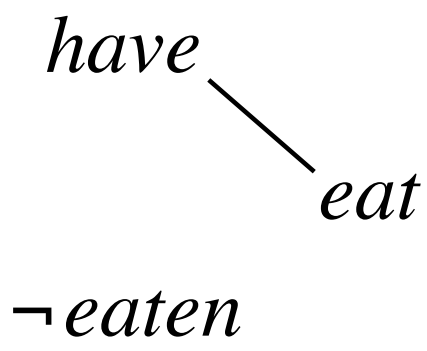
Plan graph



- *First action level: all applicable actions*
- *Linked to their preconditions*

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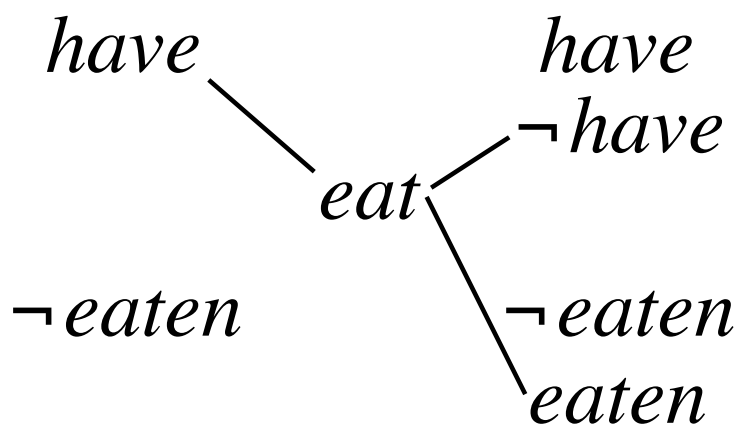
Plan graph



- *Second state level: add effects of actions to get literals that could hold at step 2*

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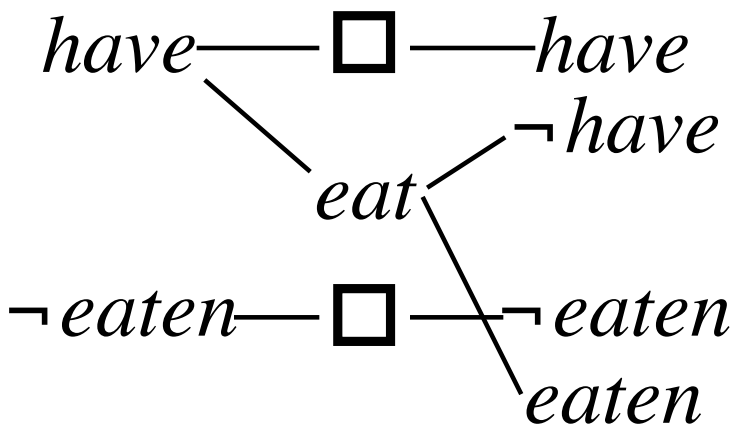
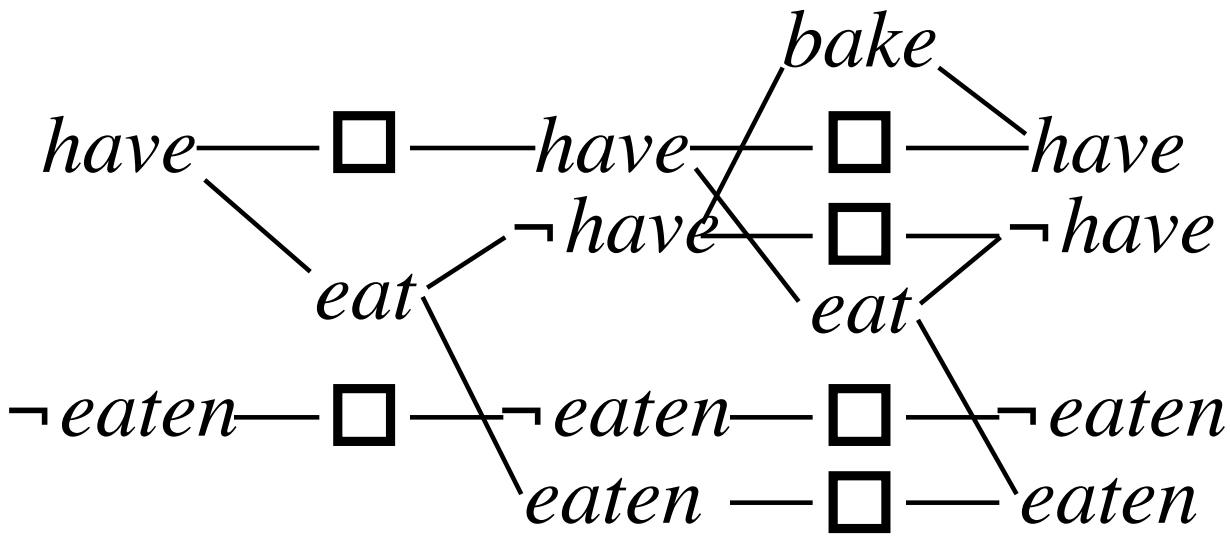
Plan graph



- *Also add* ***maintenance actions*** *to represent effect of doing nothing*

Plan graph



- *Extend another pair of levels: now bake is a possible action*

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Plan graph

- *Can extend as far right as we want*
- *Plan = subset of the actions at each action level*
- *Ordering unspecified within a level*

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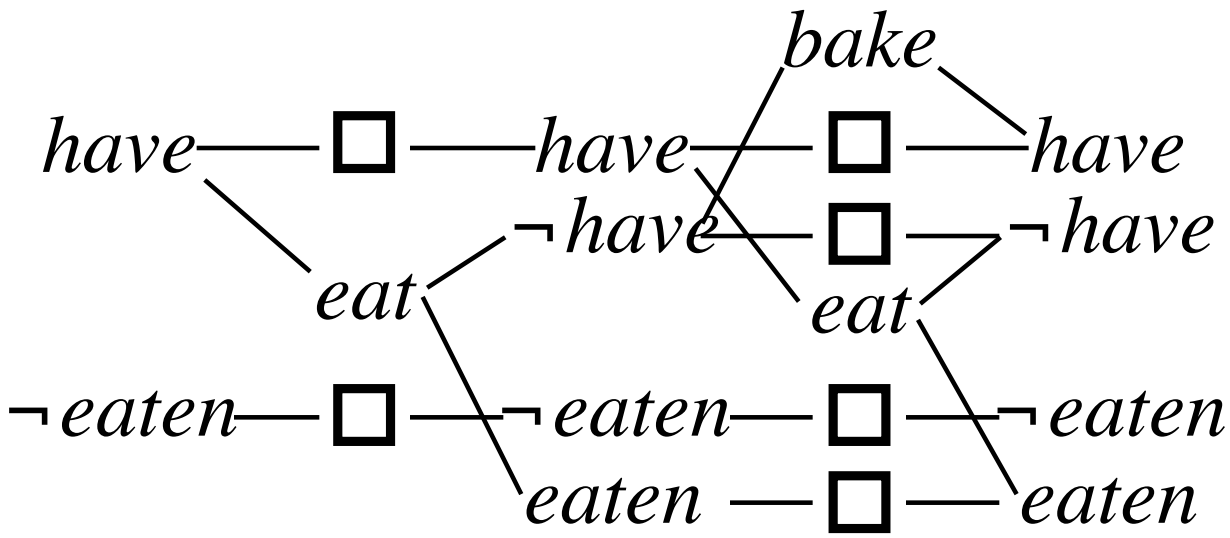
Plan graph



- *In addition to the above links, add* ***mutex*** *links to indicate mutually exclusive actions or literals*

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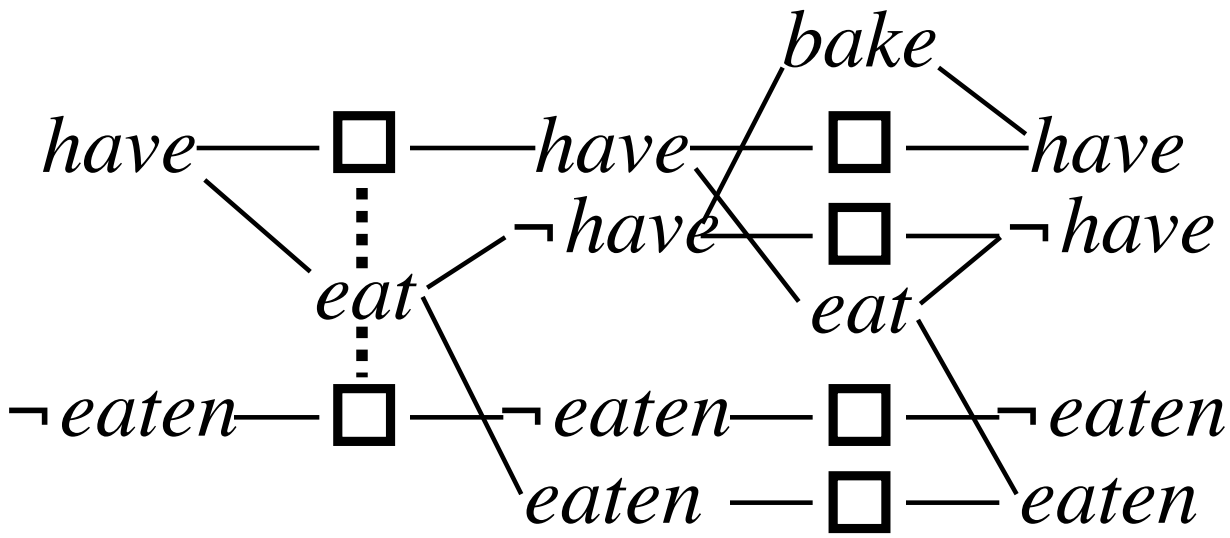
Plan graph



- *Actions which assert contradictory literals are mutex*

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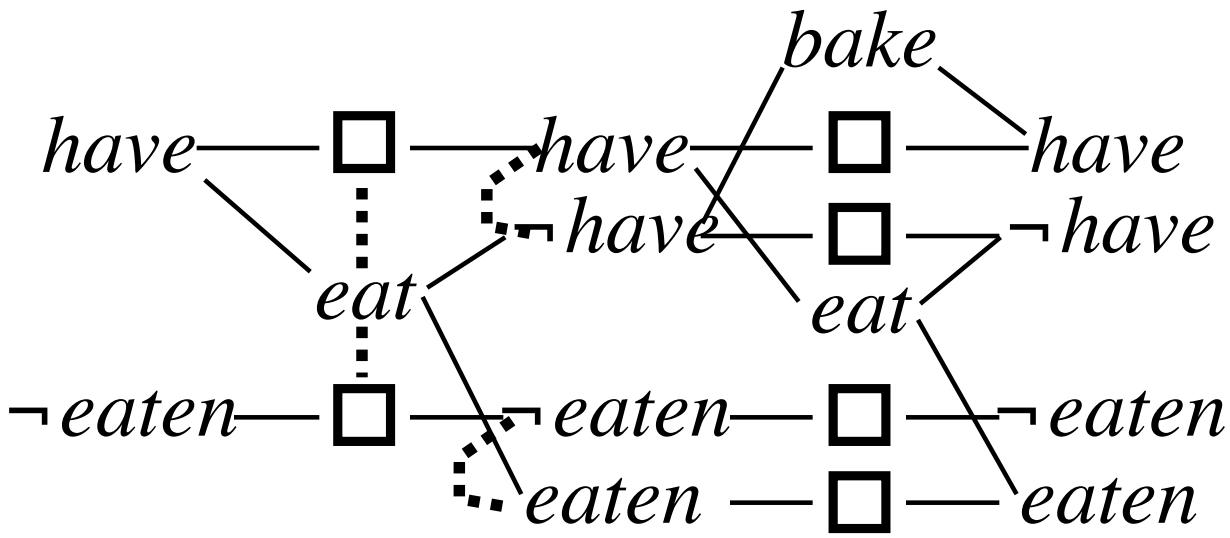
Plan graph



- *Literals are mutex if they are contradictory*

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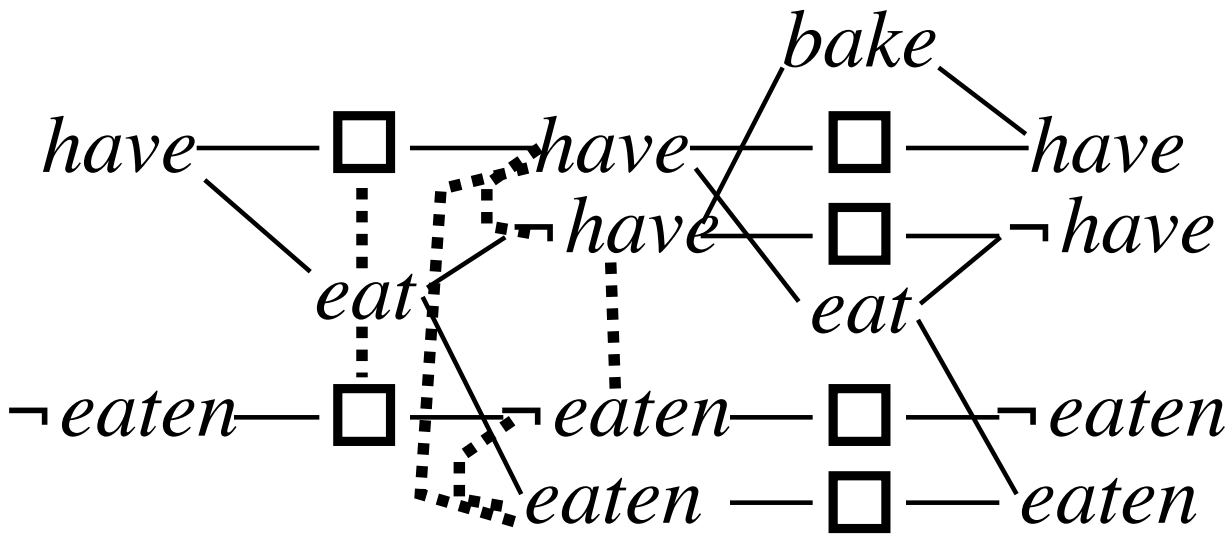
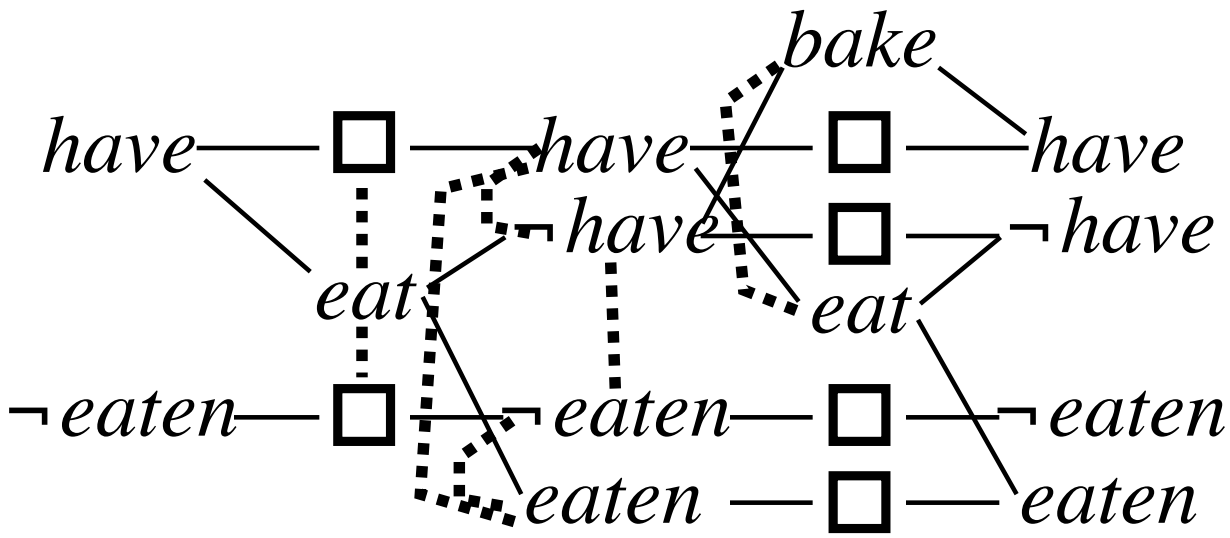
Plan graph



- *Or if there is no non-mutex set of actions that could achieve both*

Plan graph



- *Actions are also mutex if one deletes a precondition of the other, or if their preconditions are mutex*

Getting a plan

- *Build the plan graph out to some length k*
- *Translate to a SAT formula or CSP*
- *Search for a satisfying assignment*
- *If found, read off the plan*
- *If not, increment k and try again*
- *There is a test to see if k is big enough*

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Translation to SAT

- *One variable for each pair of literals in state levels*
- *One variable per action in action levels*
- *Constraints implement STRIPS semantics*
- *Solution tells us which actions are performed at each action level, which literals are true at each state level*

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Action constraints

- *Each action can only be executed if all of its preconditions are present:*

$$act_{t+1} \Rightarrow pre1_t \wedge pre2_t \wedge \ldots$$

- *If executed, action asserts its postconditions:*

$$act_{t+1} \Rightarrow post1_{t+2} \wedge post2_{t+2} \wedge \ldots$$

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Literal constraints

- *In order to achieve a literal, we must execute an action that achieves it*
  - $post_{t+2} \Rightarrow act1_{t+1} \vee act2_{t+1} \vee \ldots$
- *Might be a maintenance action*

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Initial & goal constraints

- *Goals must be satisfied at end:*

  $goal1_T \wedge goal2_T \wedge \ldots$

- *And initial state holds at beginning:*

  $init1_1 \wedge init2_1 \wedge \ldots$

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Mutex constraints

- *Mutex constraints between actions or literals: add clause* $(x \oplus y)$
- *Note: some mutexes are redundant, but help anyway*

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Plan search

- *Hand problem to SAT solver*
- *Or, simple DFS: start from last level, fill in last action set, compute necessary preconditions, fill in 2nd-to-last action set, etc.*
- *If at some level there is no way to do any actions, or no way to fill in consistent preconditions, backtrack*

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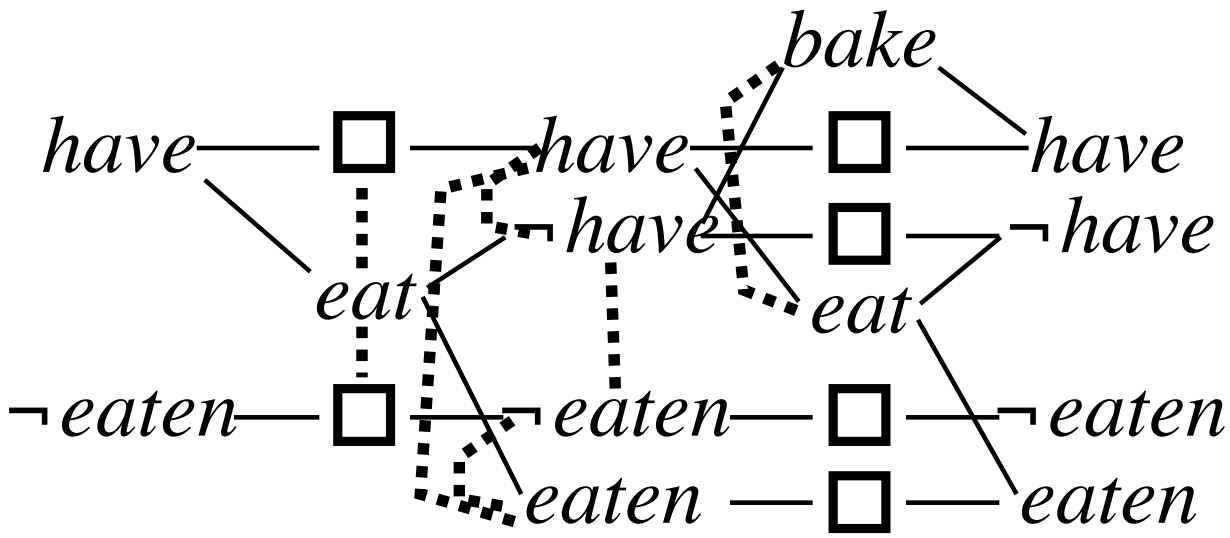
Plan search



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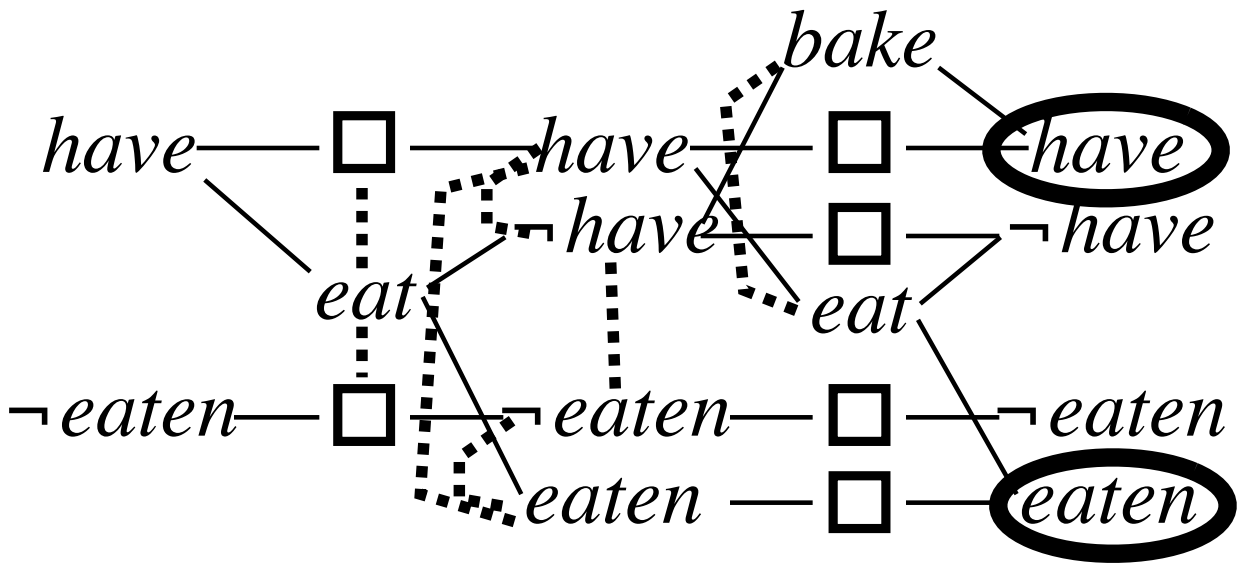
Plan search



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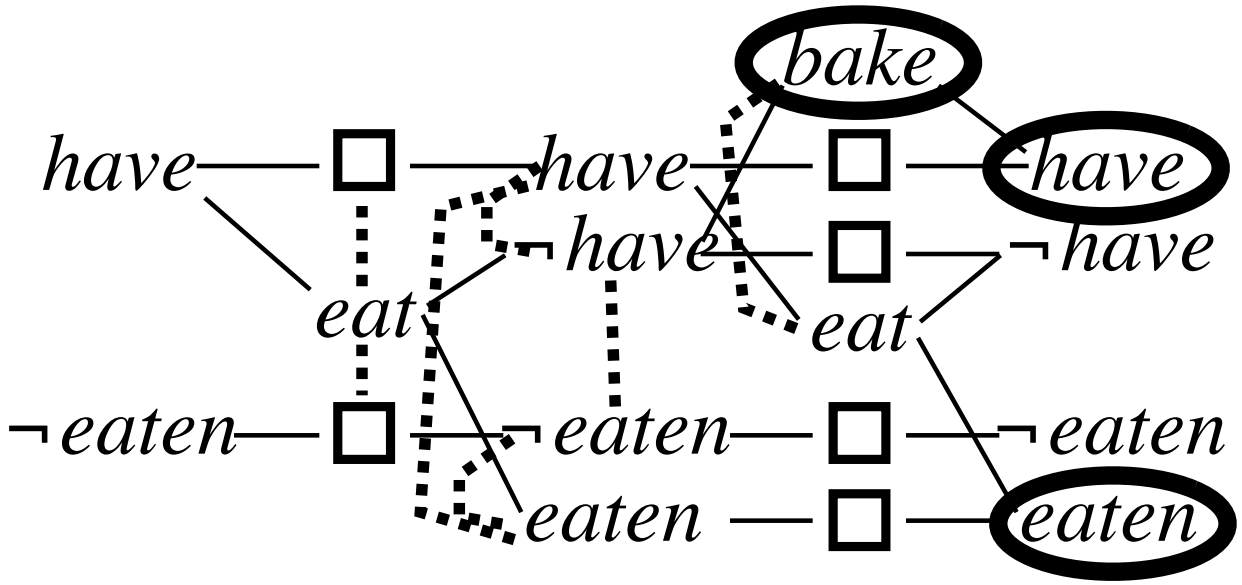
Plan search



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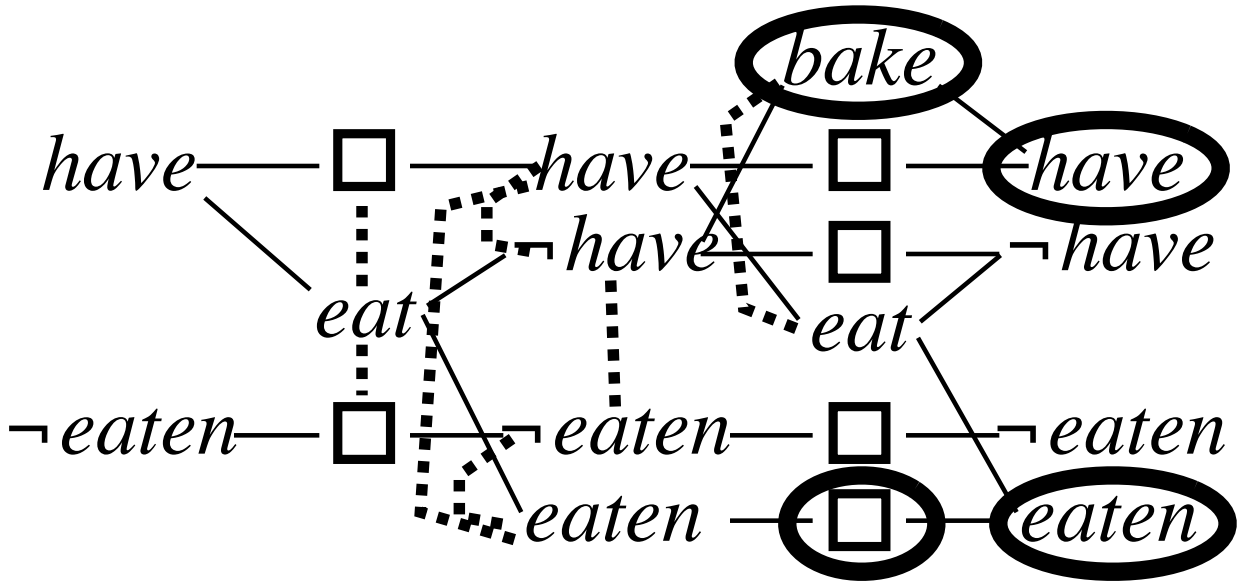
Plan search



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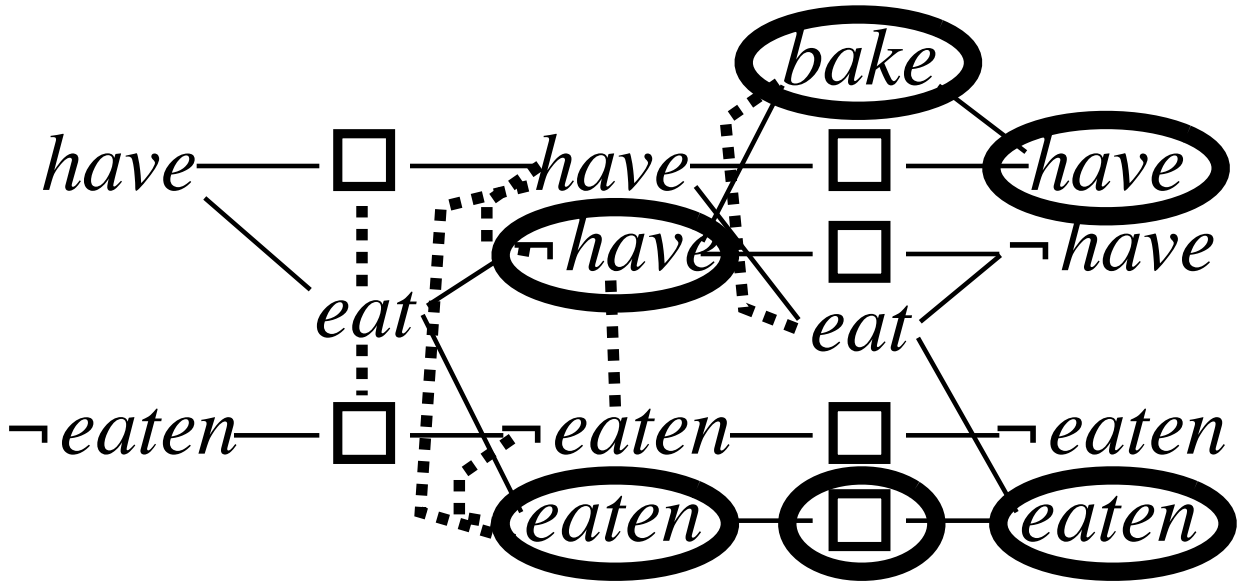
Plan search



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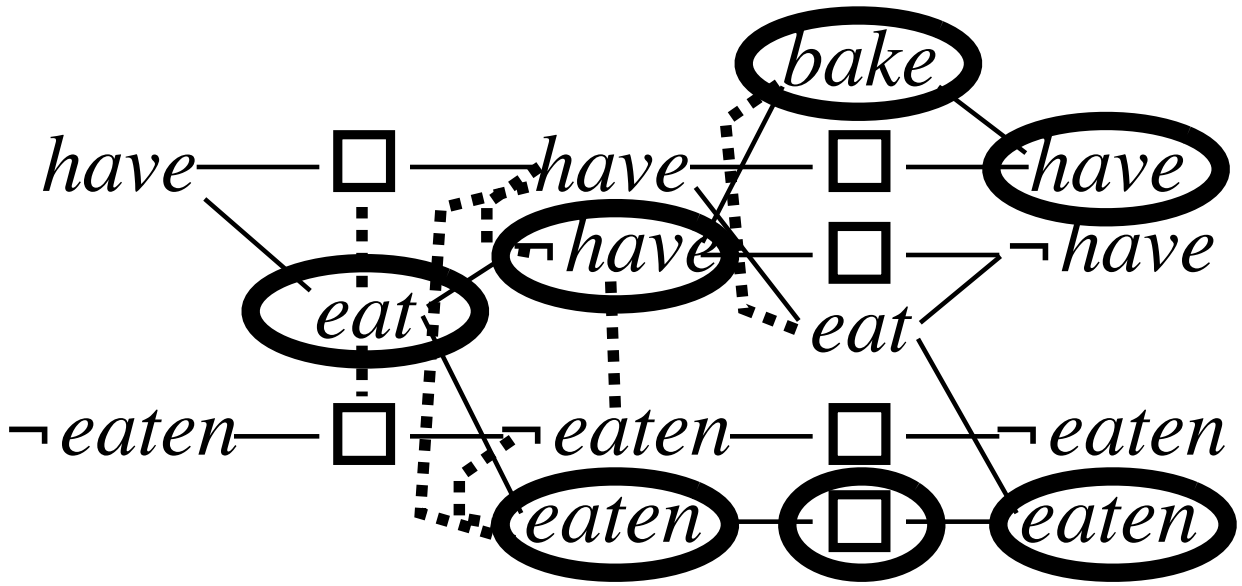

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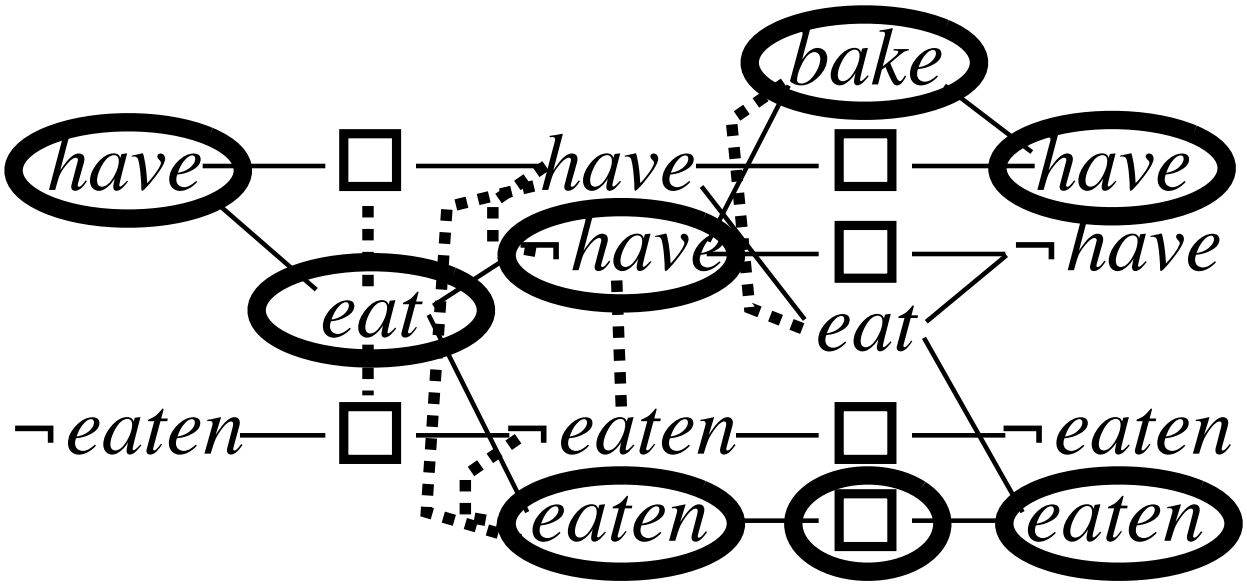
Plan search



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