Time-Space Tradeoffs for Counting NP Solutions Modulo Integers

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[Diehl-Van Melkebeek], [Viola] Lower bounds for QSAT with randomized algorithms

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Define MOD_mP, for which MOD_mSAT is a complete problem.

Recent Attention: Fueled by Valiant's *accidental algorithms*

– For some $\#$P-complete problems,

- Can determine in P if the number of solutions is divisible by 7,
- But MOD_2P-hard to determine if the number of solutions is divisible by 2.

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  RP-reduction from SAT to MOD_m SAT

- (Naik-Regan-Sivakumar)

  $O(n \cdot \text{poly}(\log n))$-time randomized reduction from SAT to MOD_2 SAT

- (Toda-Ogihara, Gupta)

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Naive Idea For MOD_m SAT Lower Bounds:

Prove lower bounds for MOD_m SAT *by applying above reduction(s) and appealing to known* SAT *lower bounds.*

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- **Derandomizations? We don't know no stinking derandomizations!**
  (But we have good reason to believe they exist [Klivans-Van Melkebeek])

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SAT cannot be solved in t time and s space,

then for every $\varepsilon > 0$ and primes $p \neq q$, there is a proof that:

One of MOD_p SAT or MOD_q SAT can't be solved
in $t^{1-\varepsilon}$ time and $s^{1-\varepsilon}$ space.

Outline of Talk

- Introduction
- Some Notation
- Some Preliminaries
- Alternation-Trading Proofs
- Transfer Principle

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Notation for Alternating Classes:

- $(\exists\ f(n))\ C$ = class of problems solved by a machine that: $\exists$-guesses $f(n)^{1+o(1)}$ bits, then runs a C-machine.
  Example: $\mathsf{NTIME}[n^{1+o(1)}] = (\exists\ n)\mathsf{DTIME}[n^{1+o(1)}]$.
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- $(\forall\ f(n))\ C$ defined similarly.

- $(\mathsf{MOD}_m\ f(n))\ C$ = class of problems solved by a machine that:
  guesses $y : |y| = f(n)^{1+o(1)}$, runs a C-machine N,
  accepts iff the number of y that make N accept is divisible by m.
  Example: $\mathsf{MOD}_m\mathsf{TIME}[n^{1+o(1)}] = (\mathsf{MOD}_m\ n)\mathsf{DTIME}[n^{1+o(1)}]$.

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Proof Sketches:

[Kannan]

Existentially guess configs $C_1, \ldots, C_{b+1}$ of a DTISP machine.

Universally guess $i \in [b]$.

Accept iff $C_i \vdash^{t/b} C_{i+1}$ and C_1 is initial and C_{b+1} is accepting.

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[Fortnow-van Melkebeek]

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Existentially guess $i \in [b]$.

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Let $c \geq 1$.

Theorem: For all $b \geq a \geq 1$,

$\mathsf{NTIME}[n] \subseteq \mathsf{DTS}[n^c] \Longrightarrow (\exists n^a)(\forall n^b)\mathsf{DTS}[n^b] \subseteq (\exists n^a)\mathsf{DTS}[n^{bc}]$

and $(\forall n^a)(\exists n^b)\mathsf{DTS}[n^b] \subseteq (\forall n^a)\mathsf{DTS}[n^{bc}]$.

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3. (Combination) $(\exists\, n^a)(\exists\, n^b)\textsf{DTS}[n^d] \subseteq (\exists\, n^a + n^b)\textsf{DTS}[n^d]$
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There is an alternation-trading proof of

$$\textsf{SAT} \notin \textsf{DTS}[n^{\sqrt{2}-\varepsilon}] \quad \text{[Lipton-Viglas'99]}$$

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Can prove $\textsf{SAT} \notin \textsf{DTS}[n^{2\cos(\pi/7)-\varepsilon}]$ using alternation-trading.

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Corollary: $\text{MOD}_6\,\text{SAT} \notin \text{DTS}[n^{1.8}]$.

Proof: If not, then both $\text{MOD}_2\,\text{SAT}$ and $\text{MOD}_3\,\text{SAT}$ are in $\text{DTS}[n^{1.8}]$, a contradiction.

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- $\mathsf{MOD}_p\,\textsc{Sat} \in \mathsf{DTS}[n^c] \Rightarrow \mathsf{MOD}_p\mathsf{TIME}[n] \subseteq \mathsf{DTS}[n^c]$

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- The alternation-trading rules have "modular counting counterparts":

For all primes $p \neq q$ and $\varepsilon > 0$,

(Speedup) $\mathsf{DTS}[n^b] \subseteq (\mathsf{MOD}_p\; n^a)(\mathsf{MOD}_q\; \log n)\mathsf{DTS}[n^{b-a+\varepsilon}]$

(Slowdown) $\mathsf{MOD}_q\mathsf{TIME}[n] \subseteq \mathsf{DTS}[n^c]$ implies

$(\mathsf{MOD}_p\; n^a)(\mathsf{MOD}_q\; n^b)\mathsf{DTS}[n^b] \subseteq (\mathsf{MOD}_p\; n^a)\mathsf{DTS}[n^{bc}]$

(Combination)

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Informally, if $\mathsf{SAT} \in \mathsf{DTS}[n^c]$ implies $\mathsf{DTS}[t] \subseteq \mathsf{DTS}[t^{1-\varepsilon}]$, then $\mathsf{MOD}_p\,\mathsf{SAT}$ and $\mathsf{MOD}_q\,\mathsf{SAT}$ are in $\mathsf{DTS}[n^{c-\varepsilon}]$ also implies it.

Speedup of DTISP via Modular Counting

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Theorem: For all $\varepsilon > 0$ and $b \geq a \geq 1$,

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Proof Steps:

1. Convert $\mathsf{DTS}[n^b]$ machine into a "canonical form" that runs in time $n^{b+\varepsilon}$ and space $n^{o(1)}$

2. By essentially replacing "$\exists$" with "MOD_p" and "$\forall$" with "MOD_q", the Alternating Speedup Theorem works on a canonical machine.

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Recall the notion of a configuration graph for M on x:

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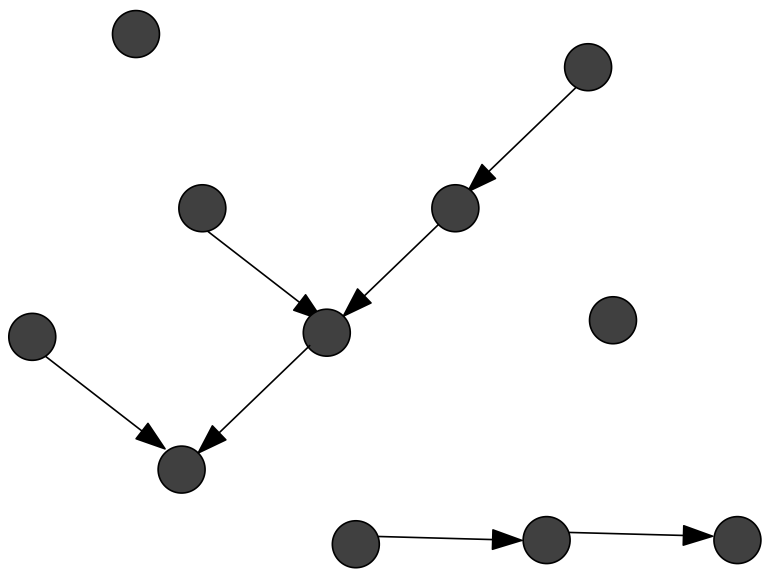

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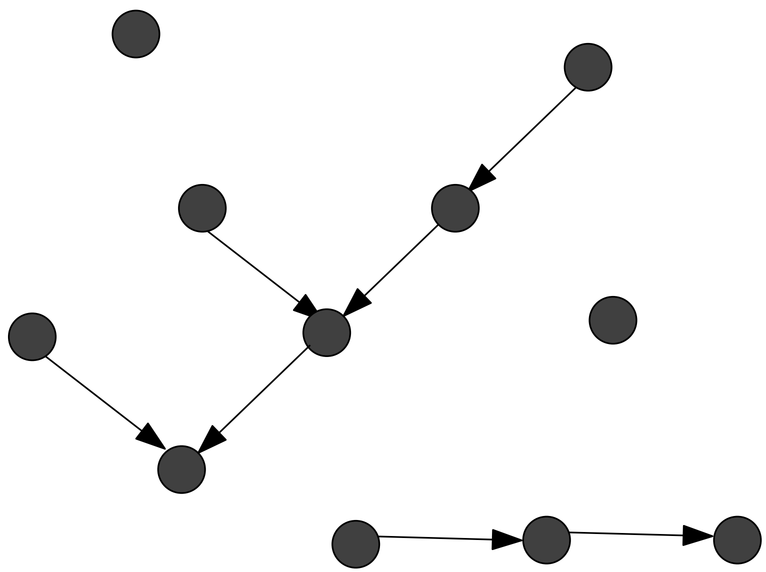

Nodes = Configurations C of $M(x)$, Edge $(C, C') \iff C \vdash C'$

M is deterministic $\Longrightarrow outdeg(G_{M,x}) \leq 1$.

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A deterministic machine M is *canonical* if, for every input x,

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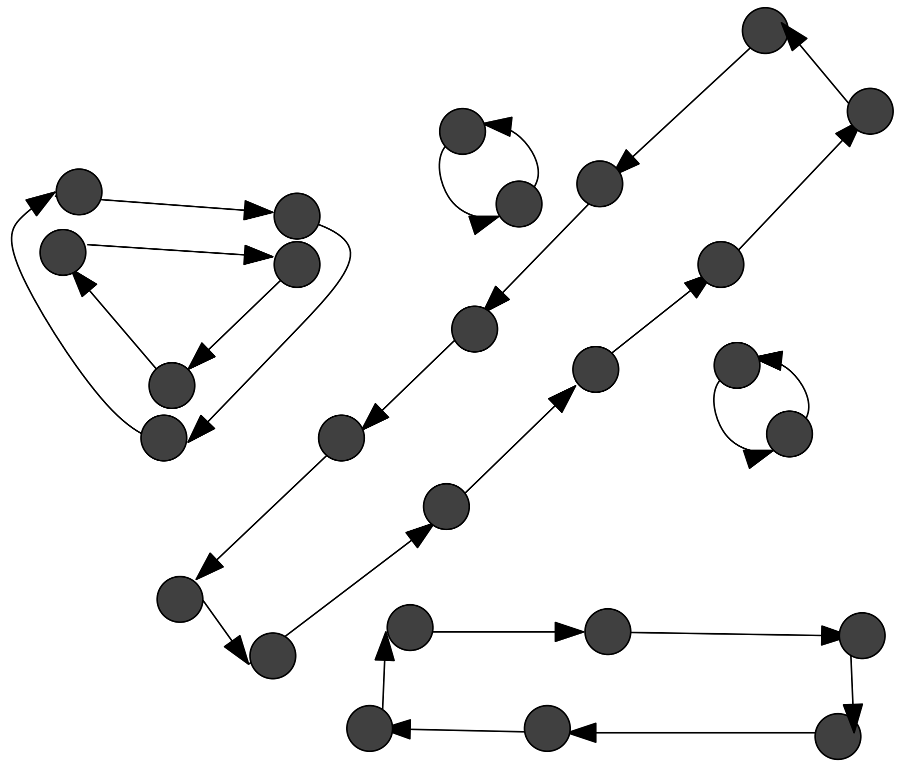

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Theorem: [Bennett'89] $\mathsf{DTISP}[t, s] \subseteq$ rTISP$[t^{1+\varepsilon}, s \log t]$.

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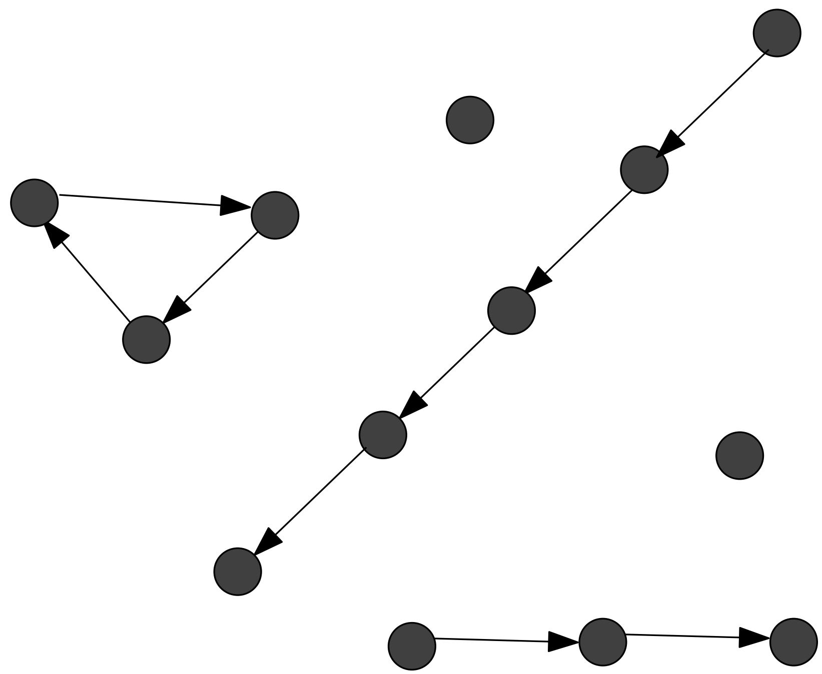
Take a $\mathsf{DTS}[n^b]$ machine M and make it reversible using Bennett.

On an input, its config graph looks like:



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From Reversible Machine to Canonical Machine

To make M canonical, we make another machine that simulates it, but starts running “backwards” when it reaches a dead end.

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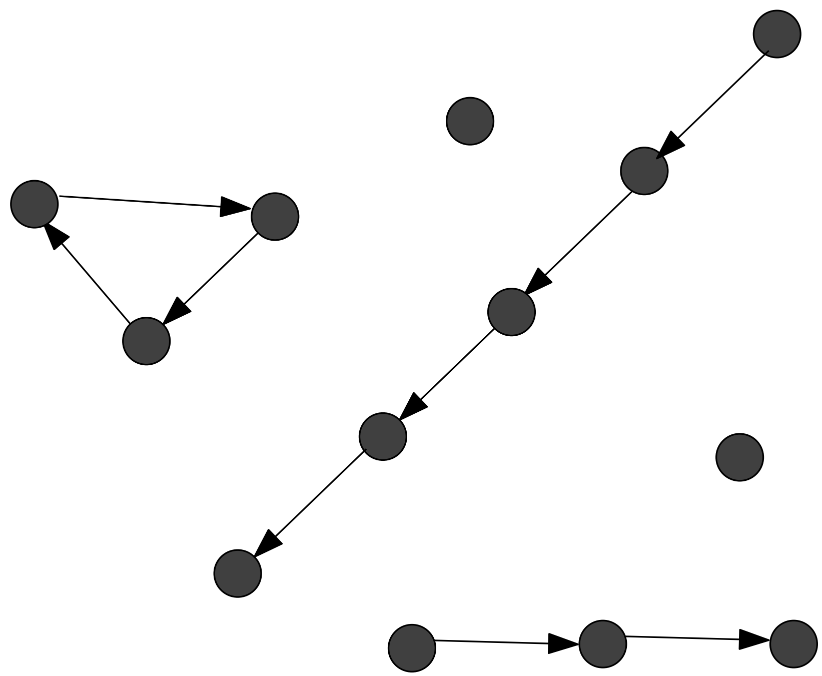

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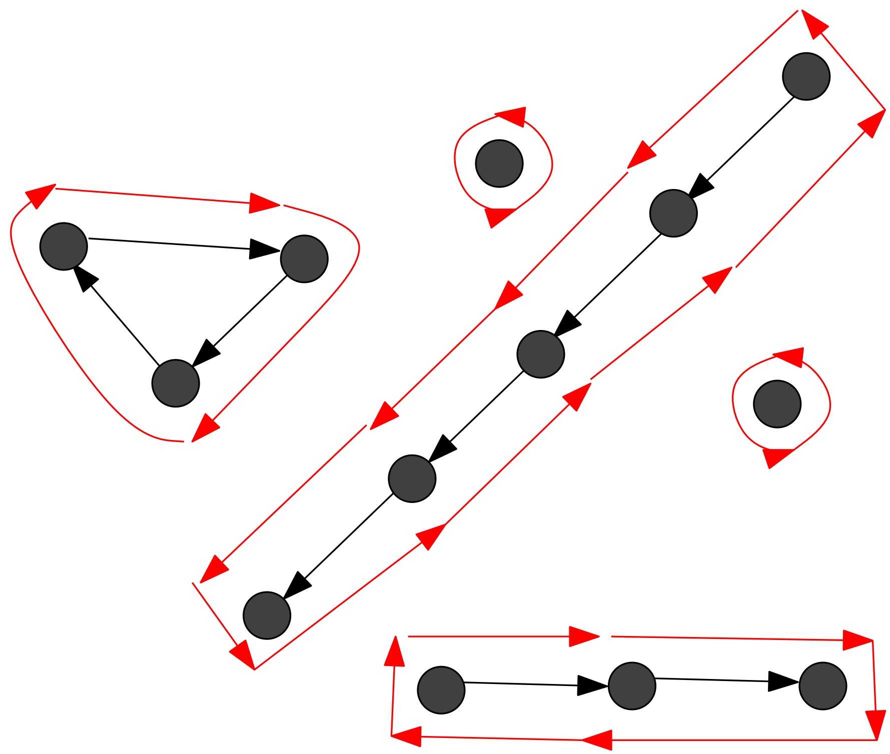

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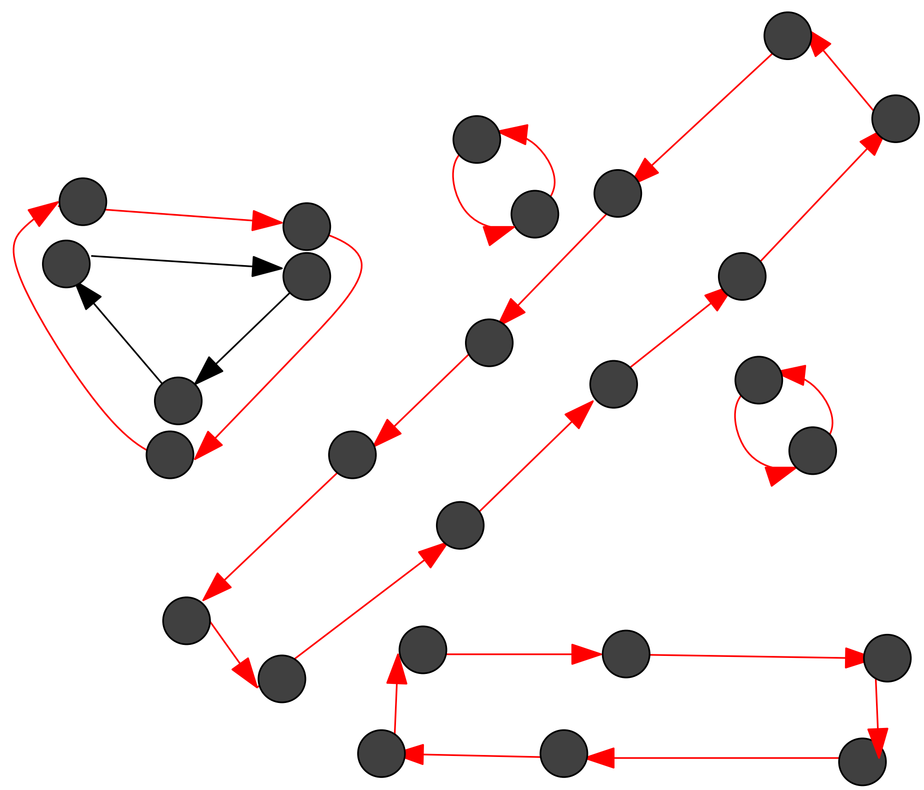

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- Can pre-compute the mod-p residue for the accepting case.

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For a canonical machine, input x, integer ℓ, and given configuration C:

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A configuration sequence $C_1, \ldots, C_{b+1}$ has k mistakes for $M(x)$ iff there are exactly k pairs (C_i, C_{i+1}) s.t. $\neg(C_i \vdash^{T/b} C_{i+1})$.

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Can show that the number of configuration sequences with k mistakes:

- depends *only* on time-space usage and accepting/rejecting condition
- *not* on subtle properties of the machine's behavior

Formal Statement

Let a *complete configuration sequence* $C_1, \ldots, C_{b+1}$ have C_1 as initial and C_{b+1} as accepting.

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Counting Lemma: Let n be an integer, let $k \in \{0, 1, \ldots, b+1\}$, and let $\hat{M}$ be canonical. Then there are positive integers $N_A(n, k, b)$ and $N_R(n, k, b)$ such that, for all inputs x of length n:

1. If $\hat{M}(x)$ accepts, then the number of complete b-configuration sequences for $\hat{M}(x)$ with k mistakes is $N_A(n, k, b)$.
2. If $\hat{M}(x)$ rejects, then the number of complete b-configuration sequences for $\hat{M}(x)$ with k mistakes is $N_R(n, k, b)$.
3. $N_A(n, k, b) - N_R(n, k, b) = (-1)^k \binom{b+1}{k}$.

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- Relies on properties of alternation-trading proofs and determinism
- Can we prove better time lower bounds for MAJORITY SAT?

  (It's PP*-complete...)*
- How far can alternation-trading proofs go?

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- **Automated Time Lower Bounds:**

  Formalization of alternation-trading proofs

  $\Longrightarrow$ Implementation of a theorem prover

  Proofs found by a combo of exhaustive search and linear programming

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If this is correct, then some REALLY NEW IDEAS will be required

to make further progress on time-space lower bounds

Thank you!