Two Notions of Beauty in Programming

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Ongoing work with Guy E. Blelloch at Carnegie Mellon.
Two Sources of Beauty in Programs

For me beauty in a program arises from two sources:

- **Structure**: code as an expression of an idea.
- **Efficiency**: code as instructions for a computer.
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This has given rise to two theories of computation.

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- **Combinatorial**: efficiency (machine effort).
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Oddly, these are largely disparate communities!
Reconciling the Two Theories

Historically,

- The logical side neglects efficiency in favor of structure.
- The combinatorial side neglects structure in favor of efficiency.
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- The logical side should pay more attention to efficiency.
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The \( \lambda \)-calculus is the key!
The Great Rift

“On the fact that the Atlantic Ocean has two sides.” [EWD]

- American theory \(\approx\) combinatorial theory.
- Euro-theory \(\approx\) semantics and logic.
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Both have had a big influence on practice:

- Efficient algorithms for a broad range of problems.
- Language design and verification tools.
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Yet these two “theories” operate largely in isolation!
Algorithm analysis is based on **machine models**:

- Turing machine (TM) or Random Access Machine (RAM).
- Low-level: no abstraction, no composition.
- Allegedly, close to the hardware.

Machine models provide natural **complexity measures**:

- **Time** = number of instructions.
- **Space** = tape or memory usage.

**Asymptotics** smoothes over differences among models.
In practice algorithms are described using C-like notation.

- Clearer than TM or RAM code.
- Analyze compiled code, rather than source code.

An improvement, but still very limited:

- ephemeral data structures.
- manual memory management.
- poor composability.
- no abstraction.
Euro theory is based on language models:

- Church’s (typed and untyped) $\lambda$-calculus.
- High-level: abstraction, composition are fundamental.
- Platform-independent.

Language models support composition via variables:

- If $\phi \text{ true} \vdash \psi \text{ true}$, then if $\phi \text{ true}$, then $\psi \text{ true}$.
- If $x : \sigma \vdash N : \tau$, then if $M : \sigma$, then $[M/x]N : \tau$.

The $\lambda$-calculus is an elegant theory of composition.
Languages based on $\lambda$-calculus stress

- **persistent** data structures.
- **automatic** memory management.
- **strong** composability.
- **abstract types**.

But there is relatively little emphasis on **efficiency**.

- No clear complexity measures.
- Few analytic results (but see Okasaki’s CMU Ph.D.).
Traditional imperative methods of programming are obsolete.

- Tedious to program, a nightmare to maintain.
- Largely incompatible with parallelism.

Functional methods are destined to dominate.

- Support verification and composition.
- Naturally accommodate parallelism.

The way forward is to synthesize Euro- and American theory.
To elevate the level of discourse we require a cost semantics.

- Define the abstract cost of execution of a language.
- Defines the parallel and sequential complexity.

Algorithm analysis is conducted at the level of the code we write.

- Cost semantics assigns a measure to each execution.
- Analyze asymptotic complexity in terms of this measure.
Cost Semantics

The abstract cost is validated by a bounded implementation.
- Transform abstract cost into concrete cost on a machine.
- Account for platform characteristics such as number of processors, cache hierarchy, and interconnect.

An end-to-end asymptotics with a clear separation of concerns.
- High-level, composable development and reasoning.
- Low-level implementation on hardware platforms.
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So simple we teach it to first-year undergraduates!
Cost Semantics for Time

Associate a **cost graph** to the evaluation of a program.

- **Dynamic**, fully accurate record of data dependencies.
- **Not** a static analysis or an approximation!

Example: function application.

\[
\begin{array}{c}
e_1 \Downarrow \lambda x.e & e_2 \Downarrow \nu_2 & [\nu_2/x]e \Downarrow \nu \\
e_1(e_2) \Downarrow \nu
\end{array}
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e_1(e_2) & \downarrow & v
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\[
e_1(e_2) \downarrow^{(g_1 \odot g_2) \oplus 1 \oplus g} \nu
\]
Series-parallel cost graphs:

- $\mathbf{1}$: one unit of computation.

Application cost $(g_1 \otimes g_2) \oplus \mathbf{1} \oplus g$ specifies that
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- \(1\): one unit of computation.
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- Function and argument are evaluated in parallel.
- Function call costs one unit.
- Function execution depends on the function and argument.
Cost Graphs
The **work** $w(g)$ of a cost graph $g$ is the size of $g$.

- $w(1) = 1$, $w(g_1 \otimes g_2) = w(g_1 \oplus g_2) = w(g_1) + w(g_2)$.
- Measures the **sequential time complexity**.

The **span** $d(g)$ of a cost graph $g$ is the critical path length of $g$.

- $d(1) = 1$, $d(g_1 \otimes g_2) = \max(d(g_1), d(g_2))$,
  $d(g_1 \oplus g_2) = d(g_1) + d(g_2)$.
- Measures the **parallel time complexity**.
Cost Graphs

Work = 11, Span = 6
fun merge xs ys = 
    case (xs, ys) of 
        ([], ys) ⇒ ys 
    | (xs, []) ⇒ xs 
    | (x::xs', y::ys') ⇒ 
        case x<y of 
            true ⇒ x :: merge xs' ys 
        | false ⇒ y :: merge xs ys'

fun sort [] = [] 
    | sort [x] = [x] 
    | sort xs = 
        let val (ys, zs) = split xs 
        in  merge (sort ys, sort zs)  end
Mergesort

The work (sequential time) is optimal, $O(n \log n)$ for $n$ items.

The span (parallel time) is sensitive to the data structure:
- For lists, $O(n)$, because splitting is slow.
- For trees, $O(\log^3 n)$, using rebalancing.

The parallelizability ratio, $w/d$, is $O(n/\log^2 n)$ for trees.

The correctness of the parallel implementation is never in question!
Bounded Implementation for Time

**Brent's Principle:** A computation with work $w$ and span $d$ can be implemented on a $p$-processor PRAM in time $O(\max(w/p, d))$.

- Work in chunks of $p$ as much as possible.
- Number of processors is chosen at run-time.
- Proof is constructive: exhibits a scheduler.

Relates abstract cost to its concrete realization. No need for pseudo-code!
Aggarwal and Vitter introduced the IO Model:

- Distinguish primary from secondary memory.
- Cache size $M = k \times B$ words.
- Evaluate algorithm efficiency in terms of $M$ and $B$.

Main result: $k$-way merge sort is optimal for the IO model:

\[ O\left(\frac{n}{B} \log_{M/B}(\frac{n}{B})\right) \]
IO Efficiency

A&V’s results can be matched in a purely functional model.

- No manual memory management.
- Natural functional programming.

Key idea: temporal locality implies spatial locality.

- Allocation order determines proximity.
- Reloading of migrated objects preserves proximity.
- Control stack specially managed to avoid cache contention.
Cost Semantics for IO

Cost semantics makes storage explicit:

$$\sigma @ e \downarrow^n \sigma' @ v$$

Store $\sigma$ has three components:
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- Unbounded main memory with blocks of size \( B \).
- Read cache of size \( M = k \times B \).
- Linearly ordered allocation cache of size \( M \).

Figure of merit: traffic between main memory and cache expressed in terms of \( M \) and \( B \).
(Simplified) Cost Semantics

\[
\begin{align*}
\sigma \circ \text{app}(e_1; e_2) &\downarrow n_1' + n_1'' + n_2 + n_2' \quad \sigma' \circ l'
\end{align*}
\]

\[
\begin{align*}
\sigma_1 \circ e_1 &\downarrow n_1' \\
\sigma_1 \circ k_1
\end{align*}
\]
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\[
\begin{aligned}
\sigma \circ \text{app}(e_1; e_2) \Downarrow & \quad n'_1 + n''_1 + n'_2 + n''_2 \quad \sigma' \circ l'' \\
\{ & \sigma' \circ l'_1 \Downarrow^{n''_1} \sigma'' \circ \lambda x. e \} \\
\{ & \sigma_1 \circ e_1 \Downarrow^{n'_1} \sigma'_1 \circ l'_1 \}
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\sigma'_1 \odot l'_1 \downarrow^{n'_1} & \quad \sigma'_1 \odot \lambda x.e \\
\sigma''_1 \odot e_2 \downarrow^{n_2} & \quad \sigma'_2 \odot l''_2 \\
\sigma \odot \text{app}(e_1; e_2) \downarrow & \quad n'_1 + n''_1 + n_2 + n'_2 \quad \sigma' \odot l''
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\sigma \circ \text{app}(e_1; e_2) \downarrow^{n_1' + n_1'' + n_2 + n_2'} \quad \sigma' \circ l'
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Thm (Bleloch & H) An evaluation of cost $n$ may be implemented on a stack machine with cache of size $4 \times M + B$ with cache complexity $k \times n$ for some small constant $k$. 
Bounded Implementation for IO

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Thus, the cost semantics is a valid basis for IO analysis.
fun merge nil ys = ys
  | merge xs nil = xs
  | merge (xs as x::xs') (ys as y::ys') =
      case compare x y of
          LESS ⇒ !a::merge xs' ys
  | GTEQ ⇒ !b::merge xs ys'
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A data structure is compact iff it may be traversed in time $O(n/B)$.

Thm: For compact inputs $xs$ and $ys$ the call $\text{merge } xs \text{ ys}$ has cache complexity $O(n/B)$.

- Recurs down lists allocating only stack $n$ frames: $O(n/B)$.
- Returns allocating $n$ list cells: $O(n/B)$.

Copying operations $!a$ and $!b$ ensure compactness (locality).
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Cost semantics supports analysis of complexity of high-level code.

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Costs can be chosen to reflect different notions of complexity:

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- Space usage of scheduling [Spoonhower, B, Gibbons, & H 09].
- Memory hierarchy effects [B & H 13, 15].
λ-calculus provides a logical model of computation.

- Inherently compositional.
\(\lambda\)-calculus provides a \textit{logical} model of computation.

- Inherently compositional.
- Mathematically sensible.
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Cost semantics integrates the combinatorial aspects:

- Enrich the tools available to algorithms designers.
- Extend complexity analysis to mathematically elegant languages.
Where From Here?

Develop new (abstract and concrete) cost measures.

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Improve both the structure and efficiency of programs!