Two Notions of Beauty in Programming

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Thanks

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Ongoing work with Guy E. Blelloch at Carnegie Mellon.

Two Sources of Beauty in Programs

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- Structure: code as an expression of an idea.
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Oddly, these are largely disparate communities!

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The λ-calculus is the key!

The Great Rift

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- Language design and verification tools.

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Yet these two "theories" operate largely in isolation!

American Theory

Algorithm analysis is based on machine models:

- Turing machine (TM) or Random Access Machine (RAM).
- Low-level: no abstraction, no composition.
- Allegedly, close to the hardware.

Machine models provide natural complexity measures:

- Time = number of instructions.
- Space = tape or memory usage.

Asymptotics smoothes over differences among models.

American Theory

In practice algorithms are described using C-like notation.

- Clearer than TM or RAM code.
- Analyze compiled code, rather than source code.

An improvement, but still very limited:

- ephemeral data structures.
- manual memory management.
- poor composability.
- no abstraction.

Euro Theory

Euro theory is based on language models:

- Church's (typed and untyped) λ-calculus.
- High-level: abstraction, composition are fundamental.
- Platform-independent.

Language models support composition via variables:

- If ϕ true $\vdash \psi$ true, then if ϕ true, then ψ true.
- If $x : \sigma \vdash N : \tau$, then if $M : \sigma$, then $[M/x]N : \tau$.

The λ-calculus is an elegant theory of composition.

Euro Theory

Languages based on λ-calculus stress

- persistent data structures.
- automatic memory management.
- strong composability.
- abstract types.

But there is relatively little emphasis on efficiency.

- No clear complexity measures.
- Few analytic results (but see Okasaki's CMU Ph.D.).

Thesis

Traditional imperative methods of programming are obsolete.

- Tedious to program, a nightmare to maintain.
- Largely incompatible with parallelism.

Functional methods are destined to dominate.

- Support verification and composition.
- Naturally accommodate parallelism.

The way forward is to synthesize Euro- and American theory.

Cost Semantics

To elevate the level of discourse we require a cost semantics.

- Define the abstract cost of execution of a language.
- Defines the parallel and sequential complexity.

Algorithm analysis is conducted at the level of the code we write.

- Cost semantics assigns a measure to each execution.
- Analyze asymptotic complexity in terms of this measure.

Cost Semantics

The abstract cost is validated by a bounded implementation.

- Transform abstract cost into concrete cost on a machine.
- Account for platform characteristics such as number of processors, cache hierarchy, and interconnect.

An end-to-end asymptotics with a clear separation of concerns.

- High-level, composable development and reasoning.
- Low-level implementation on hardware platforms.

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So simple we teach it to first-year undergraduates!

Cost Semantics for Time

Associate a cost graph to the evaluation of a program.

- Dynamic, fully accurate record of data dependencies.
- Not a static analysis or an approximation!

Example: function application.

$$\frac{e_1 \Downarrow \lambda x.e \quad e_2 \Downarrow v_2 \quad [v_2/x]e \Downarrow v}{e_1(e_2) \Downarrow v}$$

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Series-parallel cost graphs:

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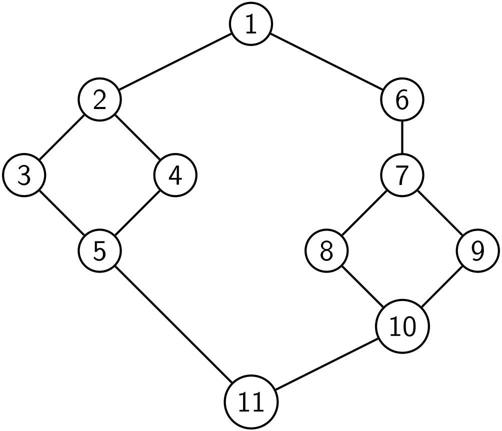

Work and Span

The work $w(g)$ of a cost graph g is the size of g.

- $w(\mathbf{1}) = 1$, $w(g_1 \otimes g_2) = w(g_1 \oplus g_2) = w(g_1) + w(g_2)$.
- Measures the sequential time complexity.

The span $d(g)$ of a cost graph g is the critical path length of g.

- $d(\mathbf{1}) = 1$, $d(g_1 \otimes g_2) = \max(d(g_1), d(g_2))$, $d(g_1 \oplus g_2) = d(g_1) + d(g_2)$.
- Measures the parallel time complexity.

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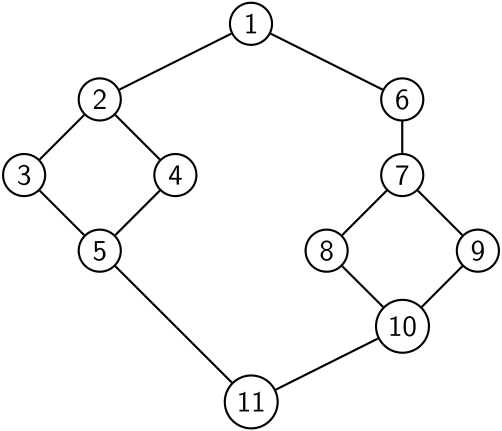

Work = 11, Span = 6

Mergesort

```
fun merge xs ys =
  case (xs, ys) of
    ([], ys) ⇒ ys
  | (xs,[]) ⇒ xs
  | (x::xs', y::ys') ⇒
    case x<y of
      true ⇒ x :: merge xs' ys
    | false ⇒ y :: merge xs ys'

fun sort [] = []
  | sort [x] = [x]
  | sort xs =
    let val (ys, zs) = split xs
    in  merge (sort ys, sort zs)  end
```

Mergesort

The work (sequential time) is optimal, $O(n \log n)$ for n items.

The span (parallel time) is sensitive to the data structure:

- For lists, $O(n)$, because splitting is slow.
- For trees, $O(\log^3 n)$, using rebalancing.

The parallelizability ratio, w/d, is $O(n/\log^2 n)$ for trees.

The correctness of the parallel implementation is never in question!

Bounded Implementation for Time

Brent's Principle: A computation with work w and span d can be implemented on a p-processor PRAM in time $O(\max(w/p, d))$.

- Work in chunks of p as much as possible.
- Number of processors is chosen at run-time.
- Proof is constructive: exhibits a scheduler.

Relates abstract cost to its concrete realization. No need for pseudo-code!

IO Efficiency

Aggarwal and Vitter introduced the IO Model:

- Distinguish primary from secondary memory.
- Cache size $M = k \times B$ words.
- Evaluate algorithm efficiency in terms of M and B.

Main result: k-way merge sort is optimal for the IO model:

$$O(n/B \log_{M/B}(n/B))$$

IO Efficiency

A&V's results can be matched in a purely functional model.

- No manual memory management.
- Natural functional programming.

Key idea: temporal locality implies spatial locality.

- Allocation order determines proximity.
- Reloading of migrated objects preserves proximity.
- Control stack specially managed to avoid cache contention.

Cost Semantics for IO

Cost semantics makes storage explicit:

$$\sigma \mathbin{@} e \Downarrow^{n} \sigma' \mathbin{@} v$$

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Figure of merit: traffic between main memory and cache expressed in terms of M and B.

(Simplified) Cost Semantics

$$\frac{\left\{ \quad \sigma_1 \mathbin{@} e_1 \Downarrow^{n_1'} \quad \sigma_1' \mathbin{@} l_1' \quad \right\}}{\sigma \mathbin{@} \mathtt{app}(e_1; e_2) \Downarrow^{n_1' + n_1'' + \quad n_2 + n_2'} \sigma' \mathbin{@} l'}$$

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Thm (Blelloch & H) An evaluation of cost n may be implemented on a stack machine with cache of size $4 \times M + B$ with cache complexity $k \times n$ for some small constant k.

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Thus, the cost semantics is a valid basis for IO analysis.

Merge, Revisited

```
fun merge nil ys = ys
  | merge xs nil = xs
  | merge (xs as x::xs’) (ys as y::ys’) =
    case compare x y of
      LESS ⇒ !a::merge xs’ ys
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Merge, Revisited

A data structure is compact iff it may be traversed in time $O(n/B)$.

Thm: For compact inputs `xs` and `ys` the call `merge xs ys` has cache complexity $O(n/B)$.

- Recurs down lists allocating only stack n frames: $O(n/B)$.
- Returns allocating n list cells: $O(n/B)$.

Copying operations `!a` and `!b` ensure compactness (locality).

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- Memory hierarchy effects [B& H 13, 15].

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Cost semantics integrates the combinatorial aspects:

- Enrich the tools available to algorithms designers.
- Extend complexity analysis to mathematically elegant languages.

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Improve both the structure and efficiency of programs!