Two Kinds of Foundations

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Edinburgh University
Thanks to Rod, Robin, and Gordon for setting up the Foundations Lab (and letting me be part of it).
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Joint work with Guy E. Blelloch and our students, past and present.
Inspiration from LFCS

LFCS has had a profound effect on theoretical computer science.

- **Mathematics** as a tool for understanding computation.
- **Application to** and **influence from** programming practice.
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Strong emphasis on **beauty** in theory and practice.

- Elegant mathematical theories (domains, logics, models).
- Elegant programming languages (HOPE, ML).
- Elegant verification tools (LEGO, CWB).
Two Sources of Beauty

For me beauty in a program arises from two sources:

- **Structure**: code as an expression of an idea.
- **Efficiency**: code as instructions for a computer.
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This has given rise to two theories of computation.

- **Logical**: compositionality (human effort).
- **Combinatorial**: efficiency (machine effort).
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• **Logical**: compositionality (human effort).
• **Combinatorial**: efficiency (machine effort).

But these are largely disparate communities, both in the US and in Europe.
Reconciling the Two Theories

Historically,

- The logical side neglects efficiency in favor of structure.
- The combinatorial side neglects structure in favor of efficiency.
Reconciling the Two Theories

Historically,
- The logical side neglects efficiency in favor of structure.
- The combinatorial side neglects structure in favor of efficiency.

Prospectively,
- The logical side should pay more attention to efficiency.
- The combinatorial side should pay more attention to structure.
The Great Rift

“On the fact that the Atlantic Ocean has two sides.” [EWD]

- **American theory** $\approx$ combinatorial theory.
- **Euro-theory** $\approx$ semantics and logic.
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Both have had a big influence on practice:

- **Efficient algorithms** for a broad range of problems.
- **Language design** and verification tools.
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Yet these two “theories” operate largely in isolation!
Algorithm analysis is based on machine models:

- Turing machine (TM) or Random Access Machine (RAM).
- Low-level: no abstraction, no composition.
- Allegedly, close to the hardware.

Machine models provide natural complexity measures:

- **Time** = number of instructions.
- **Space** = tape or memory usage.

Asymptotics smoothes over differences among models.
In practice algorithms are described using C-like notation.

- Clearer than TM or RAM code.
- Analyze compiled code, rather than source code.

An improvement, but still very limited:

- ephemeral data structures.
- manual memory management.
- poor composability.
- no abstraction.
Euro theory is based on language models:

- Church’s (typed and untyped) $\lambda$-calculus.
- High-level: abstraction, composition are fundamental.
- Platform-independent.

Language models support composition via variables:

- If $\phi \text{ true} \vdash \psi \text{ true}$, then if $\phi \text{ true}$, then $\psi \text{ true}$.
- If $x : \sigma \vdash N : \tau$, then if $M : \sigma$, then $[M/x]N : \tau$.

The $\lambda$-calculus is an elegant theory of composition.
Languages based on $\lambda$-calculus stress

- **persistent** data structures.
- **automatic** memory management.
- **strong** composability.
- **abstract types**.

But there is relatively little emphasis on **efficiency**.

- No clear complexity measures.
- Few analytic results (but see Okasaki’s CMU Ph.D.).
Traditional imperative methods of programming are obsolete.
  • Tedious to program, a nightmare to maintain.
  • Largely incompatible with parallelism.

Functional methods are destined to dominate.
  • Support verification and composition.
  • Naturally accommodate parallelism.

The way forward is to synthesize Euro- and American theory.
Cost Semantics

To elevate the level of discourse we require a cost semantics.

- Define the abstract cost of execution of a language.
- Defines the parallel and sequential complexity.

Algorithm analysis is conducted at the level of the code we write.

- Cost semantics assigns a measure to each execution.
- Analyze asymptotic complexity in terms of this measure.
The abstract cost is validated by a bounded implementation.

- Transform abstract cost into concrete cost on a machine.
- Account for platform characteristics such as number of processors, cache hierarchy, and interconnect.

An end-to-end asymptotics with a clear separation of concerns.

- High-level, composable development and reasoning.
- Low-level implementation on hardware platforms.
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So simple we teach it to first-year undergraduates!
Cost Semantics for Time

Associate a cost graph to the evaluation of a program.

- Dynamic, fully accurate record of data dependencies.
- Not a static analysis or an approximation!

Example: function application.

\[
\begin{array}{cccccc}
e_1 \downarrow & \lambda x.e & e_2 \downarrow & v_2 & [v_2/x]e \downarrow & v \\
\hline
& e_1(e_2) \downarrow & & & v \\
\end{array}
\]
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    [v_2/x]e & \Downarrow^g v \\
    e_1(e_2) & \Downarrow((g_1 \otimes g_2) \oplus 1 \oplus g) v
\end{align*}
\]
Series-parallel cost graphs:
- $\mathbf{1}$: one unit of computation.

Application cost $(g_1 \otimes g_2) \oplus \mathbf{1} \oplus g$ specifies that
Series-parallel cost graphs:

- **1**: one unit of computation.
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Application cost $(g_1 \otimes g_2) \oplus 1 \oplus g$ specifies that

- Function and argument are evaluated in parallel.
- Function call costs one unit.
- Function execution depends on the function and argument.
Cost Graphs
The work $w(g)$ of a cost graph $g$ is the size of $g$.

- $w(1) = 1$, $w(g_1 \otimes g_2) = w(g_1 \oplus g_2) = w(g_1) + w(g_2)$.
- Measures the sequential time complexity.

The span $d(g)$ of a cost graph $g$ is the critical path length of $g$.

- $d(1) = 1$, $d(g_1 \otimes g_2) = \max(d(g_1), d(g_2))$, $d(g_1 \oplus g_2) = d(g_1) + d(g_2)$.
- Measures the parallel time complexity.
Cost Graphs

Work = 11, Span = 6
fun merge \( xs \) \( ys \) = 
  \( case \ (xs, \ ys) \ of \)
  \( (\ [], \ ys) \Rightarrow ys \)
  | \( (xs, \ []) \Rightarrow xs \)
  | \( (x::xs', \ y::ys') \Rightarrow \)
    \( case \ x<y \ of \)
      \( true \Rightarrow x :: merge \ xs' \ ys \)
    | \( false \Rightarrow y :: merge \ xs \ ys' \)

fun sort \( [] \) = \( [] \)
  | sort \( [x] \) = \( [x] \)
  | sort \( xs \) = 
    let \( val \ (ys, \ zs) = split \ xs \)
    in \( merge \ (sort \ ys, \ sort \ zs) \ end \)
The work (sequential time) is optimal, $O(n \log n)$ for $n$ items.

The span (parallel time) is sensitive to the data structure:
- For lists, $O(n)$, because splitting is slow.
- For trees, $O(\log^3 n)$, using rebalancing.

The parallelizability ratio, $w/d$, is $O(n/\log^2 n)$ for trees.

The correctness of the parallel implementation is never in question!
Bounded Implementation

Brent’s Principle: A computation with work $w$ and span $d$ can be implemented on a $p$-processor PRAM in time $O(\max(w/p, d))$.

- Work in chunks of $p$ as much as possible.
- Number of processors is chosen at run-time.
- Proof is constructive: exhibits a scheduler.

Relates abstract to concrete cost.
Aggarwal and Vitter introduced the IO Model:

- Distinguish **primary** from **secondary** memory.
- Cache size $M = k \times B$ words.
- Evaluate algorithm efficiency in terms of $M$ and $B$.

Main result: $k$-way merge sort is **optimal** for the IO model:

$$O(n/B \log_{M/B}(n/B))$$
IO Efficiency

A&V’s results can be matched in a **purely functional** model.

- No manual memory management.
- Natural functional programming.

Key idea: **temporal locality implies spatial locality**.

- Allocation order determines proximity.
- Reloading of migrated objects preserves proximity.
- Control stack specially managed to avoid cache contention.
Cost Semantics for IO

Cost semantics makes storage explicit:

$$\sigma @ e \downarrow^n \sigma' @ v$$

Store $\sigma$ has three components:
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$$\sigma \odot e \Downarrow^n \sigma' \odot v$$

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- Unbounded main memory with blocks of size $B$. 
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- Read cache of size \( M = k \times B \).
- Linearly ordered allocation cache of size \( M \).
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\[ \sigma \circ e \downarrow^n \sigma' \circ v \]

Store \( \sigma \) has three components:

- Unbounded main memory with blocks of size \( B \).
- Read cache of size \( M = k \times B \).
- Linearly ordered allocation cache of size \( M \).

Figure of merit: traffic between main memory and cache expressed in terms of \( M \) and \( B \).
(Simplified) Cost Semantics

\[
\begin{align*}
\{ \\
\sigma_1 \odot e_1 \downarrow^{n'_1} & \quad \sigma'_1 \odot l'_1 \\
\sigma \odot \text{app}(e_1; e_2) \downarrow & \quad n'_1 + n''_1 + n'_2 + n''_2 \quad \sigma' \odot l''
\}
\end{align*}
\]
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\begin{align*}
\sigma_1 @ e_1 & \downarrow^{n'_1} \quad \sigma'_1 @ l'_1 \\
\{ \sigma'_1 @ l'_1 \downarrow^{n''_1} \sigma''_1 @ \lambda x. e \} \\
\sigma @ \text{app}(e_1; e_2) & \downarrow^{n'_1 + n''_1 + n'_2 + n''_2} \sigma' @ l''
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\[
\begin{aligned}
&\quad \\
&\{ \\
&\quad \sigma_1' @ l_1' \downarrow^{n_1''} \sigma_1'' @ \lambda x. e \\
&\quad \sigma''_1 @ e_2 \downarrow^{n_2} \sigma'_2 @ l_2' \\
&\} \\
&\sigma @ \text{app}(e_1; e_2) \downarrow^{n_1' + n_1'' + n_2 + n_2'} \sigma' @ l'
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Bounded Implementation

Thm (Blelloch & H) An evaluation of cost $n$ may be implemented on a stack machine with cache of size $4 \times M + B$ with cache complexity $k \times n$ for some small constant $k$. 
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Thus, the cost semantics is a valid basis for IO analysis.
fun merge nil ys = ys
| merge xs nil = xs
| merge (xs as x::xs’) (ys as y::ys’) =
  case compare x y of
    LESS \Rightarrow !a::merge xs’ ys
| GTEQ \Rightarrow !b::merge xs ys’
fun merge nil ys = ys  
  | merge xs nil = xs  
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    case compare x y of  
      LESS ⇒ !a::merge xs' ys  
    | GTEQ ⇒ !b::merge xs ys'

Merge, Revisited
A data structure is **compact** iff it may be traversed in time $O(n/B)$.

Thm: For compact inputs $xs$ and $ys$ the call `merge xs ys` has cache complexity $O(n/B)$.

- Recurs down lists allocating only stack $n$ frames: $O(n/B)$.
- Returns allocating $n$ list cells: $O(n/B)$.

Copying operations `!a` and `!b` ensure compactness (locality).
Cost semantics supports analysis of complexity of high-level code.

- **Real** code, not **pseudo**-code!
Summary

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- Sequential and parallel time [B & Greiner 96].
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- Sequential and parallel time [B & Greiner 96].
- Space usage of scheduling [Spoonhower, B, Gibbons, & H 09].
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- Sequential and parallel time [B & Greiner 96].
- Space usage of scheduling [Spoonhower, B, Gibbons, & H 09].
- Memory hierarchy effects [B& H 13, 15].
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\(\lambda\)-calculus provides a **logical** model of computation.

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Cost semantics integrates the **combinatorial** aspects:

- Enrich the tools available to algorithms designers.
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Cost semantics integrates the combinatorial aspects:

- Enrich the tools available to algorithms designers.
- Extend complexity analysis to mathematically elegant languages.
Where From Here?

Develop new (abstract and concrete) cost measures.

- Acar, Muller, & H: Latency for time-sensitive computations.
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Improve both the structure and efficiency of programs!
Guy Blelloch and John Greiner.
Parallelism in sequential functional languages.

Guy E Blelloch and Robert Harper.
Cache efficient functional algorithms.

Jan Hoffmann, Klaus Aehlig, and Martin Hofmann.
Multivariate Amortized Resource Analysis.

Space profiling for parallel functional programs.