Computational Higher Type Theory

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Thanks

Joint work with Carlo Angiuli (CMU) and Todd Wilson (CSUF).

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Two Kinds of Type Theory

Two traditions in type theory, both embodied by Martin-Löf:

- Formal, or axiomatic, as in ITT and HoTT.
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Most work in HoTT has taken place in the formal setting.

- Univalence Axiom, subsuming Function Extensionality.
- Higher Inductive Types, supporting truncation, etc.

Formal Type Theory

Martin-Löf; Coquand; HoTT

Formal type theory is inductively defined by rules:

- Formation: $\Gamma \vdash A\,\mathsf{type}$, $\Gamma \vdash M : A$.
- Definitional equivalence: $\Gamma \vdash A \equiv B$, $\Gamma \vdash M \equiv N : A$.

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Choice of rules can be delicate, eg what is definitional equivalence?

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- Associate invariants with types, eg normalization.

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Adding axioms disrupts these properties!

Semantic Type Theory

Martin-Löf; Constable, et al

Meaning explanations define types and elements semantically:

- Computational: as programs with deterministic dynamics.
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- Types are behavioral specifications.
- Types and objects are programs that execute.

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Inverts conceptual order compared to formal type theory:

- Type theory as a theory of truth.
- Proof theory accesses the truth.

Computational Meaning Explanation

Martin-Löf: Constr. Math. and Comp. Prog.

Start with computation on closed expressions (types and terms):

- Transition: $M \longmapsto M'$, one step of execution.
- Termination: M val is canonical/complete.

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Extend to open forms by functionality aka extensionality:

- Types: $a_1{:}A_1, \ldots, a_n{:}A_n \gg A \doteq B$ type $[\Psi]$.
- Terms: $a_1{:}A_1, \ldots, a_n{:}A_n \gg M \doteq N \in A$ $[\Psi]$.

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Judgments are not intended to be decidable.

- Quantifier complexity is arbitrarily high, not merely r.e.
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Two essential moves for higher-dimensionality:

- Judgmental account of identifications.
- Exact equality of types and elements at all dimensions.

Cubical Programming Language

Licata, Brunerie; Coquand, et al.

Syntax is organized cubically:

- Points correspond to ordinary terms and types.
- Lines represent identifications.
- Squares represent homotopies, etc.

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- Finite set of dimension variables $x, y, z, \ldots$.

Substitutions $\psi : \Psi' \to \Psi$ send $x \in \Psi$ to $\psi(x) = 0/1/x' \in \Psi'$.

Cubical Programming Language

Substitutions define the aspects of a cube E:

- Faces: $E\langle 0/x\rangle$, $E\langle 1/x\rangle$.
- Diagonals: $E\langle x', x'/x, y\rangle$.
- Degeneracy: silent/implicit.

$$
\begin{tikzcd}
\cdot \arrow[r, "x"] \arrow[d, "y"'] & {} \\
{} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=huge, row sep=huge]
E\langle 0/x\rangle\langle 0/y\rangle \arrow[r, "E\langle 0/y\rangle"] \arrow[d, "E\langle 0/x\rangle"'] \arrow[dr, phantom, "E"] & E\langle 1/x\rangle\langle 0/y\rangle \arrow[d, "E\langle 1/x\rangle"] \\
E\langle 0/x\rangle\langle 1/y\rangle \arrow[r, "E\langle 1/y\rangle"'] & E\langle 1/x\rangle\langle 1/y\rangle
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Unconventional functional programming constructs:

- Circle: $\mathbb{S}^1, \mathsf{base}, \mathsf{loop}_x, \mathbb{S}^1\text{-}\mathsf{elim}_{a.A}(M; M_\mathsf{b}, x.M_\mathsf{l})$.
- Negation: not_x, a type line, and glueing, $\mathsf{notel}_x(M)$.
- Kan operations: coe, hcom.

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The Kan operations are computational content of the Kan condition (cf, LB14, CCHM16).

Kan Operations

Coercion along a type line: $\mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M)$.

- Heterogeneous along line $x.A$.
- Evaluates A to effect coercion from $A\langle r/x \rangle$ to $A\langle r'/x \rangle$.

Composition: $\mathsf{hcom}_{A}^{\overrightarrow{r_i}}(r \rightsquigarrow r', M; \overrightarrow{y.N_i^{\varepsilon}})$.

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- The tubes $\overrightarrow{y.N_i^\varepsilon}$ with extent $\overrightarrow{r_i}$ in dimension $\overrightarrow{y_i}$.

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- The cap M is the starting cube.
- The tubes $\overrightarrow{y.N_i^\varepsilon}$ with extent $\overrightarrow{r_i}$ in dimension $\overrightarrow{y_i}$.
- Evaluates A to define composite, which may or may not be the hcom itself.

Two-Dimensional Compositions

$$
\begin{tikzcd}
\cdot \arrow[rr, "M"] \arrow[rr, dashed, bend right, "{\mathsf{hcom}^x_A(0 \rightsquigarrow 0, M; y.N^0, y.N^1)}"'] \arrow[d, "N^0"'] & & \cdot \arrow[d, "N^1"] \\
N^0\langle 1/y\rangle & & N^1\langle 1/y\rangle
\end{tikzcd}
$$

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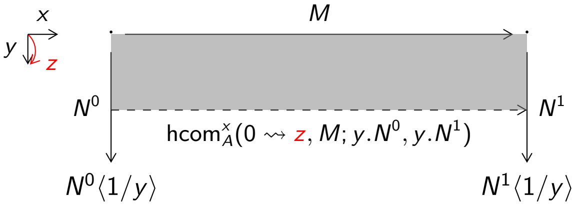

Cubical Meaning Explanation

Explanation proceeds in stages:

- Define the canonical types and their elements at each dimension Ψ.
- Define pre-types to be cubical, ie with coherent aspects.
- Define types to be Kan pre-types.

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The main criteria for a higher type system:

- All aspects of a type or element must be types or elements.
- Taking aspects must commute with evaluation.
- Equal types must have the same element equality.
- Equal types must be equally Kan.

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- $M \sim^{\Psi}_{A} N$ iff $M \longmapsto^* M_0$, $N \longmapsto^* N_0$, $A \longmapsto^* A_0$, and $M_0 \approx^{\Psi}_{A_0} N_0$.

Pre-Types: Coherent Aspects

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- Let $A\psi_1 \longmapsto^* A_1$ val, and $A_1\psi_2 \longmapsto^* A_2$ val, and $A\psi_2\psi_1 \longmapsto^* A_{12}$ val.

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- Require:

$$
\begin{tikzcd}
A \arrow[r, Rightarrow, "\psi_1"] \arrow[d, Rightarrow, "\psi_1\psi_2"'] & A_1 \arrow[d, Rightarrow, "\psi_2"] \\
A_{12} \arrow[r, phantom, "\approx^{\Psi_2}"] & A_2
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Similarly for exact equality of types and of elements: substitute-then-evaluate is functorial.

Pre-Types and Types

A pretype $[\Psi]$ is cubical: its values have coherent aspects:

- If $\psi : \Psi' \to \Psi$ and $M \approx^{\Psi'}_{A\psi} N$, then $M \doteq N \in A\psi\ [\Psi']$.

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A type is a Kan pre-type:

- Supports coercion and composition.
- Certain equational requirements are met.

Kan Conditions for Coercion

For any $\psi : (\Psi', x) \to \Psi$, if

$$M \in A\psi\langle r/x\rangle \; [\Psi'],$$

then

$$\mathsf{coe}_{x.A\psi}^{r \rightsquigarrow r'}(M) \in A\psi\langle r'/x\rangle \; [\Psi'].$$

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Constraints limit applicable substitutions; conditions can be vacuous.

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The dynamics of the conditional accounts for

- true and false, as usual.
- hcom's that are values.

Boolean Dynamics

$$\frac{}{\mathsf{bool}\ \mathsf{val}}$$

$$\frac{\overrightarrow{r_i} = x_1, \ldots, x_{i-1}, \varepsilon, r_{i+1}, \ldots, r_n}{\mathsf{hcom}^{\overrightarrow{r_i}}_{\mathsf{bool}}(r \rightsquigarrow r', M; \overrightarrow{y.N_i^\varepsilon}) \longmapsto N_i^\varepsilon\langle r'/y\rangle}$$

$$\frac{r = r'}{\mathsf{hcom}^{x_1,\ldots,x_n}_{\mathsf{bool}}(r \rightsquigarrow r', M; \overrightarrow{y.N_i^\varepsilon}) \longmapsto M}$$

$$\frac{}{\mathsf{true}\ \mathsf{val}} \qquad \frac{}{\mathsf{false}\ \mathsf{val}}$$

$$\frac{r \neq r'}{\mathsf{hcom}^{x_1,\ldots,x_n}_{\mathsf{bool}}(r \rightsquigarrow r', M; \overrightarrow{y.N_i^\varepsilon})\ \mathsf{val}}$$

Boolean Dynamics

$$\frac{M \longmapsto M'}{\mathsf{if}_{a.A}(M; T, F) \longmapsto \mathsf{if}_{a.A}(M'; T, F)} \qquad \frac{}{\mathsf{if}_{a.A}(\mathsf{true}; T, F) \longmapsto T}$$

$$\frac{}{\mathsf{if}_{a.A}(\mathsf{false}; T, F) \longmapsto F}$$

$$\frac{r \neq r' \quad H = \mathsf{hcom}_{\mathsf{bool}}^{x_1,\ldots,x_n}(r \rightsquigarrow z, M; \overrightarrow{y.N_i^{\varepsilon}})}{\begin{array}{c}\mathsf{if}_{a.A}(\mathsf{hcom}_{\mathsf{bool}}^{x_1,\ldots,x_n}(r \rightsquigarrow r', M; \overrightarrow{y.N_i^{\varepsilon}}); T, F) \\ \longmapsto \\ \mathsf{com}_{z.A[H/a]}^{x_1,\ldots,x_n}(r \rightsquigarrow r', \mathsf{if}_{a.A}(M; T, F); \overrightarrow{y.\mathsf{if}_{a.A}(N_i^{\varepsilon}; T, F)})\end{array}}$$

$$\frac{}{\mathsf{coe}_{x.\mathsf{bool}}^{r \rightsquigarrow r'}(M) \longmapsto M}$$

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Guarantees canonicity for closed points in bool: all evaluate to either true or false.

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The term $\mathsf{notel}_x(M) \in \mathsf{not}_x\ [\Psi, x]$ is a use of gluing [CCHM16]:

$$
\begin{tikzcd}
{} \arrow[r, "x"] \arrow[d, "y"'] & {} & \cdot \arrow[r, "\mathsf{bool}"] \arrow[d, maps to, "\mathsf{not}(-)"'] & \cdot \arrow[d, maps to, "\mathsf{id}"] \\
{} & & \mathsf{bool} \arrow[r, dashed, "\mathsf{not}_x"'] & \mathsf{bool}
\end{tikzcd}
$$

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Dependent function and product types (Pi's and Sigma's) with full universal properties.

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Validates expected formal rules.

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But why limit attention to these formal theories?

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- Richer notions of computation: partiality, non-determinism, recursive types, exceptions, state, ... [Constable, et al.]

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Computation model induces dynamics of explicitly typed languages.

Ongoing and Future Work

Full account of univalence for all types.

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Implementation in Sterling's RedPRL (`redprl.org`).

- NuPRL-like refinement rules.
- Richer notion of tactics.
- Name generation is primitive (cf continuity principle).

References

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