Two Notions of Beauty in Programming

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John C. Mitchell’s 60th (What?) Birthday Celebration
Thanks to Kathleen, Vitaly, Andre, Pat, and Dan for organizing.
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Joint work with Guy Blelloch and our students, past and present.
Two Sources of Beauty in Programs

For me beauty in a program arises from two sources:

- **Structure**: code as an expression of an idea.
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- **Logical**: compositionality (human effort).
- **Combinatorial**: efficiency (machine effort).
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Oddly, these are largely disparate communities.
The Great Rift

“On the fact that the Atlantic Ocean has two sides.” [EWD]

- American theory ≈ combinatorial theory.
- Euro-theory ≈ semantics and logic.
“On the fact that the Atlantic Ocean has two sides.” [EWD]

- American theory \(\approx\) combinatorial theory.
- Euro-theory \(\approx\) semantics and logic.

Both have had a big influence on practice:

- Efficient algorithms for a broad range of problems.
- Language design and verification tools.
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- **Efficient algorithms** for a broad range of problems.
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Yet these two “theories” operate largely in isolation.
American Theory

Algorithm analysis is based on machine models:

- Turing machine (TM) or Random Access Machine (RAM).
- Low-level: no abstraction, no composition.
- Allegedly, close to the hardware.

Machine models provide natural complexity measures:

- **Time** = number of instructions.
- **Space** = tape or memory usage.

Asymptotics smoothes over differences among models.
Euro theory is based on language models:

- Church’s (typed and untyped) $\lambda$-calculus.
- High-level: abstraction, composition are fundamental.
- Platform-independent.

Language models support composition via variables:

- If $\phi \text{ true} \vdash \psi \text{ true}$, then if $\phi \text{ true}$, then $\psi \text{ true}$.
- If $x : \sigma \vdash N : \tau$, then if $M : \sigma$, then $[M/x]N : \tau$.

The $\lambda$-calculus is an elegant theory of composition.
Thesis

Traditional imperative methods of programming are obsolete.
  • Tedious to program, a nightmare to maintain.
  • Largely incompatible with parallelism.

Functional methods are destined to dominate.
  • Support verification and composition.
  • Naturally accommodate parallelism.

The way forward is to synthesize Euro- and American theory.
To elevate the level of discourse we require a cost semantics.
  • Define the abstract cost of execution of a language.
  • Defines the parallel and sequential complexity.

Algorithm analysis is conducted at the level of the code we write.
  • Cost semantics assigns a measure to each execution.
  • Analyze asymptotic complexity in terms of this measure.
The abstract cost is validated by a bounded implementation.

- Transform abstract cost into concrete cost on a machine.
- Account for platform characteristics such as number of processors, cache hierarchy, and interconnect.

An end-to-end asymptotics with a clear separation of concerns.

- High-level, composable development and reasoning.
- Low-level implementation on hardware platforms.
Cost Semantics for Time

Associate a cost graph to the evaluation of a program.

- **Dynamic**, fully accurate record of data dependencies.
- **Not** a static analysis or an approximation.

Example: function application.

\[
\begin{array}{cccccc}
  e_1 \downarrow & \lambda x.e & e_2 \downarrow & v_2 & [v_2/x]e \downarrow & v \\
  e_1(e_2) \downarrow & v
\end{array}
\]
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**Example: function application.**

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e_1 \downarrow^{g_1} \lambda x. e & \quad e_2 \downarrow^{g_2} v_2 & \quad [v_2/x]e \downarrow^{g} v \\
e_1(e_2) \downarrow & \quad v
\end{align*}
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e_2 \downarrow^{g_2} & v_2 \\
[v_2/x]e & \downarrow^{g} v \\
e_1(e_2) & \downarrow^{(g_1 \otimes g_2) \oplus 1 \oplus g} v
\end{align*}
\]
Cost Graphs

Series-parallel cost graphs:
  - $\mathbf{1}$: one unit of computation.

Application cost \((g_1 \otimes g_2) \oplus \mathbf{1} \oplus g\) specifies that
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- **1**: one unit of computation.
- **$g_1 \oplus g_2$**: $g_2$ depends on result of $g_1$.

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Application cost \((g_1 \otimes g_2) \oplus 1 \oplus g\) specifies that

- Function and argument are evaluated in parallel.
- Function call costs one unit.
- Function execution depends on the function and argument.
Work and Span

The **work** $w(g)$ of a cost graph $g$ is the size of $g$.
- $w(1) = 1$, $w(g_1 \otimes g_2) = w(g_1 \oplus g_2) = w(g_1) + w(g_2)$.
- Measures the **sequential time complexity**.

The **span** $d(g)$ of a cost graph $g$ is the critical path length of $g$.
- $d(1) = 1$, $d(g_1 \otimes g_2) = \max(d(g_1), d(g_2))$,
  $d(g_1 \oplus g_2) = d(g_1) + d(g_2)$.
- Measures the **parallel time complexity**.
fun merge xs ys = 
  case (xs, ys) of 
    ([], ys) ⇒ ys 
  | (xs, []) ⇒ xs 
  | (x::xs’, y::ys’) ⇒ 
    case x<y of 
      true ⇒ x :: merge xs’ ys 
    | false ⇒ y :: merge xs ys’ 

fun sort [] = [] 
  | sort [x] = [x] 
  | sort xs = 
    let val (ys, zs) = split xs 
    in  merge (sort ys, sort zs)  end
The work (sequential time) is optimal, $O(n \log n)$ for $n$ items.

The span (parallel time) is sensitive to the data structure:
- For lists, $O(n)$, because splitting is slow.
- For trees, $O(\log^3 n)$, using rebalancing.
Bounded Implementation for Time

Brent’s Principle: A computation with work $w$ and span $d$ can be implemented on a $p$-processor PRAM in time $O(\max(w/p, d))$.

- Work in chunks of $p$ as much as possible.
- Number of processors is chosen at run-time.
- Proof is constructive: exhibits a scheduler.

No need for pseudo-code!
Aggarwal and Vitter introduced the IO Model:

- Distinguish primary from secondary memory.
- Cache size $M = k \times B$ words.
- Evaluate algorithm efficiency in terms of $M$ and $B$.

Main result: $k$-way merge sort is optimal for the IO model:

$$O\left(\frac{n}{B} \log_{M/B}(n/B)\right)$$
A&V’s results can be matched in a purely functional model.

- No manual memory management.
- Natural functional programming.

**Key idea:** temporal locality implies spatial locality.

- Allocation order determines proximity.
- Reloading of migrated objects preserves proximity.
- Control stack specially managed to avoid cache contention.
Cost Semantics for IO

Cost semantics makes storage explicit:

\[ \sigma @ e \downarrow^n \sigma' @ v \]

Store \( \sigma \) has three components:
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Store \( \sigma \) has three components:
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- Read cache of size \( M = k \times B \).
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- Read cache of size \( M = k \times B \).
- Linearly ordered allocation cache of size \( M \).
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- Read cache of size \( M = k \times B \).
- Linearly ordered allocation cache of size \( M \).

Figure of merit: traffic between main memory and cache expressed in terms of \( M \) and \( B \).
(Simplified) Cost Semantics

\[
\left\{
\begin{array}{l}
\sigma_1 \circ e_1 \Downarrow^{n_1'} \quad \sigma_1' \circ l_1'
\\
\sigma \circ \text{app}(e_1; e_2) \Downarrow^{n_1'+n_1''+n_2+n_2'} \quad \sigma' \circ l''
\end{array}
\right.
\]
(Simplified) Cost Semantics

\[
\begin{align*}
\{ & \quad \sigma_1 @ l_1 \downarrow^{n'_1} \sigma_1 \ @ \lambda x. e \\
& \quad \sigma_1 @ e_1 \downarrow^{n'_1} \quad \sigma'_1 @ l'_1 \}
\end{align*}
\]

\[
\sigma @ \text{app}(e_1; e_2) \downarrow \quad n'_1 + n''_1 + \quad n_2 + n'_2 \quad \sigma' @ l'
\]
(Simplified) Cost Semantics

\[
\left\{ \begin{array}{c}
\sigma_1' \circ l_1' \downarrow^{n_1''} \quad \sigma_1'' \circ \lambda x. e \\
\sigma_1'' \circ e_2 \downarrow^{n_2} \quad \sigma_2' \circ l_2'' \\
\end{array} \right. \\
\sigma \circ \text{app}(e_1; e_2) \downarrow^{n_1' + n_1'' + n_2 + n_2'} \sigma' \circ l''
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(Simplified) Cost Semantics

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\begin{align*}
\{ & \sigma_1' @ l_1' \downarrow^{n_1''} \sigma_1'' @ \lambda x. e \\
      & \sigma_1'' @ e_2 \downarrow^{n_2} \quad \sigma_2' @ l_2' \\
\} \quad \sigma_2 @ [l_2'/x]e \downarrow^{n_2} \sigma' @ l' \quad \sigma_1 @ e_1 \downarrow^{n_1'} \quad \sigma_1' @ l_1'
\end{align*}
\]

\[
\sigma @ \text{app}(e_1; e_2) \downarrow^{n'_1+n''_1+n_2+n'_2} \sigma' @ l'
\]
Thm (Blelloch & H) An evaluation of cost $n$ may be implemented on a stack machine with cache of size $4 \times M + B$ with cache complexity $k \times n$ for some small constant $k$. 
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- Sleator, et al.: LRU eviction policy is 2-competitive with ICM.
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- Appel: cost of copying GC is asymptotically free.
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- Sleator, et al.: LRU eviction policy is 2-competitive with ICM.
- Appel: cost of copying GC is asymptotically free.
- B&H: Stack management induces small constant overhead.
fun merge nil ys = ys
  | merge xs nil = xs
  | merge (xs as x::xs') (ys as y::ys') =
    case compare x y of
      LESS ⇒ !a::merge xs' ys
     | GTEQ ⇒ !b::merge xs ys'
fun merge nil ys = ys
  | merge xs nil = xs
  | merge (xs as x::xs’) (ys as y::ys’) =
      case compare x y of
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Merge, Revisited

A data structure is **compact** iff it may be traversed in time $O(n/B)$.

**Thm:** For compact inputs $xs$ and $ys$ the call `merge xs ys` has cache complexity $O(n/B)$.

- Recurs down lists allocating only stack $n$ frames: $O(n/B)$.
- Returns allocating $n$ list cells: $O(n/B)$.

Copying operations !a and !b ensure compactness (locality).
Cost semantics supports analysis of complexity of high-level code.

- No need for “pseudo-code”.
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Costs can be chosen to reflect different notions of complexity:
- Sequential and parallel time [B & Greiner 96].
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Costs can be chosen to reflect different notions of complexity:

- Sequential and parallel time [B & Greiner 96].
- Space usage of scheduling [Spoonhower, B, Gibbons, & H 09].
- Memory hierarchy effects [B& H 13, 15].