PFPL Supplement: Defining the Size of a λ-term Compositionally

Robert Harper

September, 2018

1 Introduction

In Harper (2018) the size of a λ-term is defined by the following rules:

\[
\begin{align*}
\frac{x \text{ var}}{\text{size}(x, 1)} & \quad (1a) \\
\frac{\text{size}(M_1, n_1) \quad \text{size}(M_2, n_2)}{\text{size}(\text{ap}(M_1, M_2), n_1 + n_2 + 1)} & \quad (1b) \\
\frac{x \text{ var} \quad \text{size}(M, n)}{\text{size}(\lambda(x).M, n + 1)} & \quad (1c)
\end{align*}
\]

This definition can be criticized on the grounds that it treats variables as “things” rather than as placeholders that may be replaced by another term. This means, in particular, that it is not easy to relate the size of \([M/x]N\) to the sizes of \(M\) and \(N\). In particular, the size of \(N\), as defined above, counts 1 for each occurrence of \(x\), possibly none, whereas the size of \([M/x]N\) counts \(m\), the size of \(M\), for each occurrence of \(x\) in \(N\). The answers will differ, in general, because \(M\) can be of arbitrary size.

2 Reformulation

If a variable \(x\) is merely a placeholder, then it does not make sense to count it as having size 1, because it can be replaced with anything! It is possible to reformulate the definition of size so that it is inherently compositional? Yes, using the concept of a hypothetical judgment. Briefly, whenever a variable is introduced, an assumption about the size of the term that will eventually be substituted for it is also introduced to account for its size before the substitution is even performed.

To do so, it suffices to give an inductive definition of judgments of the form

\[
\text{size}(x_1, m_1), \ldots, \text{size}(x_k, m_k) \vdash \text{size}(N, n)
\]

such that if \(i \neq j\), then \(x_i \neq x_j\). These judgments express entailments between the hypotheses \(\Gamma = \text{size}(x_1, m_1), \ldots, \text{size}(x_k, m_k)\) and conclusion \(\text{size}(N, n)\). The size \(n\) of \(N\) is calculated relative to the assumed sizes of the terms that will be substituted for \(x_1, \ldots, x_k\), ensuring compositionality.

\[
\Gamma, \text{size}(x, m) \vdash \text{size}(x, m) \quad (2a)
\]
The first rule states that variables have their assumed size. The second states that the size of an application is one more than the sum of the sizes of the two subterms. The third states that the size of a λ-abstraction is one more than the size of its body, calculated under the assumption that the abstracted variable $x$ has size 0. The reason for the zero size of $x$ is that it is not free in $\lambda(x.M)$, and so should not influence the size of the overall term (nothing can be substituted for it).

**Theorem 2.1.**  
1. *Weakening:* if $\Gamma \vdash \text{size}(N,n)$, then $\Gamma, \text{size}(x,m) \vdash \text{size}(N,n)$.

2. *Substitution:* if $\Gamma, \text{size}(x,m) \vdash \text{size}(N,n)$, then if $\Gamma \vdash \text{size}(M,m)$, then $\Gamma \vdash \text{size}([M/x]N,n)$.

**Proof.** Both are proved by rule induction on the major premise. □

The substitution property expresses the desired compositionality of the size calculation. Notice that the assumed size of $x$ matches the actual size $m$ of the substituting term $M$, and this information is incorporated into $n$, the size of $N$ relative to that assumption. The assumptions $\Gamma$ express "predictions" about the size of the terms that will be substituted for these variables.

**References**
