PFPL Supplement: Comparing **fix** and **self**

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General recursion, $\text{fix} \, \tau \, \text{is} \, e$, is only sensible in a by-name dynamics for **PCF**, because it steps to $[\text{fix} \, \tau \, \text{is} \, e / x] e$, which substitutes a non-value for the variable $x$ in the expression $e$. In a by-value dynamics general recursion is, for this reason, not sensible and must be replaced by type-specific forms of self-reference, such as the self-referential function form $\text{fun} \{ \tau_1 ; \tau_2 \} (f \, x \, e)$, which is postulated to be a value of type $\tau_1 \rightarrow \tau_2$. Such a function is a value that is unrolled on application, substituting the recursive function value itself for the recursive variable, $f$, in the body of the function.

In **FPC** an alternative account of self-reference is provided by the recursive type $\tau \, \text{self}$, the type of self-referential values $\text{self} \, x \, \text{is} \, e$. Within $e$ the self-reference, $x$, must be unrolled, writing $\text{unroll}(x)$, to unroll the recursion and access the underlying expression $e$. More precisely,

$$\text{unroll}(\text{self} \, x \, \text{is} \, e) \mapsto [\text{self} \, x \, \text{is} \, e / x] e,$$

which makes sense in either a by-name or a by-value dynamics.

Using **self** types the recursive factorial function has type $(\text{nat} \rightarrow \text{nat}) \, \text{self}$, which reveals in its type that it is self-referential. To call such a function, either externally or internally within its definition, it is necessary to first $\text{unroll}$ the self-reference and apply it to an argument, which may be either a value or a computation, depending on the dynamics. In any case the self-referential variable is only ever replaced by a value.

Curiously, in **FPC** it is possible to define **fix** from **self**, obtaining the expected dynamics, even by value! The “trick” is to anticipate the need to unroll any self-reference within $\text{self} \, x \, \text{is} \, e$ by forming $\hat{e}$ to be $[\text{unroll}(x) / x] e$, and then defining $\text{fix} \, \tau \, \text{is} \, e$ to be the expression $\text{unroll}(\text{self} \, x \, \text{is} \, \hat{e})$. Observe that

$$\text{fix} \, \tau \, \text{is} \, e = \text{unroll}(\text{self} \, x \, \text{is} \, \hat{e})$$

$$\mapsto [\text{self} \, x \, \text{is} \, \hat{e} / x] \hat{e}$$

$$= [\text{self} \, x \, \text{is} \, \hat{e} / x] [\text{unroll}(x) / x] e$$

$$= [\text{unroll}(\text{self} \, x \, \text{is} \, \hat{e}) / x] e$$

$$= [\text{fix} \, \tau \, \text{is} \, e / x] e.$$

The penultimate line summarizes the result of the iterated substitution; it does not arise in the dynamics itself as a substitution of a non-value for a variable, which would be disallowed in the by-value case.
References