PFPL Supplement: Comparing fix and self

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General recursion, $\text{fix} \, x : \tau \, \text{is} \, e$, is only sensible in a by-name dynamics for PCF, because it steps to $[\text{fix} \, x : \tau \, \text{is} \, e/x]e$, which substitutes a non-value for the variable $x$ in the expression $e$. In a by-value dynamics general recursion is, for this reason, not sensible and must be replaced by type-specific forms of self-reference. For example, the self-referential function form $\text{fun}\{\tau_1; \tau_2\}(f.x.e)$ is postulated to be a value of type $\tau_1 \to \tau_2$, under appropriate typing constraints. Such a function is unrolled on application, substituting the recursive function value itself for the recursive variable, $f$, in the body of the function.

In FPC an alternative account of self-reference is provided by the recursive type $\tau \, \text{self}$, the type of self-referential values $\text{self} \, x \, \text{is} \, e$. Within $e$ the self-reference, $x$, must be unrolled, writing $\text{unroll}(x)$, to unroll the recursion and access the underlying expression $e$. More precisely,

$$\text{unroll}(\text{self} \, x \, \text{is} \, e) \mapsto [\text{self} \, x \, \text{is} \, e/x]e,$$

which makes sense in either a by-name or a by-value dynamics. For example, the recursive factorial function has type $(\text{nat} \to \text{nat}) \, \text{self}$, revealing that it is self-referential. To call such a function, either from the outside or internally within its definition, it is necessary to unroll the self-reference before applying it to an argument; the self-referential variable is only ever replaced by a value.

Curiously, in FPC it is possible to define fix from self, obtaining the expected dynamics, even under a by-value interpretation! Specifically, define $\hat{e}$ to be $[\text{unroll}(x)/x]e$, and then define $\text{fix} \, x : \tau \, \text{is} \, e$ to be the expression $\text{unroll}(\text{self} \, x \, \text{is} \, \hat{e})$. Then observe the following transition:

$$\text{fix} \, x : \tau \, \text{is} \, e = \text{unroll}(\text{self} \, x \, \text{is} \, \hat{e})$$

$$\mapsto [\text{self} \, x \, \text{is} \, \hat{e}/x]\hat{e}$$

$$= [\text{self} \, x \, \text{is} \, \hat{e}/x][\text{unroll}(x)/x]e$$

$$= [\text{unroll}(\text{self} \, x \, \text{is} \, \hat{e})/x]e$$

$$= [\text{fix} \, x : \tau \, \text{is} \, e/x]e.$$ 

The penultimate line summarizes the result of the iterated substitution; it does not arise in the dynamics as a substitution of a non-value for a variable, which would be disallowed in the by-value case.

References