PFPL Supplement: Comparing fix and self

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General recursion, fix\(x:\tau\ is\ e\), is only sensible in a by-name dynamics for PCF, because it steps to \([\text{fix}\ x:\tau\ is\ e/x]e\), which substitutes a non-value for the variable \(x\) in the expression \(e\). In a by-value dynamics general recursion is, for this reason, not sensible and must be replaced by type-specific forms of self-reference, such as the self-referential function form \(\text{fun}\{\tau_1;\tau_2\}(f.x.e)\), which is postulated to be a value of type \(\tau_1 \rightarrow \tau_2\). Such a function is a value that is unrolled on application, substituting the recursive function value itself for the recursive variable, \(f\), in the body of the function.

In FPC an alternative account of self-reference is provided by the recursive type \(\tau_{\text{self}}\), the type of self-referential values \(\text{self}\ x\ is\ e\). Within \(e\) the self-reference, \(x\), must be unrolled, writing \text{unroll}(x), to unroll the recursion and access the underlying expression \(e\). More precisely,

\[\text{unroll}(\text{self}\ x\ is\ e) \mapsto [\text{self}\ x\ is\ e/x]e,\]

which makes sense in either a by-name or a by-value dynamics.

Using self types the recursive factorial function has type \((\text{nat} \rightarrow \text{nat}) \text{self}\), which reveals in its type that it is self-referential. To call such a function, either externally or internally within its definition, it is necessary to first \text{unroll} the self-reference and apply it to an argument, which may be either a value or a computation, depending on the dynamics. In any case the self-referential variable is only ever replaced by a value.

Curiously, in FPC it is possible to define fix from self, obtaining the expected dynamics, even by value! The “trick” is to anticipate the need to unroll any self-reference within \(\text{self}\ x\ is\ e\) by forming \(\hat{e}\) to be \([\text{unroll}(x)/x]e\), and then defining \(\text{fix}\ x:\tau\ is\ e\) to be the expression \(\text{unroll}(\text{self}\ x\ is\ \hat{e})\). Observe that

\[
\text{fix}\ x:\tau\ is\ e = \text{unroll}(\text{self}\ x\ is\ \hat{e}) \\
= \text{unroll}(\text{self}\ x\ is\ \hat{e}/x) \hat{e} \\
= [\text{self}\ x\ is\ \hat{e}/x][\text{unroll}(x)/x]e \\
= [\text{unroll}(\text{self}\ x\ is\ \hat{e})/x]e \\
= [\text{fix}\ x:\tau\ is\ e/x]e.
\]

The penultimate line summarizes the result of the iterated substitution; it does not arise in the dynamics itself as a substitution of a non-value for a variable, which would be disallowed in the by-value case.
References