PFPL Supplement: Comparing fix and self

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General recursion, \( \text{fix} \ x : \tau \text{is} \ e \), is only sensible in a by-name dynamics for \( \text{PCF} \), because it steps to \( [\text{fix} \ x : \tau \text{is} e/x]e \), which substitutes a non-value for the variable \( x \) in the expression \( e \). In a by-value dynamics general recursion is, for this reason, not sensible and must be replaced by type-specific forms of self-reference. For example, the self-referential function form \( \text{fun}\{\tau_1;\tau_2\}(f.x.e) \) is postulated to be a value of type \( \tau_1 \rightarrow \tau_2 \), under appropriate typing constraints. Such a function is unrolled on application, substituting the recursive function value itself for the recursive variable, \( f \), in the body of the function.

In \( \text{FPC} \) an alternative account of self-reference is provided by the recursive type \( \tau \text{self} \), the type of self-referential values \( \text{self} \ x \text{is} \ e \). Within \( e \) the self-reference, \( x \), must be unrolled, writing \( \text{unroll}(x) \), to unroll the recursion and access the underlying expression \( e \). More precisely,

\[
\text{unroll}(\text{self} \ x \text{is} \ e) \mapsto [\text{self} \ x \text{is} e/x]e,
\]

which makes sense in either a by-name or a by-value dynamics. For example, the recursive factorial function has type \( (\text{nat} \rightarrow \text{nat}) \text{self} \), revealing that it is self-referential. To call such a function, either from the outside or internally within its definition, it is necessary to \( \text{unroll} \) the self-reference before applying it to an argument; the self-referential variable is only ever replaced by a value.

Curiously, in \( \text{FPC} \) it is possible to define \( \text{fix} \) from \( \text{self} \), obtaining the expected dynamics, even under a by-value interpretation! Specifically, define \( \hat{e} \) to be \( [\text{unroll}(x)/x]e \), and then define \( \text{fix} \ x : \tau \text{is} \ e \) to be the expression \( \text{unroll}(\text{self} \ x \text{is} \hat{e}) \). Then observe the following transition:

\[
\text{fix} \ x : \tau \text{is} e = \text{unroll}(\text{self} \ x \text{is} \hat{e}) \\
\mapsto [\text{self} \ x \text{is} \hat{e}/x]\hat{e} \\
= [\text{self} \ x \text{is} \hat{e}/x][\text{unroll}(x)/x]e \\
= [\text{unroll}(\text{self} \ x \text{is} \hat{e})]/x]e \\
= [\text{fix} \ x : \tau \text{is} e/x]e.
\]

The penultimate line summarizes the result of the iterated substitution; it does not arise in the dynamics as a substitution of a non-value for a variable, which would be disallowed in the by-value case.

References