PFPL Supplement: Comparing \texttt{fix} and \texttt{self}

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September 23, 2019

General recursion, \( \texttt{fix} : \tau \texttt{is} e \), is only sensible in a by-name dynamics for \texttt{PCF}, because it steps to \( [\texttt{fix} : \tau \texttt{is} e/x]e \), which substitutes a non-value for the variable \( x \) in the expression \( e \). In a by-value dynamics general recursion is, for this reason, not sensible and must be replaced by type-specific forms of self-reference. For example, the self-referential function form \( \text{fun}\{\tau_1; \tau_2\}(f.x.e) \) is postulated to be a value of type \( \tau_1 \rightarrow \tau_2 \), under appropriate typing constraints. Such a function is unrolled on application, substituting the recursive function value itself for the recursive variable, \( f \), in the body of the function.

In \texttt{FPC} an alternative account of self-reference is provided by the recursive type \( \tau \texttt{self} \), the type of self-referential values \( \texttt{self} \texttt{is} e \). Within \( e \) the self-reference, \( x \), must be unrolled, writing \( \texttt{unroll}(x) \), to unroll the recursion and access the underlying expression \( e \). More precisely,

\[
\texttt{unroll}(\texttt{self} \texttt{is} e) \mapsto [\texttt{self} \texttt{is} e/x]e,
\]

which makes sense in either a by-name or a by-value dynamics. For example, the recursive factorial function has type \( (\texttt{nat} \rightarrow \texttt{nat}) \texttt{self} \), revealing that it is self-referential. To call such a function, either from the outside or internally within its definition, it is necessary to \texttt{unroll} the self-reference before applying it to an argument; the self-referential variable is only ever replaced by a value.

Curiously, in \texttt{FPC} it is possible to define \texttt{fix} from \texttt{self}, obtaining the expected dynamics, even under a by-value interpretation! Specifically, define \( \hat{e} \) to be \( [\texttt{unroll}(x)/x]e \), and then define \( \texttt{fix} : \tau \texttt{is} e \) to be the expression \( \texttt{unroll}(\texttt{self} \texttt{is} \hat{e}) \). Then observe the following transition:

\[
\texttt{fix} : \tau \texttt{is} e = \texttt{unroll}(\texttt{self} \texttt{is} \hat{e})
\]
\[
\mapsto [\texttt{self} \texttt{is} \hat{e}/x]e
\]
\[
= [\texttt{self} \texttt{is} \hat{e}/x][\texttt{unroll}(x)/x]e
\]
\[
= [\texttt{unroll}(\texttt{self} \texttt{is} \hat{e})/x]e
\]
\[
= [\texttt{fix} : \tau \texttt{is} e/x]e.
\]

The penultimate line summarizes the result of the iterated substitution; it does not arise in the dynamics as a substitution of a non-value for a variable, which would be disallowed in the by-value case.

References