PFPL Supplement: Comparing \textbf{fix} and \textbf{self}

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General recursion, $\textbf{fix} : \tau \text{ is } e$, is only sensible in a by-name dynamics for \textbf{PCF}, because it steps to $[\textbf{fix} x : \tau \text{ is } e/x]e$, which substitutes a non-value for the variable $x$ in the expression $e$. In a by-value dynamics general recursion is, for this reason, not sensible and must be replaced by type-specific forms of self-reference, such as the self-referential function form $\textbf{fun}\{\tau_1; \tau_2\}(f.x.e)$, which is postulated to be a value of type $\tau_1 \rightarrow \tau_2$. Such a function is a value that is unrolled on application, substituting the recursive function value itself for the recursive variable, $f$, in the body of the function.

In \textbf{FPC} an alternative account of self-reference is provided by the recursive type $\tau \text{self}$, the type of self-referential values $\text{self} x \text{ is } e$. Within $e$ the self-reference, $x$, must be unrolled, writing $\text{unroll}(x)$, to unroll the recursion and access the underlying expression $e$. More precisely,

$$\text{unroll}(\text{self} x \text{ is } e) \mapsto [\text{self} x \text{ is } e/x]e,$$

which makes sense in either a by-name or a by-value dynamics.

Using \texttt{self} types the recursive factorial function has type $(\texttt{nat} \rightarrow \texttt{nat}) \text{self}$, which reveals in its type that it is self-referential. To call such a function, either externally or internally within its definition, it is necessary to first unroll the self-reference and apply it to an argument, which may be either a value or a computation, depending on the dynamics. In any case the self-referential variable is only ever replaced by a value.

Curiously, in \textbf{FPC} it is possible to define $\textbf{fix}$ from \texttt{self}, obtaining the expected dynamics, even by value! The “trick” is to anticipate the need to unroll any self-reference within $\text{self} x \text{ is } e$ by forming $\hat{e}$ to be $[\text{unroll}(x)/x]e$, and then defining $\textbf{fix} x : \tau \text{ is } e$ to be the expression $\text{unroll}(\text{self} x \text{ is } \hat{e})$. Observe that

$$\begin{align*}
\textbf{fix} x : \tau \text{ is } e &= \text{unroll}(\text{self} x \text{ is } \hat{e}) \\
&\mapsto [\text{self} x \text{ is } \hat{e}/x]\hat{e} \\
&= [\text{self} x \text{ is } \hat{e}/x][\text{unroll}(x)/x]e \\
&= [\text{unroll}(\text{self} x \text{ is } \hat{e})/x]e \\
&= [\textbf{fix} x : \tau \text{ is } e/x]e.
\end{align*}$$

The penultimate line summarizes the result of the iterated substitution; it does not arise in the dynamics itself as a substitution of a non-value for a variable, which would be disallowed in the by-value case.
References