A Separation Logic for Concurrent Randomized Programs

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Abstract
We present a concurrent separation logic with support for probabilistic reasoning. As part of our logic, we extend the idea of coupling, which underlies recent work on probabilistic relational logics, to the setting of programs with both probabilistic and non-deterministic choice. To demonstrate our logic, we verify a variant of a recent randomized concurrent counter algorithm. All of our results have been mechanized in Coq.

Keywords separation logic, concurrency, probability

1 Introduction
Many concurrent algorithms are randomized. Randomization is useful in the concurrent setting because it reduces the need for coordination between threads. Roughly speaking, these algorithms are designed so that if each thread makes a local random choice, then on average the aggregate behavior of the whole system will have some good property.

However, concurrent randomized algorithms are difficult to write and reason about. Indeed, the use of just concurrency or randomness alone makes it hard to establish the correctness of an algorithm. For that reason, a number of program logics for reasoning about concurrent [15, 16, 18, 25, 28, 38, 40, 50] or randomized [6, 7, 29, 36, 42] programs have been developed.

But, to our knowledge, the only prior program logic designed for reasoning about programs that are both concurrent and randomized is the recent probabilistic rely-guarantee calculus developed by McIver et al. [33]. However, this work, which extends Jones’ original rely-guarantee logic [25] with probabilistic constructs, lacks many of the features of modern concurrency logics that have been developed for reasoning about fine-grained concurrent data structures. This is understandable, since the techniques that have been developed to support reasoning about these features are quite complex. Developing a logic with all of these features, in addition to support for probabilistic reasoning, is therefore difficult.

To make matters worse, each year there are new concurrency logics and probabilistic logics that extend prior work. Thus, even if one develops a logic for both concurrency and randomness, it seems as if before long it will be “obsolete”, lacking recent developments in logics targeted for just one of these two effects.

In spite of these challenges, we have developed a program logic with the expressive features of recent concurrency logics, in addition to support for probabilistic reasoning. Moreover, the metatheory of our logic is developed in a way that we believe avoids the obsolescence issue we just described. The key to our approach is several recent developments in probabilistic logics, concurrency logics, and denotational semantics. However, before we give an overview of how we build on this related work, let us describe an example of a concurrent randomized algorithm in order to illustrate some of the challenges involved in reasoning about them.

1.1 Case Study: Concurrent Approximate Counters
In many concurrent systems, threads need to keep counts of events. For example, in OS kernels, these counts can track performance statistics or reference counts. Somewhat surprisingly, Boyd-Wickizer et al. [10] have shown that maintaining such counts was a serious scalability bottleneck in a prior version of the Linux kernel. In many cases, however, there is no need for these counts to be exactly right: an estimate is good enough. Taking advantage of this, Dice et al. [13] created a scalable concurrent counter by adapting Morris’s [37] approximate counting algorithm.

In order to understand Dice et al.’s concurrent version it is helpful to understand Morris’s original work. Morris’s motivation was to be able to count up to \( n \) using fewer than \( O(\log_2(n)) \) bits. His idea was that, rather than storing the exact count \( n \), one could instead store something like \( \log_2(n) \) rounded to the nearest integer. This would require only \( O(\log_2 \log_2(n)) \) bits, at the cost of the error introduced by rounding.

Of course, if we round the stored count, then when we need to increment the counter, we do not know how to update the rounded value correctly. Instead, Morris developed a randomized increment routine: if the counter currently stores the value \( k \), then with probability \( \frac{1}{2} \) we update the stored value to \( k + 1 \) and otherwise leave it unchanged. The code for this increment function is shown in Figure 1a, written in an ML-like pseudo code, where \( \text{flip}(p) \) is a command returning True with probability \( p \) and False otherwise. The read function loads the current value \( k \) of the counter and returns \( 2^{k} - 1 \). Let \( C_n \) be the random variable giving the
value stored in the counter after \( n \) calls of the increment function. One can show that \( \mathbb{E}[2^{C_n} - 1] = n \). Thus, the expected value of the result returned by read is equal to the true number of increments, and so the counter is said to be an unbiased estimator. Flajolet \cite{flajolet1996hyperloglog} gave a very detailed analysis of this algorithm by showing that it is equivalent to a simple Markov chain, and proved that it indeed only requires \( O(\log \log (n)) \) bits with high probability.

Although this space-saving property is interesting, the key aspect of the algorithm that makes it useful for concurrent counting is that, as the stored count gets very large, the probability that an increment needs to write to memory to update the count gets smaller and smaller. A simplified version of the concurrent algorithm proposed by Dice et al. \cite{dice2016cas} is shown in Figure 1b. The increment procedure starts by generating a large number of random bits. It then calls a recursive helper function \( \text{incr\_aux} \) with the random bit vector \( b \) as an argument. This helper function reads the current value \( k \) stored in the counter and checks whether the first \( k \) bits of the bitvector are all 0 (performed in the code by the \( \text{lsb\_Zero(k, b)} \) function). If not, the increment is over. If they are all zero, then because this occurs with probability \( \frac{1}{2^k} \), the thread tries to atomically update the value stored in the counter from \( k \) to \( k + 1 \) using a compare-and-swap (CAS) operation. If this operation succeeds, it means that no other thread has intervened and modified the counter, and so the increment is finished. If the swap fails, some other thread has modified the counter, so the \( \text{incr\_aux} \) function is recursively called to try again. The read procedure is the same as for the sequential algorithm.

As the count gets larger, the probability that a thread will perform a CAS operation during the increment gets smaller, which is useful because these operations are slow. Dice et al. \cite{dice2016cas} show that this algorithm works quite well in practice, but do not give a formal argument for its correctness. Therefore, one might ask whether it is still guaranteed to give an unbiased estimate of the count. In fact, the answer is no: the scheduler can bias the count by ordering the compare-and-swap operations in a particular way.

In Figure 1c we present a new concurrent version that is statistically unbiased, yet retains the same good properties of low contention\(^1\). Our increment function reads the current value in the counter, then takes the minimum of that value and a parameter \( \text{MAX} \). If the minimum value is \( k \), then with probability \( \frac{1}{k+1} \), it uses a fetch-and-add operation (FAA) to atomically add \( k + 1 \) to the counter, otherwise it returns. In our version, the read function just returns the value in the counter. Like CAS instructions, FAA are expensive, so the reason the algorithm scales is that the probability that a FAA happens decreases as the counter value grows. The parameter \( \text{MAX} \) caps how small this probability gets, somewhat like generating only 64 random bits does in the beginning of Figure 1b.

How does one show that this algorithm is unbiased, as we have claimed? Informally, it is because in expectation, each increment adds 1 to the count, so the total expected value is equal to the number of increments. Moreover, because addition is associative and commutative, it does not matter if other threads modify the counter inbetween when a flip happens and the corresponding FAA occurs. However, it is challenging to make this argument formal. We might try to model the value of the counter as a family of Markov chains\(^2\), as Flajolet did for the sequential algorithm. But this is unwieldy because the relevant state of the chain is not just the current value stored in \( l \), but also the local state of each thread in the middle of an increment operation. Moreover, even if one could model the algorithm in this way, it is hard to

\(^1\)However, it requires \( O(\log(n)) \) bits to store the count. Nevertheless, it uses less space than other alternatives for decreasing contention (e.g., having each thread maintain its own local counter).

\(^2\)Or rather, a Markov decision process, which accounts for the nondeterminism of the ordering of operations.
justify the connection between the concrete implementation and this mathematical representation.

As we will see, the program logic we have developed makes this algorithm easy to verify.

1.2 Background from Recent Work

Our program logic is based on three ideas developed in recent work:

pRHL: probabilistic relational reasoning. In many program logics for reasoning about probabilistic programs, assertions in the logic either explicitly make statements about probabilities, or are interpreted as being true with some probability (e.g., [6, 29, 36, 42], among others). Although effective, this non-standard semantics of assertions is hard to reconcile with the semantic models used in concurrency logics.

However, Barthe et al. have shown that reasoning about probabilistic programs can often be done without explicitly reasoning about probabilities in the assertions of the logic. Using their pRHL logic [4, 7, 8], one establishes a refinement between two randomized programs using proof rules that encode a special type of simulation relation. The only time that explicit probabilities arise is a special rule for points in the simulation when both programs take a randomized step. The soundness theorem for their logic says that derivations in the logic imply the existence of a coupling, a construct from probability theory that is often used to relate two probability distributions. (We describe couplings in detail in §2.4).

Iris: a “layered” concurrency logic. As we have mentioned, recent concurrency logics are rather complex, maximal hard to adapt them to incorporate probabilistic reasoning. Recent work, however, has sought to unify and simplify these logics [15, 28, 38].

In particular, Iris [27, 28, 30], a recent higher order concurrent separation logic, is composed of two layers: a “base logic” and a derived “program logic” which is encoded on top of the base logic. Crucially, most of the difficult semantic constructions are developed in the base logic. We are able to encode probabilistic relational reasoning à la pRHL in Iris by only modifying the second layer. We therefore get all of the results developed in the base logic “for free”, and retain the expressive features of Iris.

Indexed valuations: a monadic encoding. Our logic, like pRHL, is designed for relational reasoning: it establishes a refinement between two programs. That means, if we want to prove a property about some program e, we first come up with some simpler specification program e′, use the logic to establish a refinement connecting properties of e′ to those of e, and then reason about e′.

Therefore, we need to complement our logic with a way to express the simpler program e′ and suitable tools for reasoning about it. We write these specification programs using the monad of indexed valuations developed by Varacca and Winskel [53], which makes it possible to combine operations for both probabilistic and non-deterministic choice effects. This monad has a clean equational theory that makes it possible to reason about probabilistic properties of programs expressed using it.

1.3 Our Contributions

We make several contributions:

- We develop results for reasoning about computations expressed in the monadic encoding of Varacca and Winskel. Although prior work had used this monad (and related ones) for denotational semantics [20, 21, 35, 47, 53] and to reason about small program equivalences [19], we found it necessary to develop new ways to reason about this monad. In particular, we develop an inequational theory of orderings between computations, and rewriting rules for bounding expected values. In addition, we adapt the notion of couplings to this setting.
- We extend Iris with support for probabilistic relational reasoning in the style of pRHL, which lets us establish refinements between concurrent programs and these monadic representations.
- Using our logic, we prove that the concurrent approximate counter algorithm introduced in §1.1 is unbiased. Although approximate counting algorithms have been used in a number of concurrent and distributed applications [13, 26, 48], we are not aware of any prior correctness proofs that take concurrency into account.

All of the results in this paper, including the soundness of our logic and the case study, have been mechanized in Coq.

We start by describing Varacca and Winskel’s [53] monad for probabilistic and non-deterministic choice, and our results for reasoning about computations expressed in it (§2). We then describe our program logic (§3). In §4 we apply our logic to the case study of approximate counters. Finally, we conclude (§5) by discussing related work and possible extensions.

2 Monadic Representation

A common approach to reasoning about effectful programs in a dependently typed proof assistant is to model effects using a suitable monad, M. Using this monad, one represents an effectful program that returns a value of type T as a term of type M(T). Next, one usually proves a series of equational rules for simplifying terms of type M(T), and other lemmas for reasoning about such terms. This approach has been used for reasoning about a number of effects, including state [39, 44], non-termination [12], non-determinism [19], and probabilistic choice [2, 19, 41, 51].
2.1 Monads for non-determinism or probability.

Let us recall common monadic encodings for non-deterministic and probabilistic choice (separately). For non-determinism, we can choose \( M_N(T) = \{ I : \text{List}(T) \mid I \neq \emptyset \} \). That is, such computations are represented as pure terms that return a non-empty list of results, where each element of the list represents one of the different non-deterministic outcomes. We say two terms \( A \) and \( B \) of type \( M_N(T) \) are equivalent, written \( A \equiv B \), if their sets of elements are the same: for all \( x, x' \in A \leftrightarrow x \in B \). In addition to the standard monadic operations (bind and return) we can define an operation for non-deterministic choice between two operations:

\[
A \cup B \triangleq \text{append}(A, B)
\]

That is, we simply append the list of results for computation \( B \) to those of \( A \). This operation satisfies a number of natural rules:

\[
A \cup B \equiv B \cup A \quad A \cup (B \cup C) \equiv (A \cup B) \cup C \quad A \cup A \equiv A
\]

These, in combination with the usual monad laws, can be used to prove that one non-deterministic computation is equivalent to another.

We can represent a probabilistic computation of type \( T \) as a list of pairs of values of type \( T \) along with a probability that they occur, subject to the constraint that the sum of the probabilities is equal to 1:

\[
M_P(T) \triangleq \left\{ I : \text{List}(T \times \mathbb{R}) \mid \sum_{(x, p) \in I} p = 1 \right\}
\]

Given a value \( v \) of type \( T \) and a probabilistic computation \( A \) of type \( M_P(T) \), we write \( A|v \) for the sublist of \( A \) containing only elements whose first component is equal to \( v \). The probability that a computation \( A \) returns \( v \), written \( A(v) \), is then:

\[
A(v) \triangleq \sum_{(x, p) \in A|v} p
\]

We say \( A \equiv B \) if for all \( x \), \( A(x) = B(x) \).

Given such a computation \( A \) and a real number \( p \), we write \( p \cdot A \) for the list in which we multiply the second component of each element of \( A \) by \( p \). Using this, we can define an operation which selects between a computation \( A \) with probability \( p \) and another computation \( B \) with probability \( 1 - p \):

\[
A \oplus_p B \triangleq \text{append}(p \cdot A, (1 - p) \cdot B)
\]

This operation satisfies equational rules such as:

\[
A \oplus_p B \equiv B \oplus_{1-p} A \quad A \oplus_p A \equiv A
\]

2.2 Combining effects

In order to reason about programs that use both probability and non-determinism, we would like some way to combine the monads we have just described. A number of generic ways of combining two monads have been developed [23], but these approaches do not always adequately represent the intended semantics of programs that use both effects. In our case, we want to use non-deterministic choice to represent the different outcomes arising from the ordering of concurrent operations. We might try to represent computations of type \( T \) combining both effects as terms of type \( M_N(M_P(T)) \), i.e., non-empty lists of probability distributions.

However, Varacca and Winskel [53] have shown that it is not possible to define the monad operations for such a representation in a way that has the correct equational rules. For our purposes, it is not necessary to understand the impossibility proof. What is important is that they also describe an alternative way of representing probabilistic choice, which they call the indexed valuation monad, and they then show that non-empty lists of these indexed valuations do give a suitable monad which combines both effects.

An indexed valuation of type \( T \) is a tuple \( (I, \text{ind}, \text{val}) \), where \( I \) is a finite set whose elements are called indices; \( \text{ind} \) is a function of type \( I \rightarrow T \), and \( \text{val} \) is a function of type \( I \rightarrow \mathbb{R} \) such that:

\[
\sum_{i \in I} \text{val}(i) = 1
\]

Informally, we can think of the indices as a set of “codes” or identifiers, the \( \text{val} \) function gives the probability of a particular index occurring, and \( \text{ind} \) maps these codes to values of type \( T \). Importantly, the \( \text{val} \) function is not required to be injective, so that different codes can lead to the same observable value. We write \( M_I(T) \) for the type of indexed valuations of type \( T \). The support of the valuation, notated \( \text{support}(\text{val}) \), is the set of indices \( i \) for which \( \text{val}(i) > 0 \). We say \( (I_1, \text{ind}_{I_1}, \text{val}_{I_1}) \equiv (I_2, \text{ind}_{I_2}, \text{val}_{I_2}) \) if there exists a bijection \( h : \text{support}(\text{val}_{I_1}) \rightarrow \text{support}(\text{val}_{I_2}) \) such that for all \( i \in \text{support}(\text{val}_{I_1}) \), \( \text{val}_{I_1}(i) = \text{val}_{I_2}(h(i)) \) and \( \text{ind}_{I_1}(i) = \text{ind}_{I_2}(h(i)) \). That is, the bijection can only “relabel” indices in a way that preserves their probabilities and what they decode to. Note that there is a map \( H \) which takes indexed valuations of type \( T \) to elements of \( M_P(T) \), defined by:

\[
H(I, \text{ind}, \text{val}) = [(\text{ind}(i_1), \text{val}(i_1)), \ldots, (\text{ind}(i_n), \text{val}(i_n))]
\]

where \( I = \{i_1, \ldots, i_n\} \)

It is clear that if \( I_1 \equiv I_2 \), then \( H(I_1) \equiv H(I_2) \). However, the converse is not true because \( I_1 \) and \( I_2 \) could have supports with different cardinalities.

The probabilistic choice between two indexed valuations is defined by:

\[
(i_1, \text{ind}_{I_1}, \text{val}_{I_1}) \oplus_p (i_2, \text{ind}_{I_2}, \text{val}_{I_2}) \triangleq (I_1 + I_2, \text{val}', \text{ind}')
\]

\[\text{In fact, Varacca and Winskel first define a more general structure in which the sums of val(i) do not have to equal 1, and the set of indices does not have to be finite. After working out the theory of these more general objects, they show that one can restrict to the subcategory satisfying these additional constraints.}\]
where:

\[
\text{ind}'(i) = \begin{cases} 
\text{ind}_1(i') & \text{if } i = \text{inl}(i') \\
\text{ind}_2(i') & \text{if } i = \text{inr}(i') 
\end{cases}
\]

and

\[
\text{val}'(i) = \begin{cases} 
p \cdot \text{val}_1(i') & \text{if } i = \text{inl}(i') \\
(1 - p) \cdot \text{val}_2(i') & \text{if } i = \text{inr}(i') 
\end{cases}
\]

One can show that for all indexed valuations \(I_1\) and \(I_2\) and \(0 \leq p \leq 1\), we have \(I_1 \oplus p I_2 \equiv I_2 \oplus (1 - p) I_1\). However, unlike the original probabilistic choice monad we described before, \(I \oplus p I \neq I\), unless \(p = 0\) or \(p = 1\). The reason is that, when \(p\) is neither 0 nor 1, the support of \(I \oplus p I\) will have a larger cardinality than the support of \(I\), so there can be no bijection between them.

At first this lack of equivalence seems odd: if in either case we choose the same \(I\), then the probabilistic choice was irrelevant. However, when we later add in the effect of non-determinism, the distinction between \(I \oplus p I\) and \(I\) becomes more justifiable, since it allows us to account for the fact that subsequent non-determinism in the computation can be resolved differently on the basis of this seemingly irrelevant probabilistic choice. In fact, Varacca and Winskel show that the original equational law \(A \oplus p A \equiv A\) in the probability distribution monad is precisely what prevents the \(M_N \circ M_1\) monad combination from working properly.

With this problematic equation not present for indexed valuations, it is possible to define the monad operations on \(M_N \circ M_1\). Given \(I_1\) and \(I_2\) of type \(M_N(M_1(T))\), the probabilistic choice operation \(I_1 \oplus p I_2\) is defined by taking the pairwise probabilistic choice of each indexed valuation in the respective lists:

\[
I_1 \oplus p I_2 \equiv \{[I_1 \oplus p I_2] \mid I_1 \in I_1, I_2 \in I_2\}
\]

while the non-deterministic choice is as before:

\[
I_1 \cup I_2 = \text{append}(I_1, I_2)
\]

We say \(I_1 \equiv I_2\) if for each \(I_1 \in I_1\), there exists some \(I_2 \in I_2\) such that \(I_1 \equiv I_2\), and vice versa. The full definition of the bind operation is somewhat involved, as are the proofs of the monad laws, so we refer to Varacca and Winskel [53]. What is important is the equational properties that hold, of which a selection are shown in Figure 2 (the standard monad laws are omitted).

In Figure 3 we show how to model the approximate counter code from Figure 1c using this monad. The \text{approxIncr} computation first non-deterministically selects a number \(k\) up to \(\text{MAX}\) – this models the process of taking the minimum of the value in \(l\) and \(\text{MAX}\) in the code. The non-determinism accounts for the fact that the value that will be read depends on what other threads do. The monadic encoding then makes a probabilistic choice, returning \(k + 1\) with probability \(\frac{1}{\text{MAX}}\) and 0 otherwise, which represents the probabilistic choice that the code will make about whether to do the fetch-and-add.

Finally, the process of repeatedly incrementing the counter \(n\) times is modeled by \text{approxN}. The first argument \(n\) tracks the number of pending increments to perform, while the second argument \(l\) accumulates the sum of the values returned by the calls to \text{approxIncr}.

Note that this model does not try to represent multiple threads in the middle of an increment each waiting to add its value to the shared count – rather, it is as if the actual calls to \text{incr} all happened atomically in sequential order, with the effects of concurrency captured by the non-determinism in the \text{approxIncr} computation.

Of course, we need to show that this model accurately captures the behavior of the code from Figure 1c – this is what the program logic we describe in §3 will do. First, however, we need to describe the new results we have developed to reason further about the monadic encoding itself.

### 2.3 Reasoning about Quantitative Properties

With what we have described so far, we can express computations with randomness and non-determinism and derive equivalences between them, but we do not yet have a way to talk about the standard concerns of probability theory (e.g., expected values, variances, tail bounds).

Given an indexed valuation \(I = (l, \text{ind}, \text{val})\) of type \(T\) and a function \(f : T \to \mathbb{R}\), we can define the expected value of \(f\) on \(I\) as:

\[
\mathbb{E}_f[I] \triangleq \sum_{i \in I} f(\text{ind}(i)) \cdot \text{val}(i)
\]

(this coincides with the usual notion of expected value of a random variable if we interpret the indexed valuation as...
a distribution using the map \( H \) defined above). If \( I_1 \equiv I_2 \), then \( E_f[I_1] = E_f[I_2] \). Moreover, given a value \( t \) of type \( T \), if we take \( f \) to be the indicator function that returns 1 if its input is equal to \( t \) and 0 otherwise, then \( E_f[I] \) is equal to the probability that \( I \) yields the value \( t \).

Since an \( I \) of type \( M_b(M_l(T)) \) is just a list of indexed valuations, we can apply \( E_f[-] \) to each \( I \in I \) to get the list of expected values that can arise depending on how nondeterministic choices are resolved. Generally speaking, we will be interested in bounding the smallest or largest possible value that an expected value can take. We can define the minimal and maximal expected value of \( f \) on \( I \) as:

\[
E_{\text{min}}^f[I] \triangleq \min_{i \in I} E_f[I_i] \quad E_{\text{max}}^f[I] \triangleq \max_{i \in I} E_f[I_i]
\]

Rules for calculating these extrema are given in Figure 4.

To help reason about these extrema, we introduce a partial order on terms of type \( M_b(M_l(T)) \): We say \( I_1 \subseteq I_2 \) if for each \( I_1 \in I_1 \), there exists some \( I_2 \in I_2 \) such that \( I_1 \equiv I_2 \). If \( I_1 \subseteq I_2 \) then \( E_{\text{min}}^f[I_1] \leq E_{\text{max}}^f[I_2] \) and \( E_{\text{min}}^f[I_2] \leq E_{\text{max}}^f[I_1] \). Thus, we can bound \( I_1 \)'s extrema by first finding some \( I_2 \) such that \( I_1 \subseteq I_2 \), and then bounding the latter’s extrema.

Using these rules, we can show that \( E_{\text{id}}[\text{approxN n0}] = E_{\text{id}}[\text{approxN n0}] = n \), which implies that no matter how the non-determinism is resolved in our model of the counter, the expected value of the result will be the number of increments. Let us just consider the case for the minimum, since the maximum is the same. The proof proceeds by induction on \( n \), after first strengthening the induction hypothesis to the claim that \( E_{\text{id}}[\text{approxN n l}] = n + l \). The key step of the proof is to show that \( E_{\text{id}}[\text{approxNcr}] = 1 \), i.e., each increment contributes 1 to the expected value. From the last rule in Figure 4, it suffices to show that whatever value of \( k \) is non-deterministically selected, the resulting expected value will be 1. We have that for all \( k \):

\[
E_{\text{id}}[\text{ret} (k + 1) \oplus \text{ret} 0] = \frac{1}{k + 1} \cdot (k + 1) + \left(1 - \frac{1}{k + 1}\right) \cdot 0 = 1
\]

### 2.4 Nondeterministic Couplings

Up to this point, we have only considered two kinds of relations on elements of \( M_b(M_l(T)) \), \( = \) and the \( \subseteq \) ordering. These relations are too fine to apply in many cases.

In contrast, in classical probability theory, there is a more flexible relation called a coupling [32] between two probability distributions. Recent work by Barthe et al. [3, 4, 8] has shown that this notion is fundamental for relational reasoning in probabilistic program logics. Given two distributions \( A : M_F(T_A) \) and \( B : M_F(T_B) \), a coupling between \( A \) and \( B \) is a distribution \( C : M_F(T_A \times T_B) \) such that:

1. \( \forall x : T_A, A(x) = \sum_y C(x, y) \)
2. \( \forall y : T_B, B(y) = \sum_x C(x, y) \)

That is, \( C \) is a joint distribution whose marginals equal \( A \) and \( B \). These two conditions are equivalent to requiring that:

1. \( A \equiv ((x, y) \leftarrow C ; \text{ret} x) \)
2. \( B \equiv ((x, y) \leftarrow C ; \text{ret} y) \)

Given a predicate \( P : A \times B \rightarrow \text{Prop} \), we say that \( C \) is a \( P \)-coupling, if, in addition to the above, we have:

\( \forall x, y, C(x, y) > 0 \rightarrow P(x, y) \)

i.e., all pairs \( (x, y) \) in the support of the distribution \( C \) satisfy \( P \). The existence of a \( P \)-coupling can tell us important things about the two distributions. For example, if \( P(x, y) = (x = y) \), then the existence of a \( P \)-coupling tells us the two distributions are equivalent. Moreover, there are rules for systematically constructing couplings between distributions. We will explain some of these rules once we have described how to adapt couplings to the monad \( M_b \circ M_l \).

First, using the monadic formulation of the coupling conditions, it is straightforward to define an analogous idea for \( M_l \): Given \( I_1 : M_l(T_A) \) and \( I_2 : M_l(T_B) \), a coupling between \( I_1 \) and \( I_2 \) is an \( I : M_l(T_A \times T_B) \) such that:

1. \( I_1 \equiv ((x, y) \leftarrow I ; \text{ret} x) \)
2. \( I_2 \equiv ((x, y) \leftarrow I ; \text{ret} y) \)

and \( I = (I, \text{ind}, \text{val}) \) is a \( P \)-coupling if for all \( i \) such that \( \text{val}(i) > 0, P(\text{ind}(i)) \) holds. As before, if \( P \) is the equality predicate, then the existence of a \( P \)-coupling between \( I_1 \) and \( I_2 \) implies \( I_1 \equiv I_2 \).

We can lift this to a relation between a single indexed valuation \( I \) and a list of indexed valuations \( J \): We say\(^4\) there is a non-deterministic \( P \)-coupling between \( I \) and \( J \) if there is a relation between a single indexed valuation \( I \) and a list of indexed valuations \( J \): We say\(^4\) there is a non-deterministic \( P \)-coupling between \( I \) and \( J \) if there

\(^4\) Barthe et al. [3] use “non-deterministic coupling” to refer to a particular kind of coupling which is unrelated to adversarial non-deterministic choice.
We now describe the program logic we have developed for proving that a program is modeled by the monadic specifications from the previous section.

**Figure 5.** Rules for constructing non-deterministic couplings.

exists some \( I' \in I \) and a \( P \)-coupling between \( I \) and \( I' \). We write \( \parallel I \sim I : P \parallel \) to denote the existence of such a coupling.

In this case, if \( P \) is the equality relation, then this means \( I \) is equivalent to some \( I' \in I \) in \( I \), hence to bound the range of an expected value \( E[I]\), it suffices to bound the extrema \( E_I^{\min}[I] \) and \( E_I^{\max}[I] \). More generally, we have the following:

**Theorem 2.1.** For all \( f \) and \( g \), if \( P(x, y) = (f(x) = g(y)) \), then \( \parallel I \sim I : P \parallel \) implies:

\[
E_I^{\min}[I] \leq E[I] \leq E_I^{\max}[I]
\]

Rules for constructing these couplings are shown in Figure 5. If we interpret the \( P \) in \( \parallel I \sim I : P \parallel \) as a kind of “post-condition” for the execution of the computations \( I \) and \( I' \), then these coupling rules have the structure of a Hoare-like relational logic [9], as in the work of Barthe et al. [3]: e.g., the rule Bind is analogous to the usual sequencing rule in Hoare logic.

The rule P-Choice lets us couple probabilistic choices \( \parallel I \sim I : P \parallel \) and \( \parallel I' \sim I' : P \parallel \) with post-condition \( P \) by coupling \( I \) to \( I \) and \( I' \) to \( I' \). This is somewhat surprising: we get to reason about these two probabilistic choices as if they both chose the left alternative or both chose the right alternative, rather than considering the full set of four combinations. This counter-intuitive rule is quite useful, as demonstrated in the many examples given in the work of Barthe et al. We will see an example of its use in §4.

**3 Program Logic**

We now describe the program logic we have developed for proving that a program is modeled by the monadic specifications from the previous section.

### Syntax:

- **Val:**
  \[ v ::= \lambda x. e_1 \mid (v_1, v_2) \mid () \mid n \mid b \mid \ldots \]

- **Expr:**
  \[ e ::= x \mid v \mid e_1 + e_2 \mid \text{fork}(e) \mid \text{flip}(e_1, e_2) \mid \ldots \]

- **Eval Ctx:**
  \[ K ::= [] \mid [K e] \mid V K \mid \text{flip}(K, e) \mid \text{flip}(v, K) \mid \ldots \]

- **State:**
  \[ \sigma \in \mathbb{N} \rightarrow \text{Val} \]

- **Config:**
  \[ \rho \in \{ [], [\text{List} \text{Expr}] \mid I \neq \emptyset \} \times \text{State} \]

- **Trace:**
  \[ \varphi \in \text{Trace} \rightarrow \text{Option} \mathbb{N} \]

**Per-Thread Reduction:**

- **Flip-True:**
  \[
  \begin{array}{c}
  0 \leq \frac{n_1}{n_2} \leq 1 \\
  \text{flip}(n_1, n_2); \sigma \rightarrow \text{True}; \sigma
  \end{array}
  \]

- **Flip-False:**
  \[
  \begin{array}{c}
  0 \leq \frac{n_1}{n_2} \leq 1 \\
  \text{flip}(n_1, n_2); \sigma \rightarrow \text{False}; \sigma
  \end{array}
  \]

(Standard rules omitted.)

**Concurrent Semantics:**

- **P-Choice:**
  \[ e_i; \sigma \rightarrow e'_i; \sigma' \]

- **Concurrent Semantics:**
  \[
  \begin{array}{c}
  [\ldots, K[e_1], \ldots]; \sigma \rightarrow [\ldots, K[e'_1], \ldots]; \sigma' \\
  [e_1, \ldots, e_{i-1}, K[\text{fork}(e_i)], \ldots]; \sigma \rightarrow \frac{1}{t} [e_1, \ldots, e_{i-1}, K(\{\}), \ldots, e_i]; \sigma
  \end{array}
  \]

**Trace Semantics:**

- **T → T’:**
  \[
  \begin{array}{c}
  \varphi(T, \rho) = \text{Some}(i) \\
  \rho \rightarrow P_{\varphi} \rho' \\
  \varphi(T, \rho) = \text{None} \\
  \rho \rightarrow 1_{\varphi} \rho
  \end{array}
  \]

**Figure 6.** Syntax and semantics of concurrent language.

### 3.1 Program Semantics

Our logic is parameterized by a generic probabilistic concurrent language. However, for concreteness, we instantiate it with the ML-like language used in the examples from §1. Figure 6 gives the syntax and semantics of this language. We omit the standard rules for things like tuples, recursive functions, and closures. The per-thread reduction relation \( e; \sigma \rightarrow e'; \sigma' \) is annotated with a probability \( p \) that the transition takes place.

The \( \text{flip}(n_1, n_2) \) command takes two integers as arguments and simulates a biased coin flip: it transitions to True with
probability \( \frac{p}{n} \) and False with probability \( 1 - \frac{p}{n} \). (In the introduction we somewhat informally wrote \( \text{flip}(n_1/n_2) \) as if the language had rational numbers as a primitive). There is a side condition to ensure that \( \frac{p}{n} \) actually corresponds to a valid probability. Other than this command, the per-thread transition system for this language is deterministic. 

The generic framework for our logic allows us to extend this language with other probabilistic commands, so long as they only sample from finite probability distributions. (We restrict to finite distributions so that we can connect the semantics of this language to indexed valuations, which are finite.) We say \( e \) is atomic, written \( \text{atomic}(e) \) if \( e \) reduces to a value in a single step.

This per-thread reduction relation is then lifted to a concurrent transition system. A configuration \( \rho \) is a pair consisting of a list of expressions (representing a pool of threads) and a state \( \sigma \). We say \( \rho \xrightarrow{i}{\varphi} \rho' \) when the \( i \)th thread of \( \rho \) transitions with probability \( p \) leading to a new configuration \( \rho' \). The \( \text{fork}\{e_1 \} \) command adds a new thread \( e_1 \) to the pool.

A scheduler decides which thread will get to step at each point in an execution. We model a scheduler as a function \( \varphi \) of type \( \text{Trace} \rightarrow \text{Option} \ \mathbb{N} \), where a trace is a non-empty list of configurations representing a partial execution. The scheduler is permitted to inspect the entire history and complete state of the program when deciding which thread gets to go next. Of course, a real implementation of a scheduler does not actually do this, but by conservatively considering this strong class of adversarial schedulers, results we prove will also hold for realistic schedulers. We write \( T \xrightarrow{\varphi} T' \) to indicate that the thread selected by \( \varphi(T) \) steps with probability \( p \) to a new configuration which is appended to \( T \) to obtain \( T' \). We write \( \text{curr}(T) \) for the last configuration in a trace. We permit a scheduler to return None, and if this happens, or if the scheduler returns a thread number which cannot take a step, the system takes a "stutter" step and the current configuration is repeated again at the end of the trace. We say \( \varphi \) is well-formed if, whenever \( \varphi(T) = \text{Some}(i) \), then \( \text{curr}(T) \) is of the form \( \{e_1, \ldots, e_n, \sigma\} \), such that \( e_i, \sigma \) can take a step. That is, a well-formed scheduler always selects threads that can take a step. We say \( T \) reduces to \( T' \) in \( n \) steps under \( \varphi \) if: \( T \xrightarrow{\varphi} p_1 \ldots \xrightarrow{\varphi} T' \), for some \( p_1, \ldots, p_n \) where each \( p_i > 0 \). A configuration \( \rho \) has terminated if the first thread in the pool is a value. We say that \( T \) is terminating in at most \( n \) steps under \( \varphi \), if for all \( T' \) which \( T \) reduces to under \( \varphi \) for \( n' \geq n \) steps, \( \text{curr}(T') \) has terminated.

Given a scheduler \( \varphi \) and a trace \( T \) we can interpret the trace step relation as an indexed valuation: Since the set of traces \( T' \) which \( T \) can step to under \( \varphi \) is finite, we can take the set of indices \( I \) to be any set in bijection with this set of traces. Take \( \text{ind} \) to be this bijection, and set \( \text{val}(i) \) equal to the probability \( p \) such that \( T \xrightarrow{\varphi} \text{ind}(i) \). We refer to the resulting indexed valuation \( (I, \text{ind}, \text{val}) \) as \( \text{tstep}_\varphi(T) \). For each \( n \), we define the indexed valuation \( \text{resStep}_\varphi^n(T) \) recursively by:

\[
\text{resStep}_\varphi^0(T) \triangleq \text{match} \ \text{curr}(T) \text{ with } \{e_1, \ldots, \sigma\} \Rightarrow \text{ret} \ e_1 \\
\text{end}
\]

\[
\text{resStep}_\varphi^{n+1}(T) \triangleq T' \leftarrow \text{tstep}_\varphi(T); \text{resStep}_\varphi^n(T')
\]

This corresponds to stepping the trace \( n \) times and returning the first thread from the final configuration of the resulting trace. We regard the "return value" of a concurrent program to be the value that the first thread evaluates to, so in the event that the program terminates in \( n \) steps under the scheduler, \( \text{resStep}_\varphi^n(T) \) gives this return value.

### 3.2 Background on Iris

As we have mentioned, our program logic is an extension of Iris, a recent concurrency logic with many expressive features. For reasons of space, we cannot explain all of Iris. We refer the reader to the Iris papers and manual [27, 28, 30, 46] for a full account. Instead, we will just describe some essential aspects needed to understand our extensions and our case study on approximate counters.

Figure 7 shows the basic concurrent separation logic rules of Iris. (Treat the \( \Rightarrow \) connective as just a kind of implication for now, we explain its use below.) These rules are used to establish triples of the form:

\[
\{P\} \ e \ {x. Q}
\]

which imply that if \( e \) is executed in a state that initially satisfies \( P \), and it terminates with value \( v \), then the terminating state will satisfy \( \{v/x\}Q \). Furthermore, at no point will \( e \) go wrong and reach a stuck state, and neither will any of the threads forked by \( e \) during its execution. This is a partial correctness property: the post-condition only holds under executions where \( e \) terminates.

Recall that the fundamental idea of separation logic [43] is the introduction of the separating conjunction \( P \ast Q \), which says that the program heap can be split into two disjoint pieces satisfying \( P \) and \( Q \) respectively. Thus, assertions are interpreted as claims of ownership of resources, where a "resource" is just a fragment of the heap. In particular, \( I \mapsto v \) asserts ownership of a part of the heap containing the location \( I \), and moreover says that \( I \) maps to the value \( v \). Ownership of this resource licenses a thread to access and modify this location (see \textsc{ml-load} and \textsc{ml-store} in Figure 7).

Like other recent concurrency logics [1, 15, 38], Iris lets users of the logic extend the notion of resource beyond just heap fragments. These user-defined resources let us model the complex protocols that govern how threads access shared state in a concurrent system. Instead of describing the machinery that makes this work, let us give an example of one of
A Separation Logic for Concurrent Randomized Programs

**ML-ALLOC**
\[
\{ \text{True} \} \text{ ref } v \{ x, x \leftarrow v \}
\]

**ML-LOAD**
\[
\{ l \leftarrow v \} \! l \{ x, x = v \land l \leftarrow v \}
\]

**ML-STORE**
\[
\{ l \leftarrow v \} \ l := w \{ l \leftarrow w \}
\]

**ML-FAA**
\[
\{ l \leftarrow n \} \text{ FAA}(l, k) \{ x, x = n \land l \leftarrow n + k \}
\]

**ML-FORK**
\[
P \Rightarrow Q_0 \star Q_1
\]
\[
\{ Q_0 \} e \{ \text{True} \} \quad \{ Q_1 \} e' \{ R \}
\]
\[
P \text{ fork}(e); e' \{ R \}
\]

**HT-FRAME**
\[
P \Rightarrow Q
\]
\[
\{ P \} e \{ v. Q \}
\]

**HT-CSQ**
\[
P \Rightarrow P'
\]
\[
\{ P' \} e \{ v. Q' \} \quad \forall v. Q' \Rightarrow Q
\]
\[
P \Rightarrow \{ P \} e \{ v. Q \}
\]

**Figure 7.** Selection of rules from Iris.

**COUNTGEQ**
\[
\frac{\wedge (q, n)^{\psi} + \wedge (q, n)^{\psi} \Rightarrow n \geq n'}{\wedge (q, n)^{\psi} + \wedge (q, n)^{\psi} \Rightarrow n = n'}
\]

**COUNTEQ**
\[
\frac{\wedge (1, n)^{\psi} \Rightarrow \wedge (q, n)^{\psi} \Rightarrow n \geq n'}{\wedge (q, n)^{\psi} \Rightarrow n = n'}
\]

**COUNTPERM**
\[
\frac{\wedge (q, n)^{\psi} + \wedge (q', n)^{\psi} \Rightarrow q + q' \leq 1}{\wedge (q, n)^{\psi} + \wedge (q', n)^{\psi} \Rightarrow \wedge (q + q', n)^{\psi}}
\]

**COUNTSEP**
\[
\frac{\wedge (q, n)^{\psi} + \wedge (q', n)^{\psi} \Rightarrow \wedge (q + q', n)^{\psi}}{\wedge (q, n)^{\psi} + \wedge (q', n)^{\psi} \Rightarrow \wedge (q + q', n)^{\psi}}
\]

**COUNTALLOC**
\[
\frac{\text{True} \Rightarrow \exists i, \wedge (q, n)^{\psi} + \wedge (1, n)^{\psi}}{}
\]

**COUNTUPD**
\[
\frac{\wedge (q, n)^{\psi} + \wedge (q, n)^{\psi} \Rightarrow \wedge (q + q', n)^{\psi} + \wedge (q', n)^{\psi}}{\wedge (q, n)^{\psi} + \wedge (q, n)^{\psi} \Rightarrow \wedge (q + q', n)^{\psi} + \wedge (q', n)^{\psi}}
\]

**Figure 8.** Counter resource rules.

These user-defined resources: the abstract “counter” resource. There are two types of these counter resources, represented by the following assertions:

\[
\wedge (n)^{\psi} \quad \text{and} \quad \wedge (q, n)^{\psi}
\]

where \( n \) and \( n' \) are natural numbers, \( 0 < q \leq 1 \) is a rational number, and \( y \) is an abstract name assigned to a particular counter. The \( \bullet n \) resource represents a shared counter that contains the value \( n \). If we think of such a counter as being composed of \( n \) “units”, then the resource \( \circ (q, n') \) represents a “stake” or ownership of \( n' \) of the units in the global counter. The parameter \( q \) is a fractional permission [11] that lets us track how many threads have such a stake; when \( q = 1 \), this represents full ownership, so no other threads have a stake5.

Rules for using these assertions are given in Figure 8. The rules COUNTGEQ and COUNTEQ let us conclude that the global counter value must be at least as big as any stake’s value; and when a stake’s \( q \) value is 1, we furthermore know that the counter and the stake value are the same. The rule COUNTSEP lets us join (or conversely, split) two stakes by summing their permissions and their counter values, subject to the (implicit) constraint that \( q, q' \), and \( q + q' \) all lie in the interval \((0, 1]\).

The COUNTALLOC rule lets us create a new counter with some existentially quantified name; the \( \Rightarrow \) connective here is a kind of implication in Iris which lets one modify or create a resource. Finally, COUNTUPD lets us modify a counter: if we own the global value and a stake, we can update the value and the stake, so long as we preserve the part of the counter value owned by other stakes (represented by \( k \) in the rule).

Of course, we need some way to connect these “abstract” resources to the actual state of the program. The mechanism for doing this is an invariant. An invariant is an assertion that is dynamically established at some point in the program, and then is guaranteed to hold thereafter. We write \( [P] \) for the assertion which says that the invariant \( P \) has been established with the abstract name \( i \). If we have ownership of resources satisfying \( P \), we can use the rule INV-ALLOC from Figure 9 to establish \( P \) as an invariant; we lose the resources and get back \( [P] \) with some fresh name \( i \). If we know the invariant \( [P] \) holds and we are trying to prove some Hoare triple about an expression \( e \), we can use INV-OPEN to “open” the invariant. This lets us add \( P \) to the pre-condition of the triple we are trying to prove, but we need to re-establish and give up \( P \) in the post-condition in order to “close” the invariant. Moreover, to use this rule, \( e \) must be atomic: Since \( e \) will reduce to a value in a single-step, this ensures there is no intermediate step in which the invariant did not hold.

(In the statement of INV-OPEN we have omitted certain side conditions that are used to ensure that the same invariant is not opened multiple times simultaneously.)

For example, we can create the invariant \( [3n. l \mapsto n + \bullet n] \) to ensure the physical heap location \( l \) will always store the

\[5\text{Note that the } q \text{ is not the fraction of the global counter value represented by the stake’s value.}\]
value represented by the counter resource. Now a thread that owns a stake \( \frac{\gamma}{2} \) can read from \( I \) or modify it using a compare-and-swap by opening the invariant and updating the global counter resource suitably with \texttt{CountUp}.

This resource and invariant pattern were used to verify a (non-approximate) concurrent counter in the Iris Coq development. As we shall see, we were able to use this same resource to verify the approximate counter with our extensions.

### 3.3 Probabilistic Rules

In order to extend Iris to do probabilistic relational reasoning, we start by adding a new assertion \( \text{Theorem 3.1.} \)

Let \( M \) be a finite monoid.

The following soundness theorem for the logic will guarantee a coupling between a random choice between True and False, \((weighted by n)\) and used in other separation logics \([31, 45]\).

In this case, in the post-condition we merely know that the concrete program executes a probabilistic choice yet we do not exhibit this. This rule gives us a way to relate the execution of a concrete expression \( e \) to an execution of \( I \).

Rule \( \texttt{Ht-Couple} \) lets us handle a case where the concrete program executes a probabilistic choice yet we do not want to relate this to a step in the probabilistic specification. In this case, in the post-condition we merely know that the return value was True or False.

Finally, \( \texttt{ProbEquiv} \) lets us replace our \( I \) resource with an equivalent \( I' \). We use this to manipulate the \( I \) into a form that matches the precondition required by something like \( \texttt{Ht-Couple} \).

Because \( \text{Prob}(I) \) is just an assertion like any other, we can control access to it between threads by storing it in an invariant. This idea of representing a specification computation as a resource assertion is based on work by Turon et al. \([49]\) and used in other separation logics \([31, 45]\).

### 3.4 Soundness

The following soundness theorem for the logic will guarantee that if we prove an appropriate triple involving \( \text{Prob}(I) \), the expected value of the concrete program will lie in the range of the extrema of \( I \):

**Theorem 3.1.** Let \( I : M_N(M_T(T)) \) for some type \( T \), and let \( f : \text{Val} \rightarrow \mathbb{R}, g : T \rightarrow \mathbb{R} \). Suppose

\[
\{ \text{Prob}(I) \} e \{ \exists v'. \text{Prob}(r = v') \land f(v) = g(v') \}
\]

holds. Let \( \phi \) be a well-formed scheduler such that \( ([e], \sigma) \) terminates in at most \( n \) steps under \( \phi \). Then:

\[
\mathbb{E}^\text{min}_{[I]} \leq \mathbb{E}_{f}^\text{resStep}_{\phi}([e], \sigma) \leq \mathbb{E}^\text{max}_{[I]}
\]

Let us comment on a few aspects of this result. First, it only holds for schedulers under which the program is guaranteed to terminate in some number of steps; this is not that surprising, since the original Iris is a partial correctness logic. Second, it only holds for well-formed schedulers – those that only select threads which can in fact take steps – however, since the normal soundness theorem for Iris implies that \( e \) will not get stuck, and neither will any other threads it creates, the well-formedness requirement just means that \( \phi \) will not try to step a non-existent thread id or a thread that has already terminated in a value.

To prove this soundness theorem and validate the rules we have given, we first change the definition of the Hoare triple in Iris so that if the probabilistic resource is of the form \( \text{Prob}(x \leftarrow I : F(x)) \) and the expression \( e \) takes a step, we can exhibit a coupling between \( e \)'s transition (interpreted as an indexed valuation) and \( I \). Then in the soundness proof, as \( e \) takes successive steps, we combine these couplings together using \( \texttt{Bind} \) from Figure 5; if \( e \) terminates and the post-condition matches the form stated in Theorem 3.1, then we will have have constructed a complete coupling between \( \text{resStep}_{\phi}([e], \sigma) \) and the monadic specification. Moreover, this will be an \( R \)-coupling with \( R(x, y) \triangleq f(x) = g(y) \). Hence, we can apply Theorem 2.1 to conclude the claim about the expected values.

There is one caveat: in \( \texttt{Ht-NonCoupe} \), the program takes a probabilistic step but the premise does not require exhibiting a coupling. To support this, we have to construct a kind of trivial "dummy" coupling that preserves the expected values. We describe this further in Appendix A.
4 Case Study: Approximate Counters

In this section we prove triples that relate the approximate counter algorithm from Figure 1c to the monadic specification approxN from Figure 3. The specification, along with a client using it, is given in Figure 11.

The specification uses the predicate ACounter\(p_l, y_l, y_c\)\((l, q, n)\), which can be treated by a user as an abstract predicate representing the permission to perform \(n\) increments to the counter at \(l\). The parameter \(q\) is a fractional permission that we use to track how many threads can access the counter. (Ignore the names \(y_l, y_r,\) and \(y_c\) — we will describe how they are used when we give the definition of ACounter later).

The triple ACounterNew says that we can create a new counter by allocating a reference cell containing 0. It takes the monadic specification \(\text{Prob}(\text{approxN} \ n \ 0)\) as a precondition, and returns the full ACounter permission for \(n\) increments. The rule ACounterSep lets us split or join this ACounter permission into pieces. If we have permission to perform at least one increment, we can use ACounterIncr, which gives us back ACounter with permission to do one fewer increment. Finally, if we have ACounter with the full fractional permission 1, and there are 0 pending increments, we can use ACounterRead. In the postcondition we get back \(\text{Prob}(\text{ret } v)\), where \(v\) is the value that the call to read returns.

At first this specification seems weak, but this is exactly what we need for Theorem 3.1. To see this, consider the client on the right hand side of Figure 11. We start with a helper function countTrue, which takes a counter \(c\) and a list of booleans \(lb\), and counts the number of times True occurs in \(lb\) using the counter. The client begins by creating a new counter \(c\). It then runs two threads in parallel that run countTrue on two lists \(lb_1\) and \(lb_2\), using the shared counter \(c\) — we denote this parallel composition using \(\parallel\). The parent blocks until both threads finish and then reads from the counter.

Refer to this client code as \(e\). If we write \(|lb_1|\) for the logical function giving the number of times True occurs in \(lb\), then we would like to show that in expectation, \(e\) returns \(|lb_1| + |lb_2|\). The derivation in Figure 11 shows that the triple

\[
\{\text{Prob}(\text{approxN} \ (|lb_1| + |lb_2|) \ 0)\} e \\
\{v. \exists v'. \text{Prob}(\text{ret } v') \land v = v'\}
\]

holds, so by Theorem 3.1 we have:

\[
\mathbb{E}_{\text{id}}^{\text{min}}[\text{approxN} \ (|lb_1| + |lb_2|)] \leq \mathbb{E}_{\text{id}}[\text{resStep}_\varphi^\sigma([e], \sigma)]
\]

(and similarly for \(\mathbb{E}_{\text{id}}^{\text{max}}\)) for suitable \(\varphi\) and \(n\). And, we have shown that \(\mathbb{E}_{\text{id}}^{\text{min}}[\text{approxN} \ (|lb_1| + |lb_2|)] = |lb_1| + |lb_2|\), so we are done.

Proof set-up. The definition of ACounter and the invariants used in the proof are given in Figure 12. The proof uses three counter resources to track: (1) the number of increments left to perform in the monadic specification, (2) the accumulated count in the monadic specification, and (3) the actual count currently stored in the concrete program. We use two invariants to connect the counter resources to these intended interpretations. First, we have LocInvt\(n\) which says that the counter resource named \(y_l\) stores some value \(n\) and the physical location \(l\) points to that same value. Then, assertion ProbInvt\(n, y_r, y_c\) says that there are two counter resources containing some \(n_1\) and \(n_2\), and the invariant either contains (a) the monadic specification resource \(\text{Prob}(\text{approxN} \ n_1, n_2)\) (i.e., there are \(n_1\) further increments to perform, and the monadic counter has accumulated a value of \(n_2\)) or (b) it contains the complete stake for one of the counter resources. Then ACounter says that these two invariants have been set up with some names, and we own a stake in the \(y_r\) permission corresponding to the number of increments this permission allows. Further, for some \(n'\) there is a stake in the \(y_l\) and \(y_c\) counters both equal to \(n'\), which represents the total amount that this permission has been used to add to the counter.

Proof of ACounterIncr. We will focus on the proof of ACounterIncr, because this where we need to use the Ht-Couple rule — the other parts of the specification are straightforward.

Eliminating the existentials in the definition of ACounter, we get that the appropriate invariants have been set up and there is some \(n'\)-stake in \(y_l\) and \(y_c\), along with the \(n + 1\) stake in \(y_r\). The first step of incr \(l\) reads the value of \(l\) to perform this read the thread needs to own \(l \leftarrow v\) for some \(v\). To get this resource, it opens the LocInvt\(n\)\((l)\) invariant; after completing the read, the \(l \leftarrow v\) resource is returned to close the invariant. The code then takes the minimum of the value read and MAX, and binds this value to \(k\).

It then performs flip\((1, k + 1)\). We want to use Ht-Couple to couple this flip with the monadic code. To do so, we first open the invariant ProbInvt\(n, y_r, y_c\). We know this will contain \(\otimes_{n_1, n_2}^\varphi\), \(\otimes_{n_1, n_2}^\psi\), for some \(n_1\) and \(n_2\), and either \(\text{Prob}(\text{approxN} \ n_1, n_2)\) or a full stake \(\otimes(1, n_k)\). However, the latter is impossible because the ACounter\(p_l, y_r, y_c\)\((l, q, n + 1)\) resource entails ownership of \(\otimes(q, n + 1)p\), but \(q + 1 > 1\), contradicting COUNT-PERM. So, we obtain \(\text{Prob}(\text{approxN} \ n_1, n_2)\). Now, by CountEq\(n\) we know that \(n_\ell \geq n + 1\), hence we can unfold \(\text{approxN} \ n_1, n_2\) to obtain \(\text{Prob}(k \leftarrow \text{approxIncr} ; \text{approxN} \ (n_1 - 1, n_2))\).

We can now use Ht-Couple so long as we can exhibit a coupling between the concrete program’s coin flip and approxIncr. First, since \(0 \leq k \leq \text{MAX}\), we can show that:

\[
\{\text{ ret } k + 1 \oplus_{\text{MAX}} (0)\} \leq (x \leftarrow \text{ ret } 0 \cup \cdots \cup \text{ ret } \text{MAX} ; (\text{ret } x + 1 \oplus_{\text{MAX}} (0))
\]

\[
\equiv \text{approxIncr}
\]
We have developed a concurrent program logic that can be used for probabilistic relational reasoning. Our logic is expressive enough to verify a variant of a recently proposed algorithm for scalable approximate counters, for which there was previously not even an informal correctness argument.

Figure 11. Specification for approximate counter and example client.

Moreover, we have mechanized all the results described here in Coq by modifying the prior Coq formalization of Iris. The development is included in our supplementary material.

As we have described, the key to our approach has been to synthesize ideas from several lines of related work. We now mention further related work and possible extensions.

Non-deterministic choice in probabilistic logics. Several program logics for probabilistic reasoning (e.g., [36]) are designed to reason about languages that have primitives for both probabilistic choice and (demonic) non-deterministic choice. However, in that work, non-determinism was not used for modelling concurrency, but rather a program which might be “underspecified” and have multiple possible implementations of a component, of which one is selected non-deterministically, as in Dijkstra’s [14] work on GCL.

An exception is the work by McIver et al. [33], which develops a probabilistic rely-guarantee logic. However, we believe it would be difficult to verify an example like the approximate counter using their logic because it lacks the features of recent concurrency logics.

Coupling in program logics. Barthe et al. [3] were the first to connect the idea of coupling to the kind of probabilistic relational reasoning done in pRHL, an earlier logic by Barthe et al. [7]. Since then, different results from the theory of coupling and variants of couplings have been used to extend pRHL [4, 5, 8, 22]. It would be interesting to try to develop these further extensions in the concurrent setting.

5 Conclusion

We have developed a concurrent program logic that can be used for probabilistic relational reasoning. Our logic is expressive enough to verify a variant of a recently proposed algorithm for scalable approximate counters, for which there was previously not even an informal correctness argument.

Figure 12. Invariants and definitions for proof.

hence by EQUIV, it suffices to exhibit a coupling between (ret True @ γ ret False) and (ret k + 1 @ γ ret 0). Take R(x, y) to be (x = True ∧ y = k + 1) ∨ (x = False ∧ y = 0), then we can use P-CHOICE and RET to prove the existence of an R-coupling.

Applying Ht-Couple with this coupling, we then have Prob(approxN (n’1 - 1) (n’2 + v’)) where v’ and the return value v of the flip(1, k) are related by R. We use COUNTUPD to update the thread’s stake in yγ resources to n, and the global value to n’1 - 1 (to record that a simulated increment has performed), similarly, we update the thread’s stake in the yγ counter to n’ + v’ and the global value to n’2 + v’ (to record the new total) and then close the ProbInvγ invariant.

The code then cases on the value v returned by the flip. If it is false, then v’ is 0, the code returns, and the post condition holds. If v is true, then v’ = k + 1, the amount that the code adds using a fetch-and-add. We therefore open the LocInvγ(l) invariant again to get access to l, perform the increment and update the yγ counter and stake using COUNTUPD to record the fact that we are adding k + 1.
Denotational semantics. A number of denotational models combining probabilistic and non-deterministic choice have been developed [21, 24, 34, 47, 52, 53]. Our soundness theorem considers a scheduler which deterministically selects which thread to run next. Varacca and Winskel [53] showed that their monadic encoding, which we have used in our work, gives an adequate semantic model for an imperative language with this kind of deterministic scheduler. An alternative is to permit the scheduler to also make random choices when selecting which thread to run. Varacca and Winskel show that in this case, an alternative monad developed by Mislove [34] and Tix et al. [47] gives an adequate model. It would be interesting to use this latter monad in our program logic to reason about behavior under probabilistic schedulers.

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A Appendix

In this appendix, we describe in more detail how we modify the definition of Hoare triples in Iris in order to support our probabilistic extensions. Recall from the description in the paper that the main idea is to augment the definition of Hoare triples so that a derivation of a Hoare triple will encode a coupling between each step of the concrete program $e$ and the monadic specification.

As we alluded to in the body of the paper, one complexity that arises is that we want to support rules like $\text{Ht-NonCouple}$ where no coupling is exhibited in the premise. In these cases, we would like to show that there is a “dummy” coupling we can insert.

**Couplings up to Irrelevance.** Unfortunately, the analogy between the classical theory of couplings and the indexed valuation formulation breaks down in one key respect that complicates this. In the classical definition, there is always at least a trivial coupling between two distributions. Given $A$ and $B$, we can take the coupling $C$ to be the distribution:

$$x \leftarrow A; \ y \leftarrow B; \ \text{ret} (x, y)$$

One can show that this satisfies the two coupling conditions. However, the proof relies on the fact that in the probability distribution monad, $(x \leftarrow A; \ D) \equiv D$ if $x$ does not appear free in $D$, which in turn is connected to the fact that $A @ p_A \equiv A$. As we have noted, an essential fact about the indexed valuation monad is that this equivalence does not hold there.

To deal with the fact that there is not always a coupling between indexed valuations, we start by introducing a coarser notion of equivalence between two indexed valuations. The relation $\equiv I_1$ which we call equivalence up to irrelevant choices, is inductively defined by the rules in Figure 13. The key rule says $x \leftarrow I_1; \ I_2 \equiv I_2$, which captures the idea that if the outcome of $I_1$ is not used subsequently, the computation is equivalent to a version in which $I_1$ is never executed. We can similarly define an ordering $I_1 \lesssim I_2$, which extends the $\subseteq$ ordering up to irrelevant choices; the rules defining this relation are given in Figure 14.

**Definition A.1.** We say $I \approx I : P$ if there exists $I'$ and $I'$ such that: (1) $I \approx I' ;$ (2) $I' \lesssim I$; and (3) $I' \sim I' : P$. The latter coupling is called the witness.

We call this a “$P$-coupling up to irrelevance”. The following theorem is an analogue of Theorem 2.1 which shows that the
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\[
I \preceq I \\
\frac{I_i \preceq I_j \quad I_2 \preceq I_3}{I_1 \preceq I_3}
\]

\[
\begin{aligned}
&I_i \preceq I_j \\
&I_1 \leq I_2 \\
&I_1 \leq I_2' \\
&I_1 \leq I_2 \\
&I_1' \preceq I_2'
\end{aligned}
\]

Figure 14. Rules for \(\preceq\) relation.

\[
\begin{array}{ll}
\text{Irrel-Equiv} & I \approx I' \quad I \preceq I' \quad I \approx I : P \\
& I' \approx I' : P
\end{array}
\]

\[
\begin{array}{ll}
\text{Irrel-Conseq} & I \sim I : P \\
& \forall x, y, P(x, y) \rightarrow P'(x, y)
\end{array}
\]

\[
\begin{array}{ll}
\text{Irrel-Bind} & I \approx I : P \\
& \forall x, y, P(x, y) \rightarrow F(x) \sim F'(y) : Q
\end{array}
\]

\[
\begin{array}{ll}
\text{Irrel-Regular} & I \sim I : P \\
& \text{Irrel-Trivial}
\end{array}
\]

\[
\begin{array}{ll}
\text{Irrel-Support} & I \approx I : P
\end{array}
\]

Figure 15. Rules for constructing couplings up to irrelevance.

Existence of this weaker kind of coupling is still sufficient to relate the expected value of \(I\) to that of \(I'\):

**Theorem A.2.** For all \(f\) and \(g\), if \(P(x, y) = (f(x) = g(y))\), then \(I \sim I : P\) implies:

\[
\mathbb{E}^\min_g [I] \leq \mathbb{E}_f[I] \leq \mathbb{E}^\max_g[I]
\]

**Proof:** If \(I \approx I'\), then \(\mathbb{E}_f[I] = \mathbb{E}_f[I']\), and if \(I' \preceq I\):

\[
\begin{aligned}
\mathbb{E}^\min_g[I] & \leq \mathbb{E}^\min_g[I'] \\
\mathbb{E}^\max_g[I'] & \leq \mathbb{E}^\max_g[I]
\end{aligned}
\]

So the desired bound follows from Theorem 2.1.

Rules for constructing these couplings are shown in Figure 15. **Irrel-Regular** lets us derive a coupling up to irrelevance from a non-deterministic coupling. **Irrel-Trivial** shows there is always a coupling up to irrelevance between any \(I\) and \(I'\). To explain **Irrel-Support**, we need to introduce additional terminology: We say \(x : T\) is in image((\(I\), ind, val)) if there exists some \(i \in \text{support(val)}\) such that \(\text{ind}(i) = x\). Similarly, \(x \in \text{image}(I)\) if there exists some \(I \in \mathcal{I}\) such that \(x \in \text{image}(I)\). Then **Irrel-Support** says we can strengthen the post-condition of a coupling to assume that the returned values are in the image of \(I\) and \(I'\).

Suppose we have \(\bar{I} : M_1(T_1)\) and \(\bar{I} : M_2(N(T_2))\). Remember that a non-deterministic coupling \(\bar{I} \sim \bar{I} : R\) consists of an \(I' \in \bar{I}\), and an \(R\)-coupling between \(I\) and \(I'\). Since this latter coupling is itself just some indexed valuation \(C : M_1(T_1 \times T_2)\), we can meaningfully talk about its support and image. Given \(x \in T_1\), we say \(y \in \text{image}(C, x)\) if \((x, y) \in \text{image}(C)\). We will simply write \(C : \bar{I} \sim \bar{I} : R\) to refer to this underlying coupling. Similarly, since the witness for a coupling up to irrelevance is a non-deterministic coupling, we will write \(C : \bar{I} \approx \bar{I} : R\) to denote the witness.

**Modifying Weakest-Precondition.** From here on, we assume familiarity with the Iris model.

Let \(\text{ProbState}\) be the type \(\Sigma T : \text{Type} \times M_1(M_2(T))\). Given two terms \((T_1, I_1)\) and \((T_2, I_2)\) of type \(\text{ProbState}\), we say \((T_1, I_1) \equiv (T_2, I_2)\). Using this equivalence relation, we impose a discrete OFE structure on \(\text{ProbState}\). Here, we will often simply omit the type \(T\) when writing an element of \(\text{ProbState}\).

The monad specification resource is handled much in the same way that physical state is in Iris. We represent the monoid specification code using the “authoritative exclusive resource” construction. Define:

\[
\begin{aligned}
\text{Plinterp}(I) & \triangleq \bullet \text{Ex } I \\
\text{Prob}(I) & \triangleq \circ \text{Ex } I
\end{aligned}
\]

In Iris, Hoare triples are defined in terms of weakest-preconditions, so we actually need to modify the definition of the latter. Recall the following definition of the weakest-precondition, which is taken from the Iris 3.0 documentation [46] (with a typo corrected):

\[
\begin{aligned}
\text{wp} & \triangleq \mu \text{wp}. \lambda \mathcal{E}, e, \varphi. \\
(\exists \mathcal{E}. \text{expr2val}(e) = v \land \exists \varphi(e, \varphi(v))) & \lor \\
\left(\text{expr2val}(e) = \bot \land \forall \sigma. I(\sigma) \mathcal{E} \neq \emptyset \land \\
\exists \mathcal{E}. \mathcal{E} = \emptyset \land \exists \mathcal{E}. \mathcal{E} \neq \emptyset \\
\mathcal{E} & \neq \emptyset \land \mathcal{E} \neq \emptyset \\
\mathcal{E} & \neq \emptyset \land \mathcal{E} \neq \emptyset \\
\mathcal{E} & \neq \emptyset \land \mathcal{E} \neq \emptyset
\end{aligned}
\]

The definition is a guarded fixed-point, composed of a disjunction which handles two cases: (1) either the expression \(e\) is value, in which case the post-condition \(\varphi\) should hold for that value, or (2) it is not a value, in which case for each possible state \(\sigma\), given the interpretation of \(\varphi\), we need to show \(e\) is reducible, and then recursively show that for each thing which \(e\) could step to, we will be able to update
the state interpretation appropriately and recursively prove
weakest-precondition for the reduct.

We write $\text{primStep}(e, \sigma)$ for the indexed valuation of type
$\text{Option}(\text{Expr} \times \text{State} \times \text{List Expr})$ which returns reducts of
$e; \sigma$ or None if $e; \sigma$ is not reducible.

In our version, the definition of weakest precondition
becomes:

\[
\begin{align*}
wp & \triangleq \mu \wp. \lambda E, e, \varphi.
\end{align*}
\]

\[
\begin{align*}
(\exists v. \text{expr2val}(e) = v \land \exists \varphi(v)) \lor
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{l}
\text{expr2val}(e) = \bot \land \forall \sigma, \mathcal{I}. \text{I}(\sigma) \equiv \text{PInterp}(\mathcal{I}) \overset{E}{\Rightarrow} \emptyset \\
\text{red}(e, \sigma) \Rightarrow \exists R, \mathcal{I}', F, (C : \text{primStep}(e, \sigma) \approx \mathcal{I}' : R), \\
\mathcal{I} \equiv x \leftarrow \mathcal{I}'; \ F(x) \land \forall e', \sigma', \bar{e}, v'. \\
(v' \in \text{rImage}(C, (e', \sigma', \bar{e})) \land (e, \sigma \rightarrow e', \sigma', \bar{e})) \\
\emptyset \overset{E}{\Rightarrow} \mathcal{I}(\sigma') \equiv \text{PInterp}(F(v')) \equiv \wp(E, e', \varphi) \\
* \overset{E}{\Rightarrow} \wp(\emptyset, e'', \lambda, \text{True}) \end{array} \right)
\end{align*}
\]

The value case is the same as in the original definition.
In the non-value case, in addition to quantifying over the
physical state ($\sigma$) we also quantify over the probabilistic
state ($\mathcal{I}$). Given the interpretations of both forms of state,
we need to show not only is $e; \sigma$ reducible, but also, we
must show that $\mathcal{I} \equiv x \leftarrow \mathcal{I}'; F(x)$ for some $\mathcal{I}'$ and $F,$ and
moreover we must exhibit an $R$-coupling up to irrelevance
between $\text{primStep}(e, \sigma)$ and this $\mathcal{I}'$, for some choice of $R.$
Then, for each thing $e; \sigma$ could reduce to, and for each $v'$ in
the corresponding rimage of the coupling, we have to update
the physical state appropriately and the probabilistic state
to $F(v')$, and recursively prove the weakest precondition for
the reduct.