Termination for Gödel's $\mathbf{T}$

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1 The Language

The syntax of Gödel's $\mathbf{T}$ is given by the following grammar:

$$\begin{array}{llll} \textit{Types} & \tau & ::= & \mathsf{N} \mid \tau_1 \to \tau_2 \\ \textit{Terms} & t & ::= & x \mid \mathbf{0} \mid \mathbf{s}(t) \mid \mathbf{R}_\tau(t; t_0; x, y.t_1) \mid \lambda x{:}\tau.t \mid t_1\, t_2 \end{array}$$

The variable x is bound in t in $\lambda x{:}\tau.t$, and the variables x and y are bound in t_1 in $\mathbf{R}_\tau(t; t_0; x, y.t_1)$.

The typing judgement $\Gamma \vdash t : \tau$ is defined as in class. The transition system $t_1 \mapsto t_2$ is defined as in class (using a "by name" rule for function application and a "lazy" rule for successors).

2 Hereditary Termination

The type-indexed family of predicates, $\mathsf{HT}_\tau(t)$, is defined by induction on the structure of τ by the following equations:

$$\begin{array}{rll} \mathsf{HT}_{\mathsf{N}}(t) & \textit{iff} & t \mapsto^* \mathbf{0},\ \textit{or}\ t \mapsto^* \mathbf{s}(t')\ \textit{and}\ \mathsf{HT}_{\mathsf{N}}(t') \\ \mathsf{HT}_{\tau_1 \to \tau_2}(t) & \textit{iff} & t \mapsto^* \lambda x{:}\tau_1.t_2\ \textit{and}\ \mathsf{HT}_{\tau_1}(t_1)\ \textit{implies}\ \mathsf{HT}_{\tau_2}([t_1/x]t_2) \end{array}$$

Suppose that Γ is a typing context, and that γ is a finite mapping from variables to closed terms such that $\operatorname{dom}\gamma = \operatorname{dom}\Gamma$ and for every $x \in \operatorname{dom}\Gamma$, $\gamma(x) : \Gamma(x)$.

$$\mathsf{HT}_\Gamma(\gamma)\ \textit{iff}\ \forall x \in \operatorname{dom}\Gamma\ \mathsf{HT}_{\Gamma(x)}(\gamma(x)).$$

We write $\hat{\gamma}(t)$ to denote the simultaneous substitution of $\gamma(x)$ for free occurrences of x in t. No capture can occur, because the range of γ consists of closed terms.

The predicate $\mathsf{HT}_{\mathsf{N}}(t)$ is itself inductively defined to be the strongest predicate P such that $P(t)$ holds iff either

1. $t \mapsto^* \mathbf{0}$, or
2. $t \mapsto^* \mathbf{s}(t')$ and $P(t')$.

This gives rise to the following induction principle. To show that $\mathsf{HT}_{\mathsf{N}}(t)$ implies that $P(t)$ (for any property P), it is enough to show that

1. if $t \mapsto^* \mathbf{0}$, then $P(t)$; and
2. if $t \mapsto^* \mathbf{s}(t')$ with $P(t')$, then $P(t)$.

Lemma 1

1. *If $\mathsf{HT}_\tau(t)$, then there exists a value v such that $t \searrow v$.*
2. *If $\mathsf{HT}_\tau(t)$, $t' : \tau$, and $t' \mapsto t$, then $\mathsf{HT}_\tau(t')$.*
3. *If $\mathsf{HT}_\tau(t)$ and $t \mapsto t'$, then $\mathsf{HT}_\tau(t')$.*

Proof: Each part is proved by induction on the structure of τ.

1. Immediate, from the definitions.
2. Closure of evaluation under head expansion, plus the fact that if $t' \mapsto^* t$, then $t'\,t_1 \mapsto^* t\,t_1$.
3. If $t \searrow v$ and $t \mapsto t'$, then $t' \searrow v$, by determinacy of evaluation.

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3 Fundamental Lemma

Lemma 2
If $\Gamma \vdash t : \tau$ and $\mathsf{HT}_\Gamma(\gamma)$, then $\mathsf{HT}_\tau(\hat{\gamma}(t))$.

Proof: By induction on typing.

1. Suppose that $\Gamma \vdash x : \tau$ because $\Gamma(x) = \tau$. Then by assumption $\mathsf{HT}_{\Gamma(x)}(\gamma(x))$, which is to say $\mathsf{HT}_\tau(\hat{\gamma}(x))$.
2. Suppose that $\Gamma \vdash \lambda x{:}\tau_1.t_2 : \tau_1 \to \tau_2$ from the premise $\Gamma, x{:}\tau_1 \vdash t_2 : \tau_2$. Suppose $\mathsf{HT}_\Gamma(\gamma)$; we are to show $\mathsf{HT}_{\tau_1\to\tau_2}(\hat{\gamma}(\lambda x{:}\tau_1.t_2))$. Note that $\hat{\gamma}(\lambda x{:}\tau_1.t_2) = \lambda x{:}\tau_1.\hat{\gamma}(t_2)$, so the first requirement for being hereditarily terminating at function type is met. Now suppose that $\mathsf{HT}_{\tau_1}(t_1)$, with a view toward showing $\mathsf{HT}_{\tau_2}([t_1/x]t_2)$. Observe that $\mathsf{HT}_{\Gamma,x:\tau_1}(\gamma, x{=}t_1)$, and so by induction we have $\mathsf{HT}_{\tau_2}(\widehat{\gamma, x{=}t_1}(t_2))$. But $\widehat{\gamma, x{=}t_1}(t_2) = [t_1/x]\hat{\gamma}(t_2)$, which completes the proof.
3. Suppose that $\Gamma \vdash t_1\,t_2 : \tau$ from the premise $\Gamma \vdash t_1 : \tau_2 \to \tau$ and $\Gamma \vdash t_2 : \tau_2$. By induction $\mathsf{HT}_{\tau_2\to\tau}(\hat{\gamma}(t_1))$ and $\mathsf{HT}_{\tau_2}(\hat{\gamma}(t_2))$. Consequently, $\hat{\gamma}(t_1) \mapsto^* \lambda x{:}\tau_2.t$, and hence

$$\begin{array}{rcl} \hat{\gamma}(t_1\,t_2) & = & \hat{\gamma}(t_1)\,\hat{\gamma}(t_2) \\ & \mapsto^* & (\lambda x{:}\tau_2.t)\,\hat{\gamma}(t_2) \\ & \mapsto & [\hat{\gamma}(t_2)/x]t. \end{array}$$

But $\mathsf{HT}_\tau([\hat{\gamma}(t_2)/x]t)$ from the first and second inductive hypotheses, and hence the result follows by closure under head expansion.

4. Suppose $\Gamma \vdash \mathbf{0} : \mathsf{N}$. Then $\mathsf{HT}_{\mathsf{N}}(\mathbf{0})$, and $\hat{\gamma}(\mathbf{0}) = \mathbf{0}$, which is enough for the result.

5. Suppose that $\Gamma \vdash \mathbf{s}(t) : \mathsf{N}$ from the premise $\Gamma \vdash t : \mathsf{N}$. By induction $\mathsf{HT}_{\mathsf{N}}(\hat{\gamma}(t))$, and hence $\mathsf{HT}_{\mathsf{N}}(\mathbf{s}(\hat{\gamma}(t)))$, as required, noting that $\mathbf{s}(\hat{\gamma}(t)) = \hat{\gamma}(\mathbf{s}(t))$.

6. Suppose that $\Gamma \vdash \mathbf{R}_\tau(t; t_0; x, y.t_1) : \tau$ from the premises

   (a) $\Gamma \vdash t : \mathsf{N}$;

   (b) $\Gamma \vdash t_0 : \tau$;

   (c) $\Gamma, x{:}\mathsf{N}, y{:}\tau \vdash t_1 : \tau$.

   Let $\hat{t} = \hat{\gamma}(t)$, $\widehat{t_0} = \widehat{\gamma}(t_0)$, and $\widehat{t_1} = \hat{\gamma}(t_1)$. We have by the "outer" induction on typing that

   (a) $\mathsf{HT}_{\mathsf{N}}(\hat{t})$;

   (b) $\mathsf{HT}_\tau(\widehat{t_0})$;

   (c) if $\mathsf{HT}_{\mathsf{N}}(u)$ and $\mathsf{HT}_\tau(u')$, then $\mathsf{HT}_\tau([u/x, u'/y]\widehat{t_1})$.

   We show by the "inner" induction principle associated with the hereditary termination predicate at type N, that if $\mathsf{HT}_{\mathsf{N}}(u)$, then $\mathsf{HT}_\tau(\mathbf{R}_\tau(u; \widehat{t_0}; x, y.\widehat{t_1}))$. This decomposes into two parts.

   (a) If $u \mapsto^* \mathbf{0}$, then $\mathbf{R}_\tau(u; \widehat{t_0}; x, y.\widehat{t_1}) \mapsto^* \mathbf{R}_\tau(\mathbf{0}; \widehat{t_0}; x, y.\widehat{t_1}) \mapsto \widehat{t_0}$. The result follows by the second inductive hypothesis and closure under head expansion.

   (b) Suppose that $u \mapsto^* \mathbf{s}(u')$ with the inner inductive hypothesis that $\mathsf{HT}_\tau(\mathbf{R}_\tau(u'; \widehat{t_0}; x, y.\widehat{t_1}))$. Then

   $$\begin{array}{rcl} \mathbf{R}_\tau(u; \widehat{t_0}; x, y.\widehat{t_1}) & \mapsto^* & \mathbf{R}_\tau(\mathbf{s}(u'); \widehat{t_0}; x, y.\widehat{t_1}) \\ & \mapsto & [u'/x, \mathbf{R}_\tau(u'; \widehat{t_0}; x, y.\widehat{t_1})/y]\widehat{t_1}. \end{array}$$

   The result follows from the third outer inductive hypothesis together with the assumption that $\mathsf{HT}_{\mathsf{N}}(u')$ and the inner inductive hypothesis that $\mathsf{HT}_\tau(\mathbf{R}_\tau(u'; \widehat{t_0}; x, y.\widehat{t_1}))$. To finish the proof, note that by the first outer inductive hypothesis we may take $u = \hat{t}$ to obtain the desired result.

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Corollary 3 (Termination)
If $t : \tau$, then there exists v such that $t \searrow v$.