Homework 1: Heyting Algebra and IPL

15-819 Homotopy Type Theory

Out: 19/Sep/13
Due: 3/Oct/13

CHANGE (2013/10/1 10PM): There was a 24-hour extension. The new deadline is shown above.

This is 15-819's first homework assignment!

1 Heyting Meets Boole

The goal of this section is to reason about an algebra, for example a Boolean algebra, through abstract properties, without appealing a particular model, such as `true` and `false`. In particular, meets and joins are defined by their *universal properties* instead of truth tables.

As mentioned in the lectures, with implications one can show distributiveness of any Heyting algebra. The following is one of the most interesting parts in the proof.

Task 1. *Show that $A \wedge (B \vee C) \leq (A \wedge B) \vee (A \wedge C)$ in any Heyting algebra.*
(Hint) *You might find Yoneda's Lemma useful, which says (in this particular context) $A \leq B$ iff for all C, $B \leq C$ implies $A \leq C$. There is a short proof with Yoneda's, and another short proof without.*

In class we also gave two definitions of negations $\neg A$, one with explicit construction and the other through universal properties. The next task is to show that these two definitions are equivalent.

Task 2. *Show that in any Heyting algebra, $A \supset \bot$ is one of the largest elements inconsistent with A, and is equivalent to any largest inconsistent one.*

CHANGE (2013/9/26 4PM): The above task has been reworded for clarification.

Finally, with the introduction of mighty complements, exponentials become definable if distributiveness is assumed. As a corollary, in Boolean algebras negations and complements collide. (Please refer to the lecture note for the correct definition of complements. There was a mistake in the definition given in class.)

Task 3. *Show that in any Boolean algebra (complemented distributive lattice), $\overline{A} \vee B$ is a valid implementation of $A \supset B$. That is, it satisfies all properties of $A \supset B$.*

2 IPL Structural Engineering

Here we will explore structural properties of IPL, among which one of the most important is *transitive* shown below.

Task 4. *Show that IPL is transitive, which is to say*

$$\frac{\Gamma, \Gamma' \vdash P \ \textsf{true} \qquad \Gamma, P \ \textsf{true}, \Gamma' \vdash Q \ \textsf{true}}{\Gamma, \Gamma' \vdash Q \ \textsf{true}}$$

is admissible. You only have to consider the case that the last rule applied in the right sub-derivation (of $\Gamma, P \ \textsf{true}, \Gamma' \vdash Q \ \textsf{true}$) is either the primitive reflexivity or rules in the negative fragment. You may assume weakening and exchange as admissible rules.

3 Semantical Analysis of IPL

Any Heyting algebra can be a model of IPL. In fact, $\Gamma \vdash P \ \textsf{true}$ is provable iff $\Gamma^+ \leq P^*$ in any Heyting algebra, where $(-)^*$ is the straightforward lifting of any evaluation function from atomic propositions to elements in the Heyting algebra in question, and Γ^+ is defined as the comprehension of $(-)^*$ through the following equations:

1. $\cdot^+ = \top$.

2. $(\Gamma, P\ \mathsf{true})^{+} = P^{*} \wedge \Gamma^{+}$.

Task 5. *Show that for any Heyting algebra and any evaluation function on atoms, if $\Gamma \vdash P$ true then $\Gamma^{+} \leq P^{*}$. You only have to consider the cases in which the last rule applied is ($\supset$I) or ($\supset$E).*

> CHANGE (2013/10/1 10PM): The task is intended to be a general statement about any Heyting algebra and any evaluation function on atoms. The evaluation function on propositions is always lifted (in the way described in class) from the evaluation function on atoms.

Task 6. *Consider the Lindenbaum algebra of IPL where the elements are all propositions in IPL (with the translation $(-)^{*}$ being the identity function) and the relationship $\leq$ is defined by provability in IPL.[1] That is, $A \leq B$ iff A true $\vdash B$ true. Show that this is a Heyting algebra. You only have to prove the transitivity. You may assume weakening and exchange of IPL, or cite previous tasks as lemmas.*

> REMARK (2013/9/25 9AM): To really complete the theorem, you need to show that $\Gamma \vdash A$ true iff Γ^{+} true $\vdash A$ true. You do not have to prove this for this homework.

[1] To simplify the problem, we avoid taking quotients by interprovability, but one must consider that if a partial order is desired.