Termination for Natural and Unnatural Numbers

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1 Introduction

The extension of the termination proof given in Harper (2020) to account for natural and co-natural numbers illustrates the extension of Tait's method to inductive and co-inductive types.

The typing rules for these constructs is given in Figure 1. The transition system for evaluation is extended to include the rules given in Figure 2.

2 Termination for Natural Numbers

The termination proof for the natural numbers hinges on the definition of hereditary termination at the type N. Specifically, $\mathsf{HT}_{\mathsf{N}}(-)$ is *inductively defined* to be the *strongest*, or *least*, property $\mathcal{P}$ of closed terms of type N such that if either

1. $M \longmapsto^* \mathsf{zero}$, or

2. $M \longmapsto^* \mathsf{succ}(M')$ and $\mathcal{P}(M')$

then $\mathcal{P}(M)$. Being the strongest such predicate, it follows that $\mathsf{HT}_{\mathsf{N}}(M)$ holds *if and only if* either $M \longmapsto^* \mathsf{zero}$, or $M \longmapsto^* \mathsf{succ}(M')$ and $\mathsf{HT}_{\mathsf{N}}(M')$. The sufficiency of these conditions is immediate; their necessity follows from hereditary termination being the strongest property satisfying them.

It remains to validate the rules for the introduction and elimination forms given in the previous section. The introduction rules follow directly from the definition. The eliminatory rule is proved using the minimality of hereditary termination. For convenience, define $R(-) \triangleq \mathsf{rec}_A(-\,;\hat{M}_0\,;x.\hat{M}_1)$, where $\hat{M}_0$ and $\hat{M}_1$ are the appropriate substitution instances of the premises of the rule. The goal is to show that $\mathsf{HT}_A(R(\hat{M}))$, for a corresponding instance of the premise of the rule. It suffices to show that the property $\mathcal{P}(-) \triangleq \mathsf{HT}_A(R(-))$ satisfies the defining conditions for hereditary termination.

1. If $M \longmapsto^* \mathsf{zero}$, then $R(\mathsf{zero}) \longmapsto^* \hat{M}_0$, where $\mathsf{HT}_A(M_0)$ is given by induction. It follows by head expansion that $\mathcal{P}(M)$.

2. If $M \longmapsto^* \mathsf{succ}(M')$ with $\mathcal{P}(M')$, then $R(M) \longmapsto^* [R(M')/x]M_1$. By the second premise, $\mathcal{P}([R(M')/x]\hat{M}_1)$, from which it follows that $\mathcal{P}(\mathsf{succ}(M'))$, and so $\mathcal{P}(M)$ by head expansion.

As would be expected, the proof is essentially by mathematical induction, but in the form induced by the inductive definition of hereditary termination at the type N. Note that a closed term M of type N need not literally be a numeral, but must successively evaluate to some number of successors of zero.

$$\frac{}{\Gamma \vdash \mathsf{zero} : \mathsf{N}}\;\textsc{zero} \qquad \frac{\Gamma \vdash M : \mathsf{N}}{\Gamma] \vdash \mathsf{succ}(M) : \mathsf{N}}\;\textsc{succ} \qquad \frac{\Gamma \vdash M : \mathsf{N} \qquad \Gamma \vdash M_0 : A \qquad \Gamma, x : A \vdash M_1 : A}{\Gamma \vdash \mathsf{rec}_A(M\,;M_0\,;x.M_1) : A}\;\textsc{rec}$$

$$\frac{\Gamma \vdash M : \mathsf{И}}{\Gamma \vdash \mathsf{pred}(M) : \mathbf{1} + \mathsf{И}}\;\textsc{pred} \qquad \frac{\Gamma \vdash M : A \qquad \Gamma, x : A \vdash N : \mathbf{1} + A}{\Gamma \vdash \mathsf{gen}_A(M\,;x.N) : \mathsf{И}}\;\textsc{gen}$$

Figure 1: Typing Rule for Natural and Co-Natural Numbers

$$\frac{}{\mathsf{zero}\ \mathsf{val}}\;\textsc{zero-val} \qquad \frac{}{\mathsf{succ}(M)\ \mathsf{val}}\;\textsc{succ-val} \qquad \frac{M \longmapsto M'}{\mathsf{rec}_A(M\,;M_0\,;x.M_1) \longmapsto \mathsf{rec}_A(M'\,;M_0\,;x.M_1)}\;\textsc{rec-step}$$

$$\frac{}{\mathsf{rec}_A(\mathsf{zero}\,;M_0\,;x.M_1) \longmapsto M_0}\;\textsc{rec-zero} \qquad \frac{}{\mathsf{rec}_A(\mathsf{succ}(M)\,;M_0\,;x.M_1) \longmapsto [\mathsf{rec}_A(M\,;M_0\,;x.M_1)/x]M_1}\;\textsc{rec-succ}$$

$$\frac{}{\mathsf{gen}_A(M\,;x.N)\ \mathsf{val}}\;\textsc{gen-val} \qquad \frac{M \longmapsto M'}{\mathsf{pred}(M) \longmapsto \mathsf{pred}(M')}\;\textsc{pred-step}$$

$$\frac{}{\mathsf{pred}(\mathsf{gen}_A(M\,;x.N)) \longmapsto \mathsf{case}_A([M/x]N\,;\,\_.\mathsf{1}\cdot\star\,;\,y.\mathsf{gen}_A(y\,;x.N))}\;\textsc{pred-gen}$$

Figure 2: Transition Rules for Natural and Co-Natural Numbers

3 Termination for Co-Natural Numbers

The termination proof for co-natural numbers hinges on the *co-inductive definition* of $\mathsf{HT}_{\mathsf{И}}(-)$ as the *weakest*, or *largest*, property $\mathcal{P}$ of closed terms M of type $\mathsf{И}$ such that if $\mathcal{P}(M)$, then either

1. $\mathsf{pred}(M) \longmapsto^* \mathsf{1} \cdot \star$, or

2. $\mathsf{pred}(M) \longmapsto^* \mathsf{2} \cdot M'$ and $\mathcal{P}(M')$.

Consequently, $\mathsf{HT}_{\mathsf{И}}(M)$ iff either $\mathsf{pred}(M) \longmapsto^* \mathsf{1} \cdot \star$, or $\mathsf{pred}(M) \longmapsto^* \mathsf{2} \cdot M'$ and $\mathsf{HT}_{\mathsf{И}}(M')$.

Dually to the case for natural numbers, the proof of hereditary termination for the elimination rule for $\mathsf{И}$ is immediate from the definition. For the introduction rule, assume given that $\mathsf{HT}_A(P)$ implies $\mathsf{HT}_A([P/x]\hat{N})$. Writing $G(-) \triangleq \mathsf{gen}_A(- \,;\, x.\hat{N})$; the objective is to show $\mathsf{HT}_{\mathsf{И}}(G(\hat{M}))$. It suffices to find a property $Q(-)$ of closed terms of type $\mathsf{И}$ that is consistent with the defining properties of hereditary termination at type $\mathsf{И}$ and is such that $Q(G(\hat{M}))$.

To discover a suitable $\mathcal{Q}(-)$, define $M_0 = \hat{M}$, and suppose that $\mathcal{Q}(G(M_0))$. Now $\mathsf{pred}(G(M_0)) \longmapsto^*$ $[M_0/x]N$. By inductive assumptions on the premises of the rule, $\mathsf{HT}_{\mathbf{1}+A}([M_0/x]\hat{N})$. By the definition of hereditary termination at sum type, there are two cases:

1. $[M_0/x]\hat{N} \longmapsto^* \mathsf{1} \cdot \star$. Nothing to be done.

2. $[M_0/x]\hat{N} \longmapsto^* \mathsf{2} \cdot M_1$ with $\mathsf{HT}_A(M_1)$. It is necessary that $\mathcal{Q}(G(M_1)$.

Proceeding similarly from $\mathcal{Q}(G(M_1))$, it is necessary that $\mathcal{Q}(G(M_2))$ for an analogously chosen $\mathsf{HT}_A(M_2)$, and so on. The sequence of terms $G(M_0), G(M_1), \ldots$ may be finite or infinite, according to whether $\hat{N}$ ever indicates a final state. Working forward, define $\mathcal{Q}$ to hold of each element of this sequence. By the above argument, the consistency of $\mathcal{Q}$ is justified for each element by it holding for the next element. Thus, the entire sequence is consistent with the defining properties of hereditary termination at type $\mathsf{И}$. Moreover, by construction, $\mathcal{Q}(G(M_0))$, and so $\mathsf{HT}_{\mathsf{И}}(G(M_0))$, which is to say $\mathsf{HT}_{\mathsf{И}}(G(\hat{M}))$, as desired.

References

Robert Harper. How to (re)invent Tait's method. Unpublished lecture note, Spring 2020. URL `https://www.cs.cmu.edu/~rwh/courses/chtt/pdfs/tait.pdf`.