Data Structures and Algorithms
Solving Recurrence Relations

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4-0: Algorithm Analysis

```
for (i=1; i<=n*n; i++)
  for (j=0; j<i; j++)
    sum++;
```

4-1: Algorithm Analysis

```
for (i=1; i<=n*n; i++)    Executed n*n times
  for (j=0; j<i; j++)      Executed <= n*n times
    sum++;                  O(1)
```

Running Time: $O(n^4)$

But can we get a tighter bound?

```
for (i=1; i<=n*n; i++)
  for (j=0; j<i; j++)
    sum++;
```

Exact # of times `sum++` is executed:

$$\begin{aligned}\sum_{i=1}^{n^2} i &= \frac{n^2(n^2+1)}{2} \\ &= \frac{n^4+n^2}{2} \\ &\in \Theta(n^4)\end{aligned}$$

```
long power(long x, long n)
  if (n == 0)
    return 1;
  else
    return x * power(x, n-1);
```

How many times is this executed?

4-4: *Recurrence Relations*

$T(n)$ = Time required to solve a problem of size n

Recurrence relations are used to determine the running time of recursive programs – recurrence relations themselves are recursive

$$\begin{aligned} T(0) = &\quad \text{time to solve problem of size 0} \\ &\quad \text{– Base Case} \\ T(n) = &\quad \text{time to solve problem of size } n \\ &\quad \text{– Recursive Case} \end{aligned}$$

4-5: Recurrence Relations

```
long power(long x, long n)
  if (n == 0)
    return 1;
  else
    return x * power(x, n-1);
```

$T(0) = c_1$ for some constant c_1

$T(n) = c_2 + T(n-1)$ for some constant c_2

4-6: *Solving Recurrence Relations*

$$T(0) = c_1$$
$$T(n) = T(n-1) + c_2$$

If we knew $T(n-1)$, we could solve $T(n)$.

$T(n)$ = $T(n-1) + c_2$

4-7: Solving Recurrence Relations

$$T(0) = c_1$$
$$T(n) = T(n-1) + c_2$$

If we knew $T(n-1)$, we could solve $T(n)$.

$$\begin{aligned} T(n) &= T(n-1) + c_2 \qquad\qquad T(n-1) = T(n-2) + c_2 \\ &= T(n-2) + c_2 + c_2 \\ &= T(n-2) + 2c_2 \end{aligned}$$

4-8: Solving Recurrence Relations

$$T(0) = c_1$$
$$T(n) = T(n-1) + c_2$$

If we knew $T(n-1)$, we could solve $T(n)$.

$$\begin{aligned} T(n) &= T(n-1) + c_2 & T(n-1) &= T(n-2) + c_2 \\ &= T(n-2) + c_2 + c_2 & & \\ &= T(n-2) + 2c_2 & T(n-2) &= T(n-3) + c_2 \\ &= T(n-3) + c_2 + 2c_2 & & \\ &= T(n-3) + 3c_2 & & \end{aligned}$$

4-9: *Solving Recurrence Relations*

$$T(0) = c_1$$
$$T(n) = T(n-1) + c_2$$

If we knew $T(n-1)$, we could solve $T(n)$.

$$\begin{aligned} T(n) &= T(n-1) + c_2 & \qquad T(n-1) = T(n-2) + c_2 \\ &= T(n-2) + c_2 + c_2 \\ &= T(n-2) + 2c_2 & \qquad T(n-2) = T(n-3) + c_2 \\ &= T(n-3) + c_2 + 2c_2 \\ &= T(n-3) + 3c_2 & \qquad T(n-3) = T(n-4) + c_2 \\ &= T(n-4) + 4c_2 \end{aligned}$$

4-10: Solving Recurrence Relations

$T(0) = c_1$
$T(n) = T(n-1) + c_2$

If we knew $T(n-1)$, we could solve $T(n)$.

$$
\begin{aligned}
T(n) &= T(n-1) + c_2 & \qquad T(n-1) &= T(n-2) + c_2 \\
&= T(n-2) + c_2 + c_2 \\
&= T(n-2) + 2c_2 & T(n-2) &= T(n-3) + c_2 \\
&= T(n-3) + c_2 + 2c_2 \\
&= T(n-3) + 3c_2 & T(n-3) &= T(n-4) + c_2 \\
&= T(n-4) + 4c_2 \\
&= \ldots \\
&= T(n-k) + kc_2
\end{aligned}
$$

4-11: Solving Recurrence Relations

$$T(0) = c_1$$
$$T(n) = T(n-k) + k * c_2 \quad \text{for all } k$$

If we set $k = n$, we have:

$$\begin{aligned} T(n) &= T(n-n) + nc_2 \\ &= T(0) + nc_2 \\ &= c_1 + nc_2 \\ &\in \Theta(n) \end{aligned}$$

4-12: Building a Better `Power`

Can we avoid making a linear number of function calls?

```
long power(long x, long n)
  if (n==0) return 1;
  if (n==1) return x;
  if ((n % 2) == 0)
    return power(x*x, n/2);
  else
    return power(x*x, n/2) * x;
```

4-13: *Building a Better* `Power`

```
long power(long x, long n)
  if (n==0) return 1;
  if (n==1) return x;
  if ((n % 2) == 0)
    return power(x*x, n/2);
  else
    return power(x*x, n/2) * x;
```

$$T(0) = c_1$$
$$T(1) = c_2$$
$$T(n) = T(n/2) + c_3$$

(Assume n is a power of 2)

4-14: Solving Recurrence Relations

$$T(n) = T(n/2) + c_3$$

4-15: Solving Recurrence Relations

$$
\begin{aligned}
T(n) &= T(n/2) + c_3 \\
&= T(n/4) + c_3 + c_3 \\
&= T(n/4) + 2c_3
\end{aligned}
\qquad T(n/2) = T(n/4) + c_3
$$

4-16: Solving Recurrence Relations

$$
\begin{aligned}
T(n) &= T(n/2) + c_3 \\
&= T(n/4) + c_3 + c_3 \\
&= T(n/4) + 2c_3 \\
&= T(n/8) + c_3 + 2c_3 \\
&= T(n/8) + 3c_3
\end{aligned}
$$

$$T(n/2) = T(n/4) + c_3$$

$$T(n/4) = T(n/8) + c_3$$

4-17: Solving Recurrence Relations

$$
\begin{aligned}
T(n) &= T(n/2) + c_3 \\
&= T(n/4) + c_3 + c_3 \\
&= T(n/4) + 2c_3 \\
&= T(n/8) + c_3 + 2c_3 \\
&= T(n/8) + 3c_3 \\
&= T(n/16) + c_3 + 3c_3 \\
&= T(n/16) + 4c_3
\end{aligned}
$$

$$T(n/2) = T(n/4) + c_3$$

$$T(n/4) = T(n/8) + c_3$$

$$T(n/8) = T(n/16) + c_3$$

4-18: Solving Recurrence Relations

$$
\begin{aligned}
T(n) &= T(n/2) + c_3 & T(n/2) = T(n/4) + c_3 \\
&= T(n/4) + c_3 + c_3 \\
&= T(n/4)2c_3 & T(n/4) = T(n/8) + c_3 \\
&= T(n/8) + c_3 + 2c_3 \\
&= T(n/8)3c_3 & T(n/8) = T(n/16) + c_3 \\
&= T(n/16) + c_3 + 3c_3 \\
&= T(n/16) + 4c_3 & T(n/16) = T(n/32) + c_3 \\
&= T(n/32) + c_3 + 4c_3 \\
&= T(n/32) + 5c_3
\end{aligned}
$$

4-19: Solving Recurrence Relations

$$
\begin{aligned}
T(n) &= T(n/2) + c_3 & \qquad T(n/2) = T(n/4) + c_3 \\
&= T(n/4) + c_3 + c_3 & \\
&= T(n/4)2c_3 & \qquad T(n/4) = T(n/8) + c_3 \\
&= T(n/8) + c_3 + 2c_3 & \\
&= T(n/8)3c_3 & \qquad T(n/8) = T(n/16) + c_3 \\
&= T(n/16) + c_3 + 3c_3 & \\
&= T(n/16) + 4c_3 & \qquad T(n/16) = T(n/32) + c_3 \\
&= T(n/32) + c_3 + 4c_3 & \\
&= T(n/32) + 5c_3 & \\
&= \ldots & \\
&= T(n/2^k) + kc_3 &
\end{aligned}
$$

4-20: Solving Recurrence Relations

$$T(0) = c_1$$
$$T(1) = c_2$$
$$T(n) = T(n/2) + c_3$$

$$T(n) = T(n/2^k) + kc_3$$

We want to get rid of $T(n/2^k)$. We get to a relation we can solve directly when we reach $T(1)$

$$\begin{aligned} n/2^k &= 1 \\ n &= 2^k \\ \lg n &= k \end{aligned}$$

4-21: Solving Recurrence Relations

$$T(0) = c_1$$
$$T(1) = c_2$$
$$T(n) = T(n/2) + c_3$$

$$T(n) = T(n/2^k) + kc_3$$

We want to get rid of $T(n/2^k)$. We get to a relation we can solve directly when we reach $T(1)$
$\lg n = k$

$$\begin{aligned} T(n) &= T(n/2^{\lg n}) + \lg n c_3 \\ &= T(1) + c_3 \lg n \\ &= c_2 + c_3 \lg n \\ &\in \Theta(\lg n) \end{aligned}$$

4-22: `Power` Modifications

```
long power(long x, long n)
  if (n==0) return 1;
  if (n==1) return x;
  if ((n % 2) == 0)
    return power(x*x, n/2);
  else
    return power(x*x, n/2) * x;
```

4-23: `Power` *Modifications*

```
long power(long x, long n)
  if (n==0) return 1;
  if (n==1) return x;
  if ((n % 2) == 0)
    return power(power(x,2), n/2);
  else
    return power(power(x,2), n/2) * x;
```

This version of power will not work. Why?

4-24: `Power` Modifications

```
long power(long x, long n)
  if (n==0) return 1;
  if (n==1) return x;
  if ((n % 2) == 0)
    return power(power(x,n/2), 2);
  else
    return power(power(x,n/2), 2) * x;
```

This version of power also will not work. Why?

4-25: `Power` *Modifications*

```
long power(long x, long n)
  if (n==0) return 1;
  if (n==1) return x;
  if ((n % 2) == 0)
    return power(x,n/2) * power(x,n/2);
  else
    return power(x,n/2) * power(x,n/2) * x;
```

This version of power does work.

What is the recurrence relation that describes its running time?

4-26: `Power` Modifications

```
long power(long x, long n)
  if (n==0) return 1;
  if (n==1) return x;
  if ((n % 2) == 0)
    return power(x,n/2) * power(x,n/2);
  else
    return power(x,n.2) * power(x,n/2) * x;
```

$$
\begin{aligned}
T(0) &= c_1 \\
T(1) &= c_2 \\
T(n) &= T(n/2) + T(n/2) + c_3 \\
&= 2T(n/2) + c_3
\end{aligned}
$$

(Again, assume n is a power of 2)

4-27: Solving Recurrence Relations

$$
\begin{aligned}
T(n) &= 2T(n/2) + c_3 \\
&= 2[2T(n/4) + c_3]c_3 \\
&= 4T(n/4) + 3c_3 \\
&= 4[2T(n/8) + c_3] + 3c_3 \\
&= 8T(n/8) + 7c_3 \\
&= 8[2T(n/16) + c_3] + 7c_3 \\
&= 16T(n/16) + 15c_3 \\
&= 32T(n/32) + 31c_3 \\
&\ldots \\
&= 2^k T(n/2^k) + (2^k - 1)c_3
\end{aligned}
$$

$$T(n/2) = 2T(n/4) + c_3$$

$$T(n/4) = 2T(n/8) + c_3$$

4-28: Solving Recurrence Relations

$$T(0) = c_1$$
$$T(1) = c_2$$
$$T(n) = 2^k T(n/2^k) + (2^k - 1)c_3$$

Pick a value for k such that $n/2^k = 1$:

$$
\begin{aligned}
n/2^k &= 1 \\
n &= 2^k \\
\lg n &= k
\end{aligned}
$$

$$
\begin{aligned}
T(n) &= 2^{\lg n} T(n/2^{\lg n}) + (2^{\lg n} - 1)c_3 \\
&= nT(n/n) + (n-1)c_3 \\
&= nT(1) + (n-1)c_3 \\
&= nc_2 + (n-1)c_3 \\
&\in \Theta(n)
\end{aligned}
$$