Display Logics for Classical and Boolean Bunched Implication

Presentation by Robert J. Simmons

Separation Logic, April 8, 2009

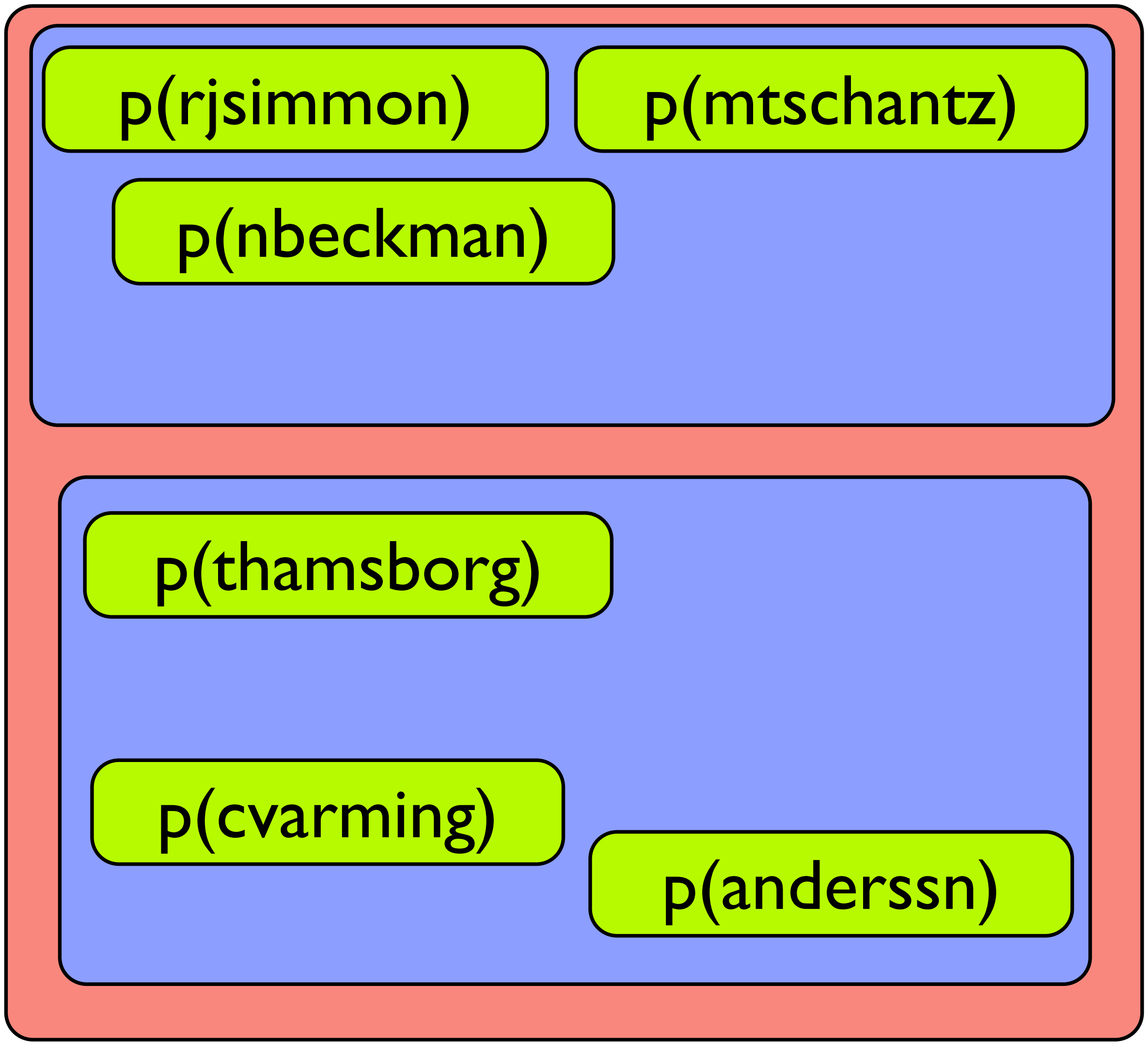
Last time:

- Bunchy contexts
- Bunchy “Natural Deduction”
- Bunchy Sequent Calculus

This time:

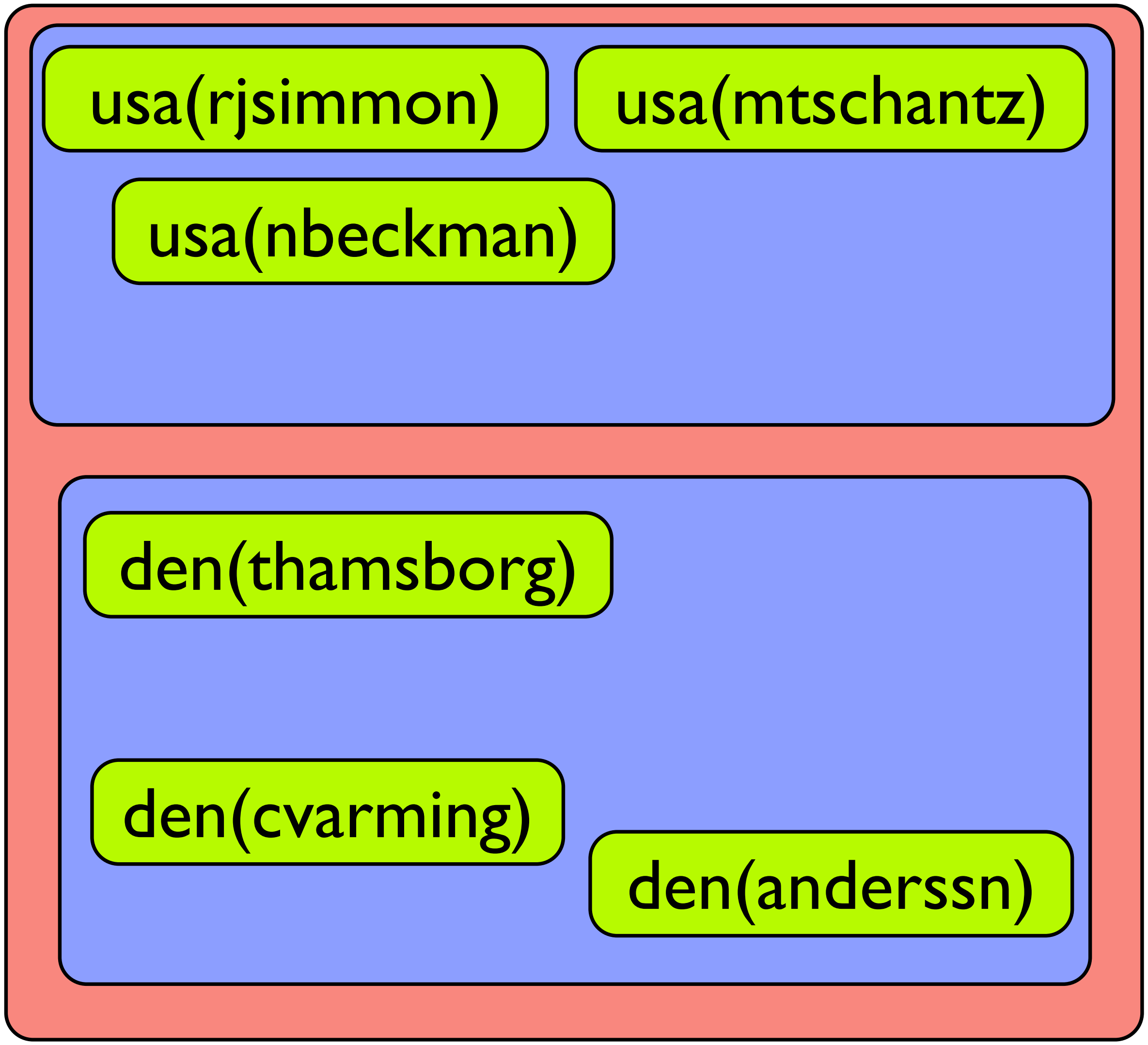
- Multiple-conclusion (classical) sequent calculi
- Display Logic
- Classical Bunched Implication
- Boolean Bunched Implication

First, international collaboration



$$\vdash \exists x,y.\ p(x) * p(y) * \top$$

First, international collaboration

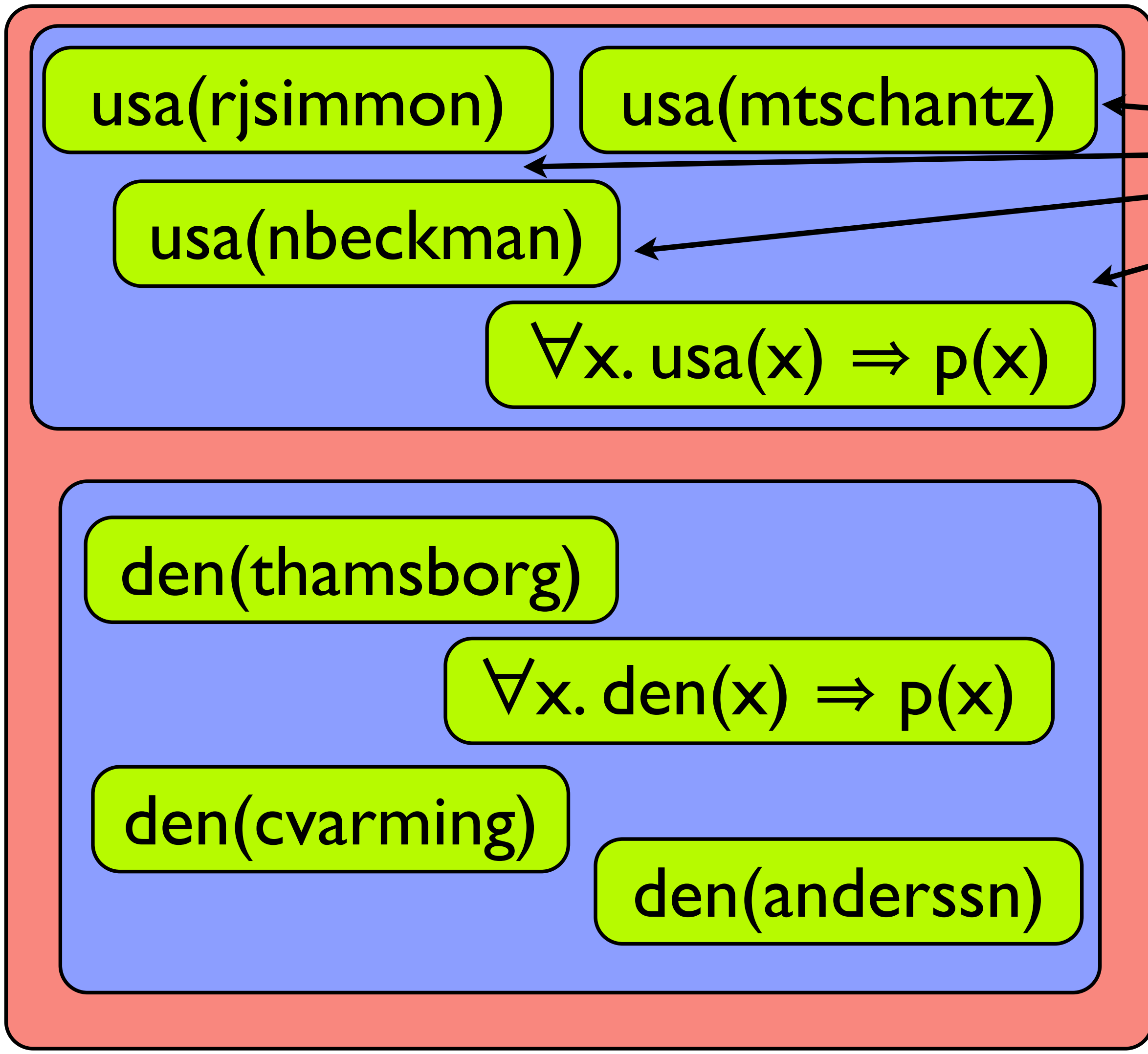


$\forall x.\ usa(x) \Rightarrow p(x)$

$\forall x.\ den(x) \Rightarrow p(x)$

$\vdash \exists x,y.$
$p(x) * p(y) * \top$

First, international collaboration



implication-ey stuff needs to be stuck in the same (additive) bunch as the stuff it will modify

$\vdash \exists x,y.\ p(x) * p(y) * \top$

(possibility: treat these things as *pure*, or perhaps notice they already are?)

Multiple conclusion sequent calculi

$$\underbrace{A; B; C; D} \vdash \underbrace{E; F; G; H}$$

Assuming all of these are true (conjunction)...

We can conclude that one of these are true (disjunction)...

$$\underbrace{A, B, C, D} \vdash \underbrace{E, F, G, H}$$

Assuming we’ve got all of these (conjunction)...

We can get... one of these also using the dual of the resources from the other ones?

(“Par exists to confuse us” - Neel K.)

Multiple conclusion sequent calculi

$$\underbrace{A; B; C; D}_{\text{Assuming all of these are true (conjunction)...}} \vdash \underbrace{E; F; G; H}_{\text{We can conclude that one of these are true (disjunction)...}}$$

$$\frac{\Gamma; A \vdash \Delta}{\Gamma \vdash \Delta; \neg A} \qquad \frac{\Gamma; \text{true} \vdash \Delta}{\Gamma \vdash \Delta; \text{false}} \qquad \frac{\Gamma; \text{false} \vdash \Delta}{\Gamma \vdash \Delta; \text{true}}$$

$$\frac{\Gamma \vdash \Delta; A}{\Gamma; \neg A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta; \text{true}}{\Gamma; \text{false} \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta; \text{false}}{\Gamma; \text{true} \vdash \Delta}$$

Multiple conclusion sequent calculi

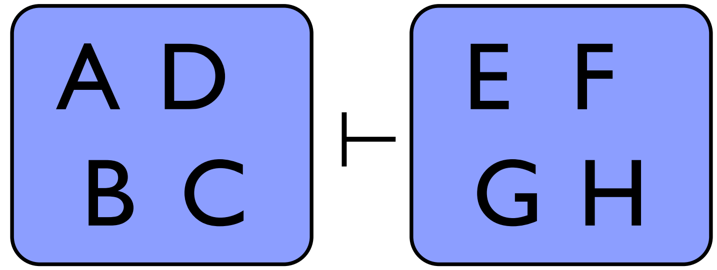
$$\underbrace{A; B; C; D}_{\text{Assuming all of these are true (conjunction)...}} \vdash \underbrace{E; F; G; H}_{\text{We can conclude that one of these are true (disjunction)...}}$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{id}}{\varnothing_a \vdash A; \neg A}\,\neg R}{\varnothing_a \vdash A \vee \neg A}\,\vee R$$

Display logic: Extend negation to contexts

$A; B; C; D \vdash E; F; G; H$

$A; D; C; B \vdash G; E; H; F$



$\Gamma; \Delta \vdash \Psi$

$\Gamma \vdash \#\Delta; \Psi$

$\Gamma \vdash \Delta; \Psi$

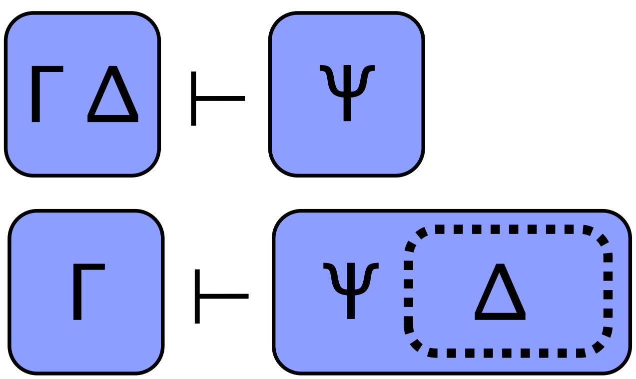
$\Gamma; \#\Delta \vdash \Psi$

$\Gamma \vdash \Psi; \Delta$

$\Gamma \vdash \Delta$

$\#\Delta \vdash \#\Gamma$

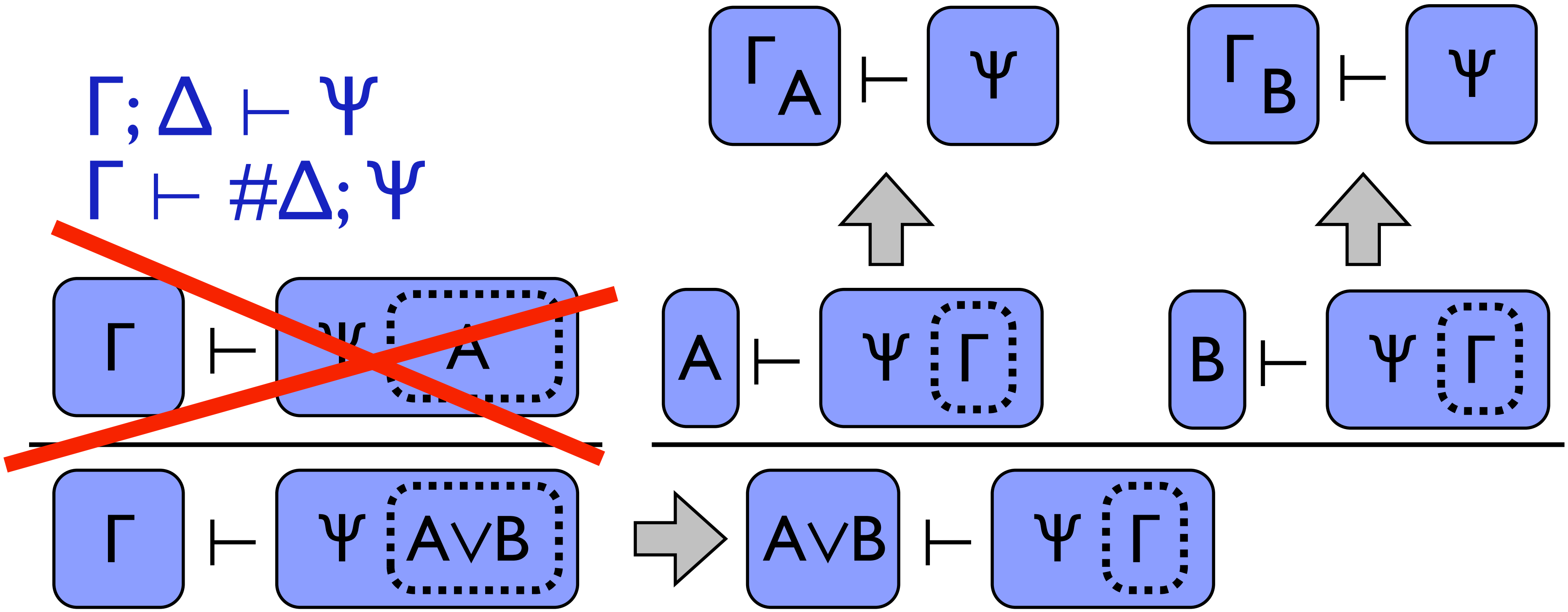
$\#\#\Gamma \vdash \Delta$



Display logic:
Extend negation to contexts

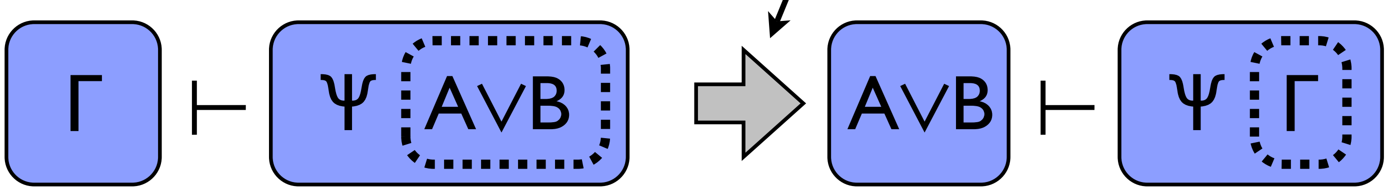
$\Gamma; \Delta \vdash \Psi$

$\Gamma \vdash \#\Delta; \Psi$

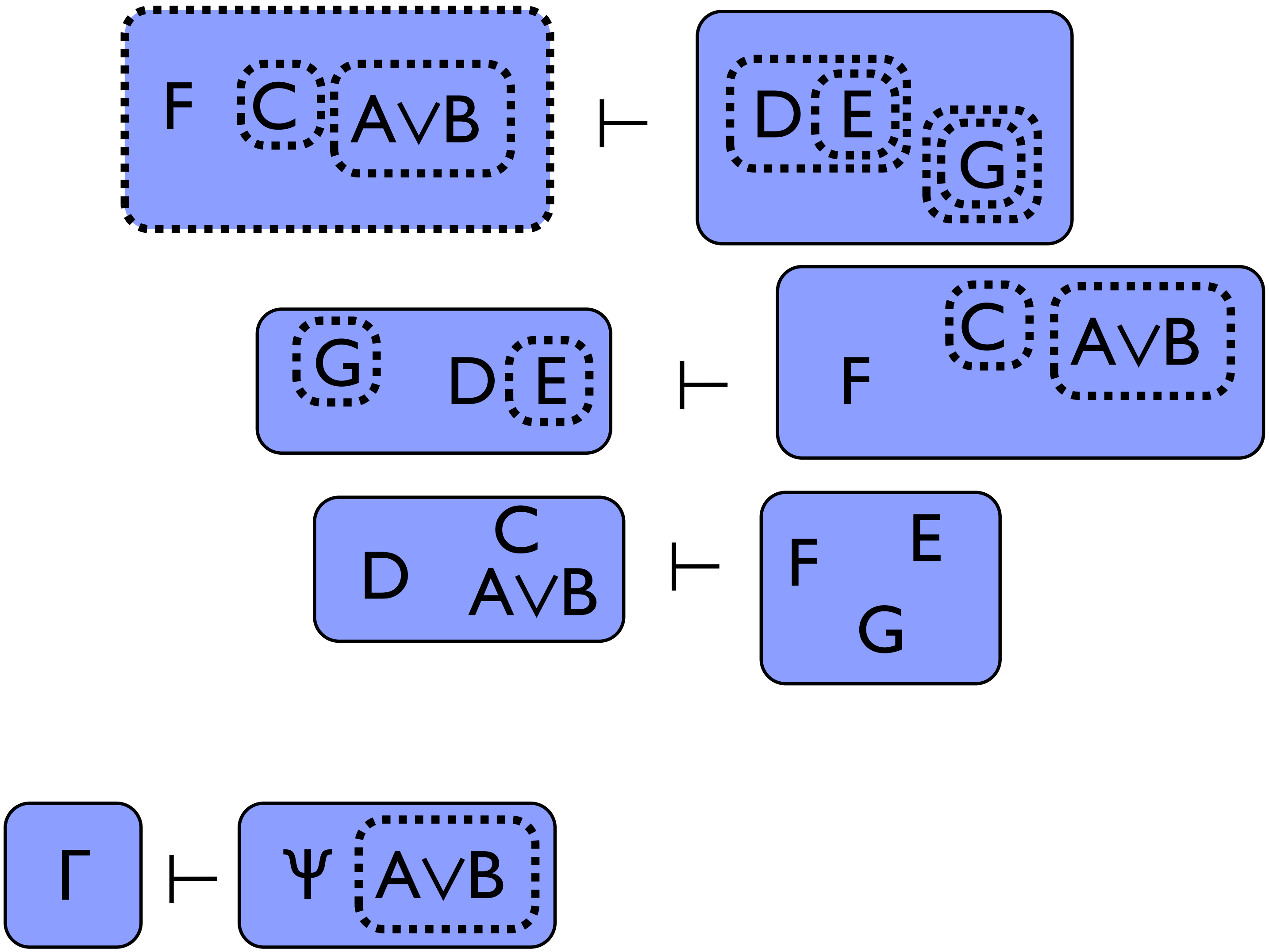


Display logic: Extend negation to contexts

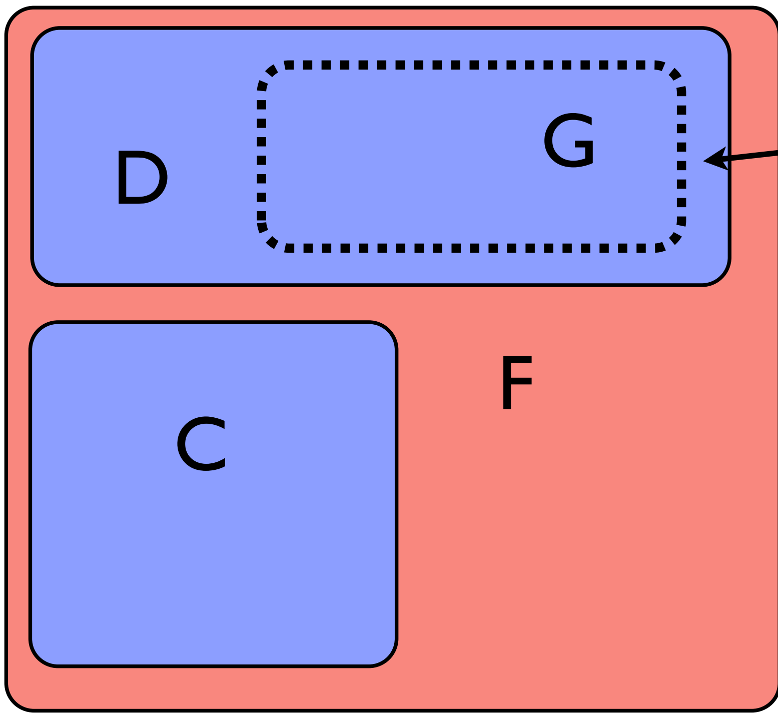
display theorem says this always works



Display logic is overkill



Display logic is ¬overkill



negated contexts will be stuck in the same bunch they are formed in

Classical BI

Straightforward BI-like display logic with both forms of negation (the additive one and the multiplicative one)

$$\frac{\sharp A \vdash \Gamma}{\neg A \vdash \Gamma} \qquad \frac{\Gamma \vdash \sharp A}{\Gamma \vdash \neg A} \qquad \frac{A ; B \vdash \Gamma}{A \wedge B \vdash \Gamma} \qquad \frac{\Delta \vdash A \quad \Gamma \vdash B}{\Delta ; \Gamma \vdash A \wedge B}$$

$$\frac{\flat A \vdash \Gamma}{\sim A \vdash \Gamma} \qquad \frac{\Gamma \vdash \flat A}{\Gamma \vdash \sim A} \qquad \frac{A , B \vdash \Gamma}{A * B \vdash \Gamma} \qquad \frac{\Delta \vdash A \quad \Gamma \vdash B}{\Delta , \Gamma \vdash A * B}$$

$$\frac{\Gamma \vdash A ; B}{\Gamma \vdash A \vee B} \qquad \frac{A \vdash \Delta \quad B \vdash \Gamma}{A \vee B \vdash \Delta ; \Gamma} \qquad \frac{\Gamma ; A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Delta \vdash A \quad B \vdash \Gamma}{A \Rightarrow B \vdash \# \Delta ; \Gamma}$$

$$\frac{\Gamma \vdash A , B}{\Gamma \vdash A \overset{*}{\vee} B} \qquad \frac{A \vdash \Delta \quad B \vdash \Gamma}{A \overset{*}{\vee} B \vdash \Delta , \Gamma} \qquad \frac{\Gamma , A \vdash B}{\Gamma \vdash A \mathrel{-\!\!*} B} \qquad \frac{\Delta \vdash A \quad B \vdash \Gamma}{A \mathrel{-\!\!*} B \vdash \flat \Delta , \Gamma}$$

Why are both $\flat$ and $\sharp$ needed?

Imagine there's only $\sharp$ that acts as both

Using display and commutative monoid equations, we can get a contradiction that " $\varnothing_a, \varnothing_a$" implies "$\varnothing_m$", which is wrong

$\varnothing_a \vdash \varnothing_a$

$\sharp\sharp\varnothing_a \vdash \varnothing_a$

$\varnothing_a; \sharp\sharp\varnothing_a \vdash \varnothing_a$

$\varnothing_a \vdash \sharp\varnothing_a; \varnothing_a$

$\varnothing_a \vdash \sharp\varnothing_a$

$\varnothing_a \vdash \sharp\varnothing_a, \varnothing_m$

$\varnothing_a, \varnothing_a \vdash \varnothing_m$

Classical BI (Figure 5 in CBI paper)

$$\boxed{\sim\neg A} \vdash \boxed{\neg\sim A}$$

Classical BI

$$\dfrac{\boxed{\sim\neg A} \;\vdash\; \boxed{\boxed{\sim A}}}{\boxed{\sim\neg A} \;\vdash\; \boxed{\neg\sim A}}\;\neg R$$

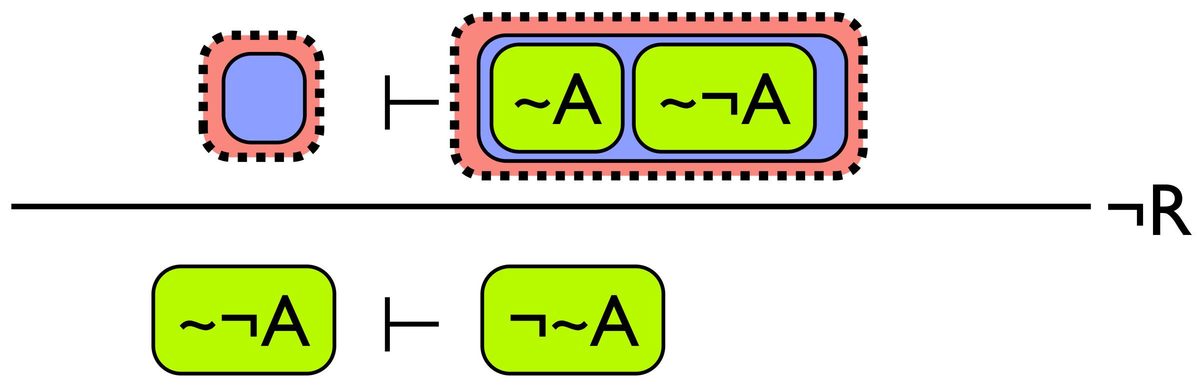
Classical BI

$$\dfrac{\boxed{\sim\neg A} \;\vdash\; \boxed{\boxed{\sim A}}}{\boxed{\sim\neg A} \;\vdash\; \boxed{\neg\sim A}}\;\neg R$$

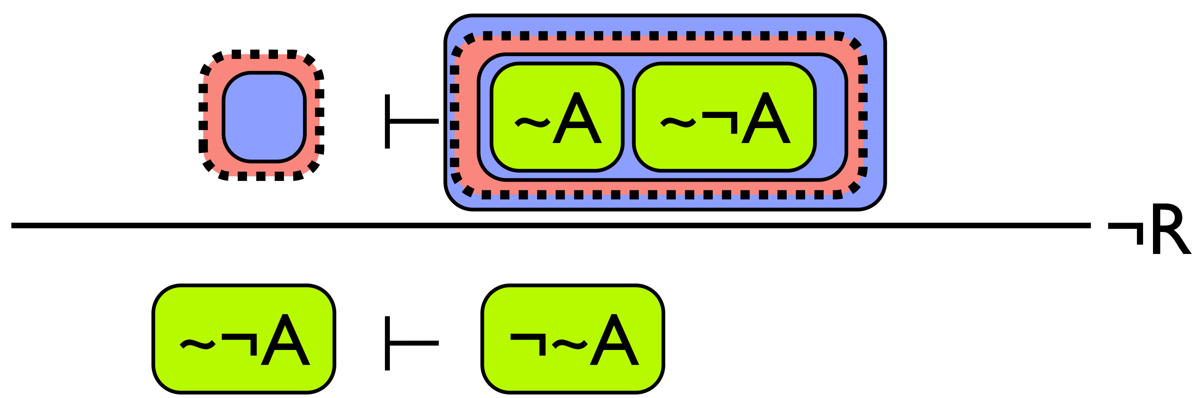
Classical BI

$$\frac{\sim A, \sim\neg A \vdash \quad}{\sim\neg A \vdash \neg\sim A}\;\neg R$$

Classical BI



Classical BI



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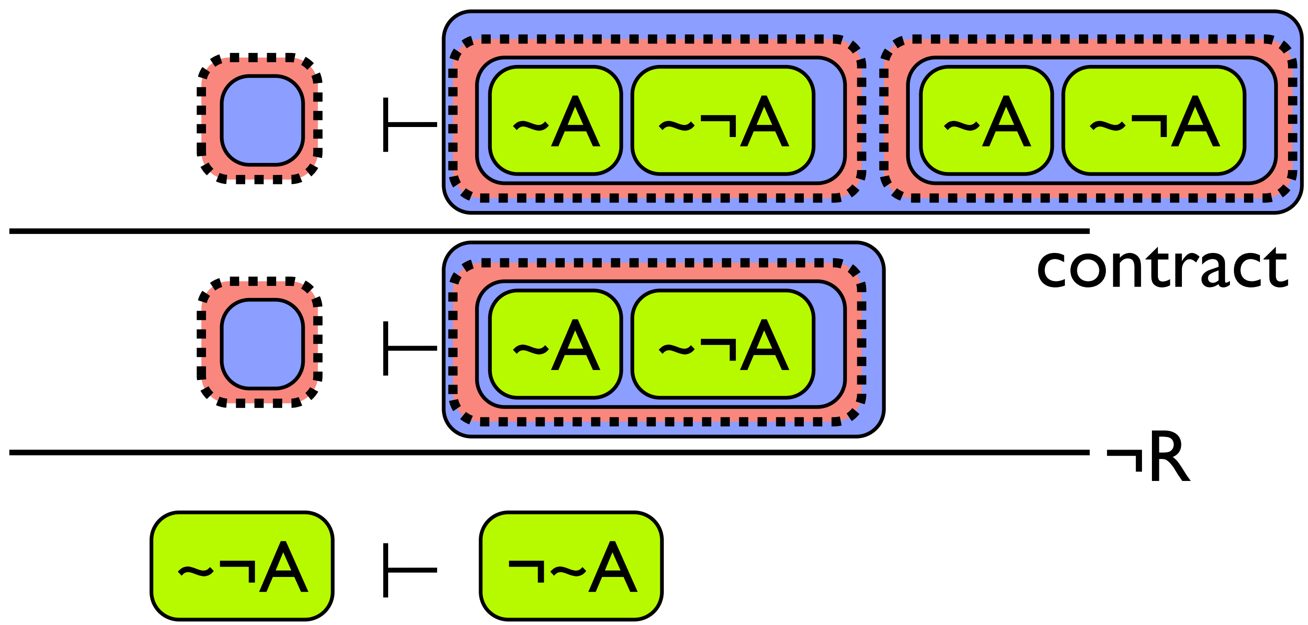
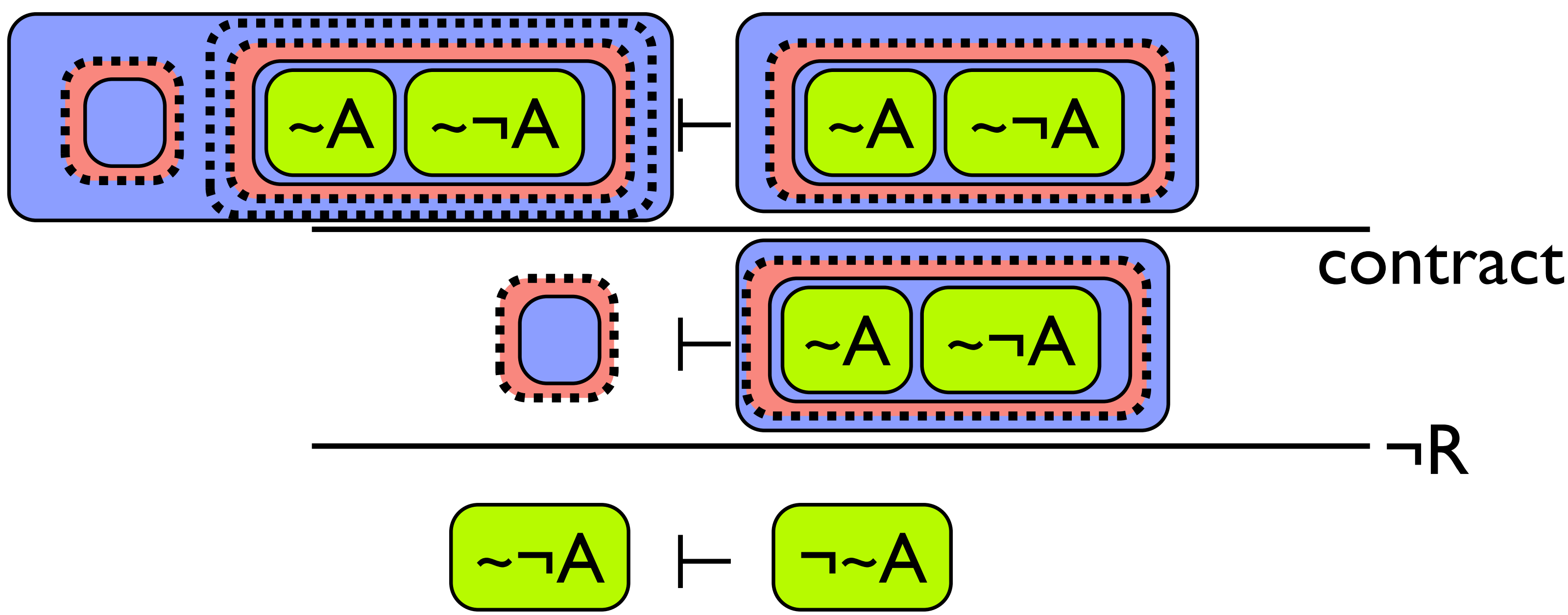


Classical BI



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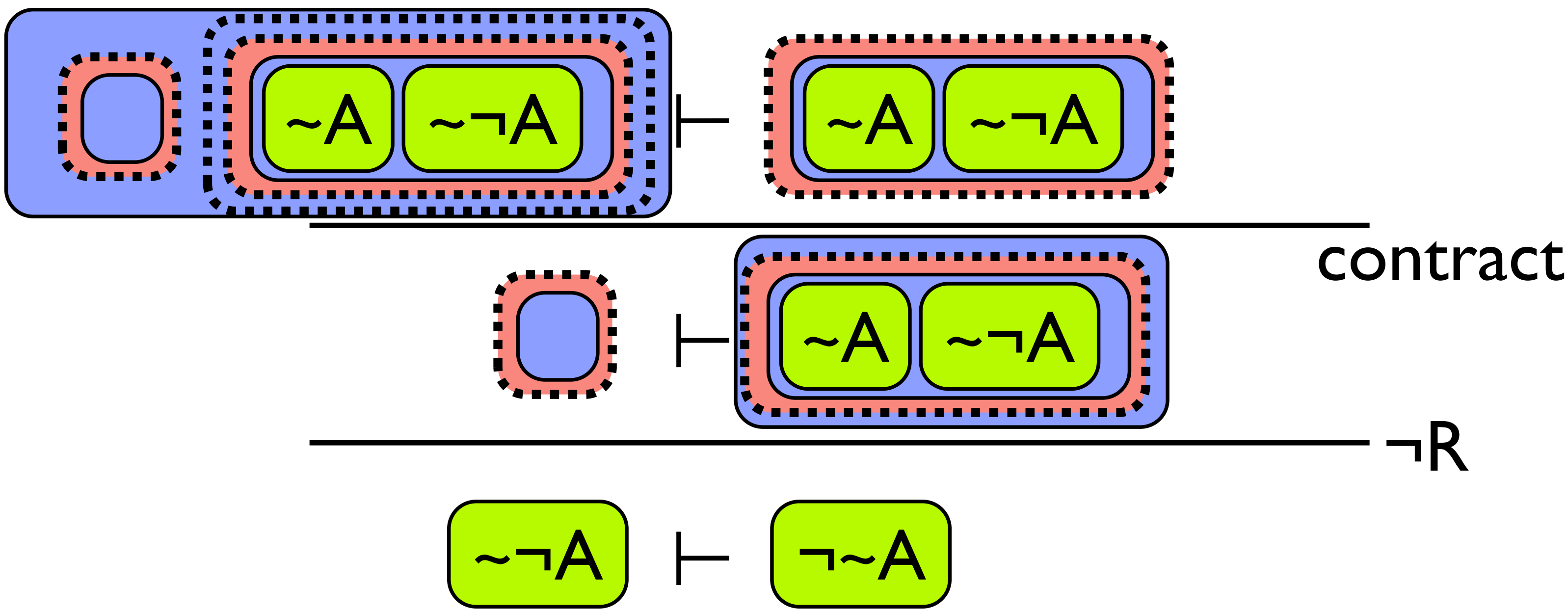

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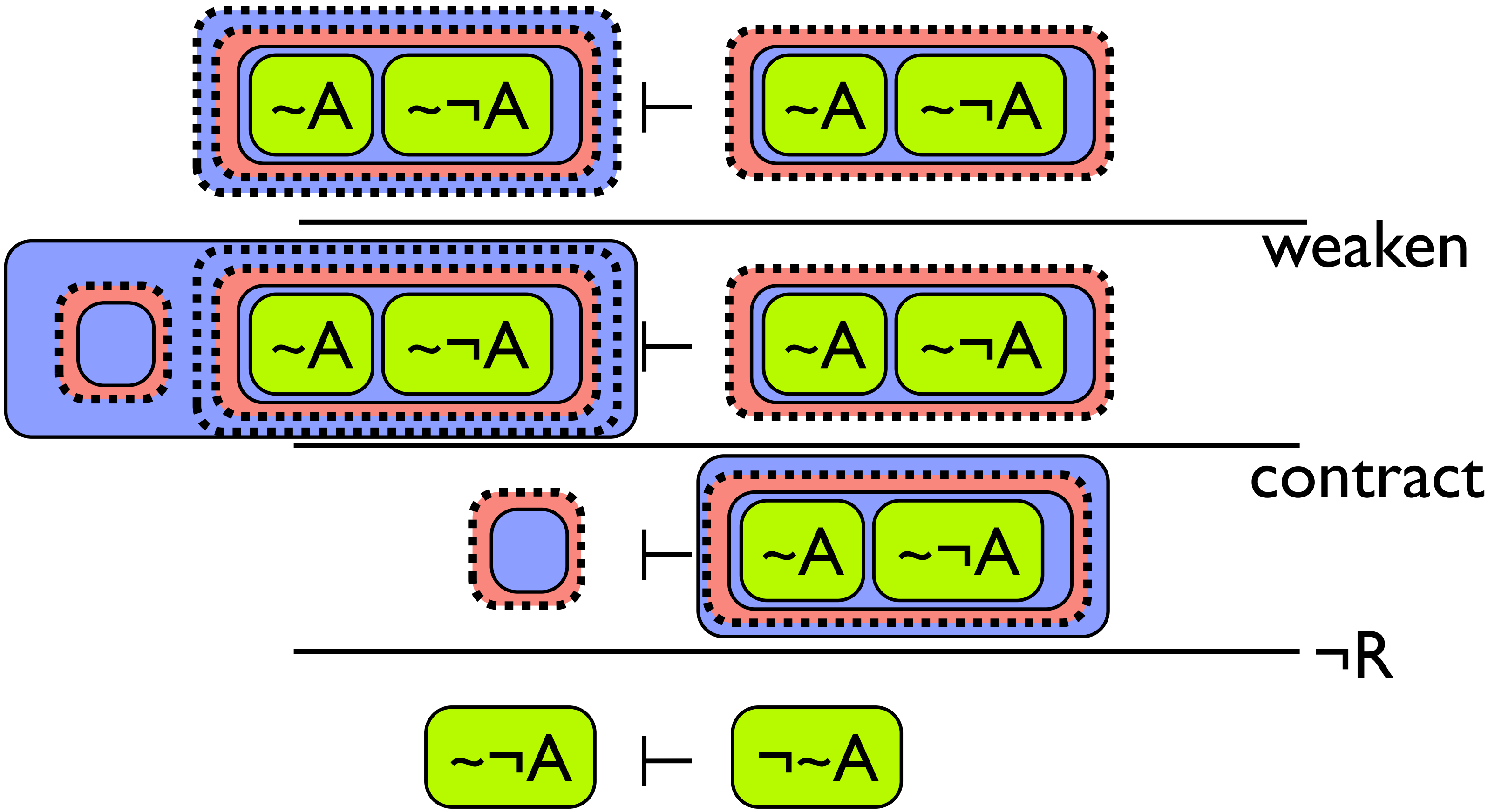
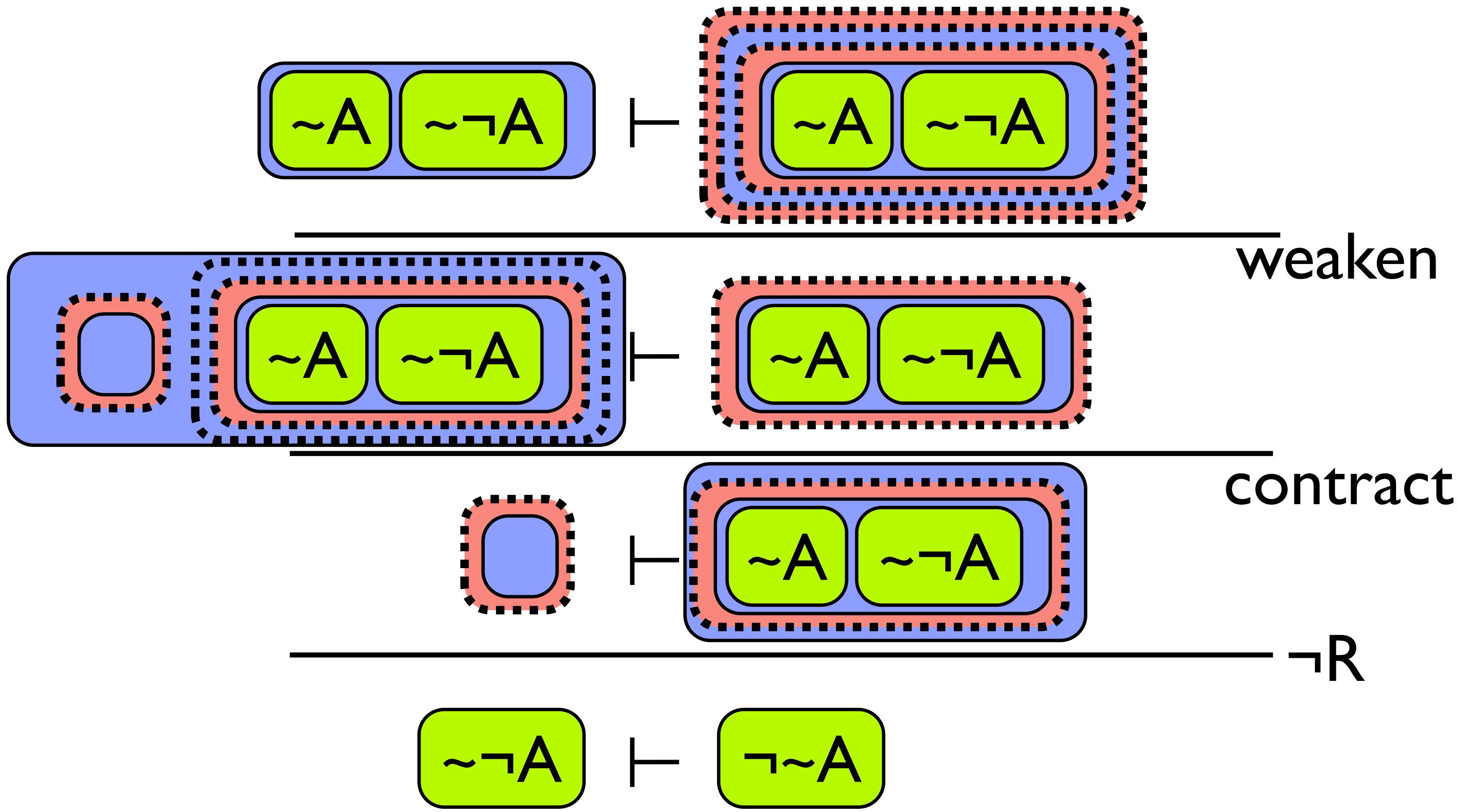
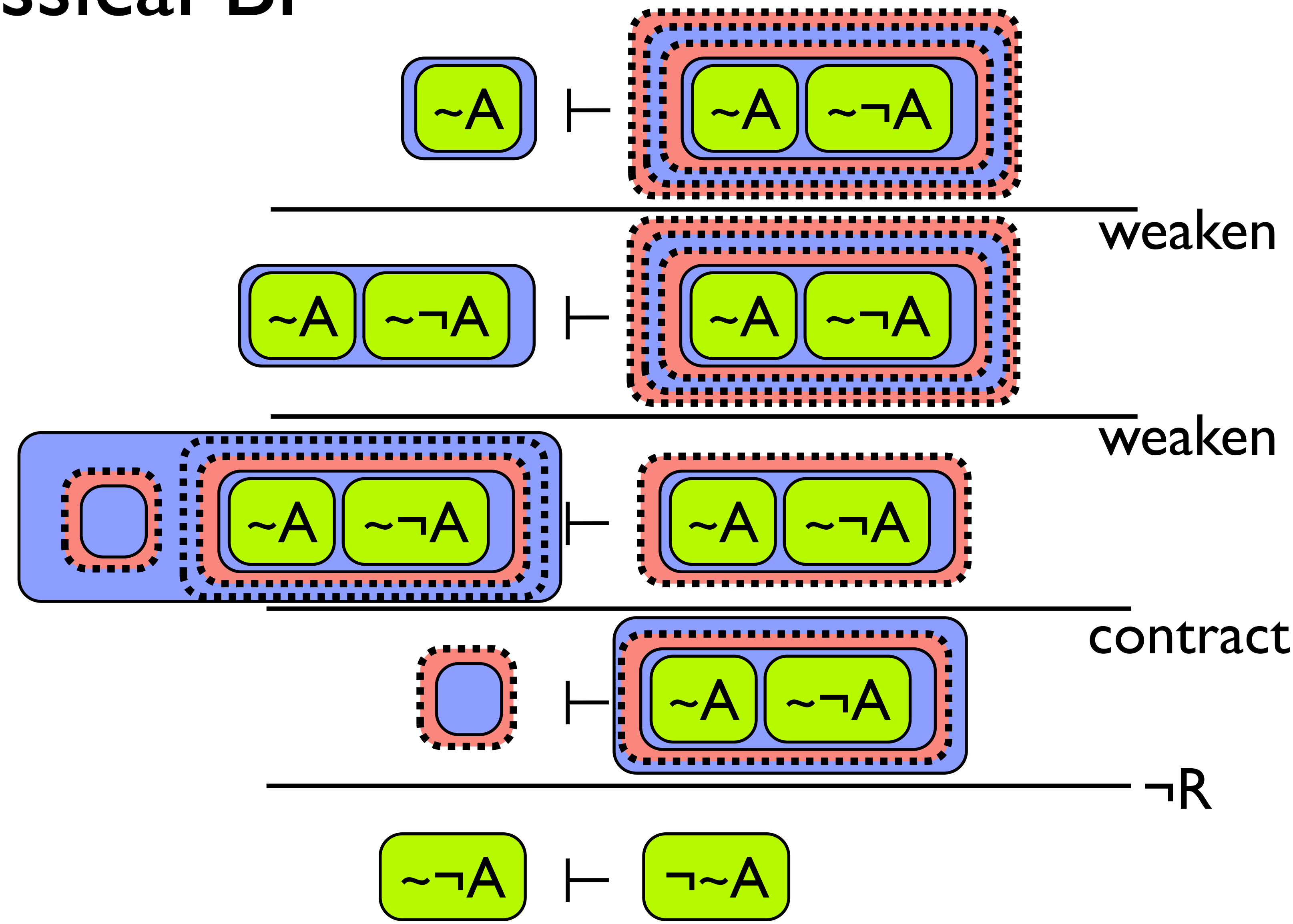


Classical BI



Classical BI



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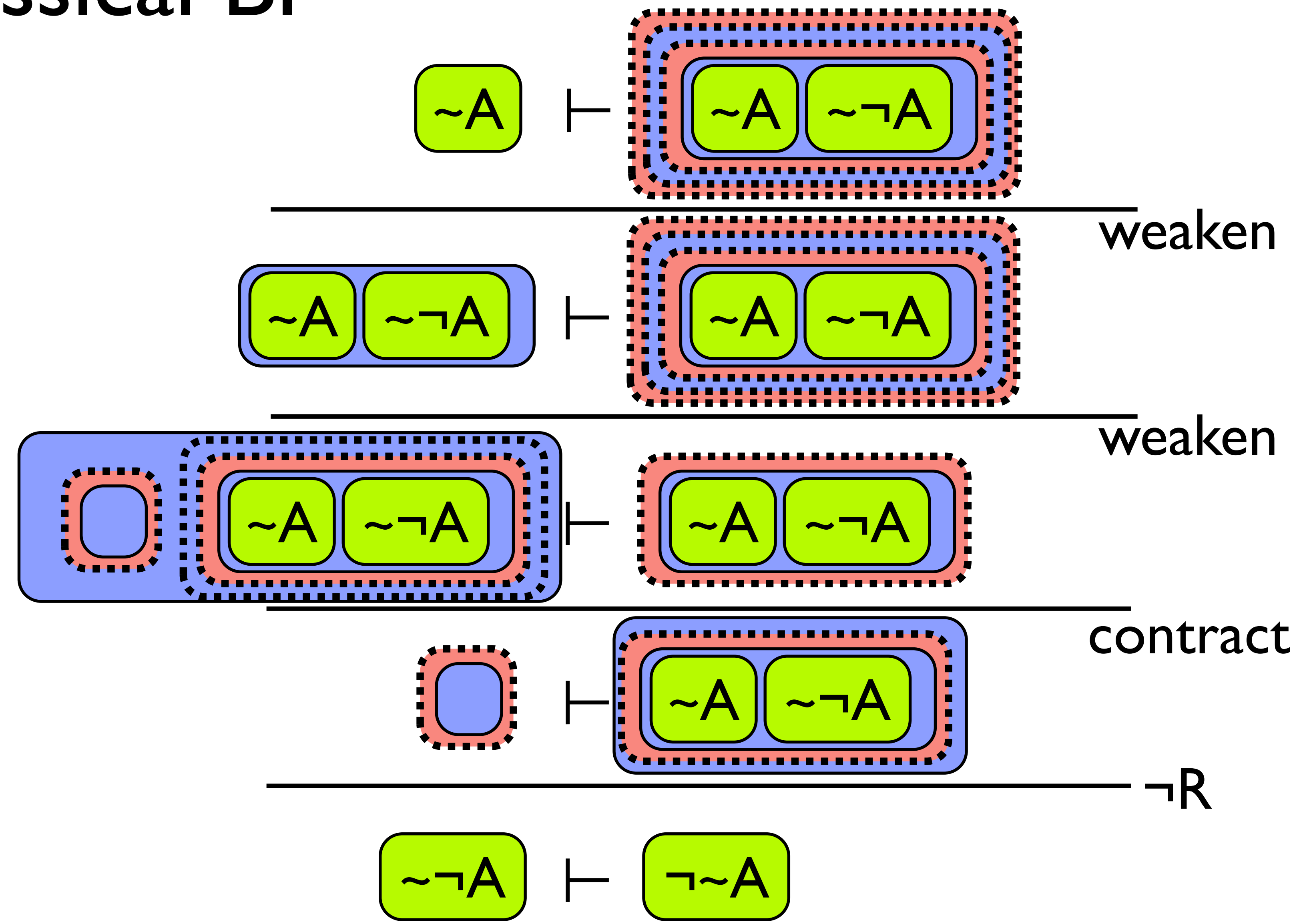
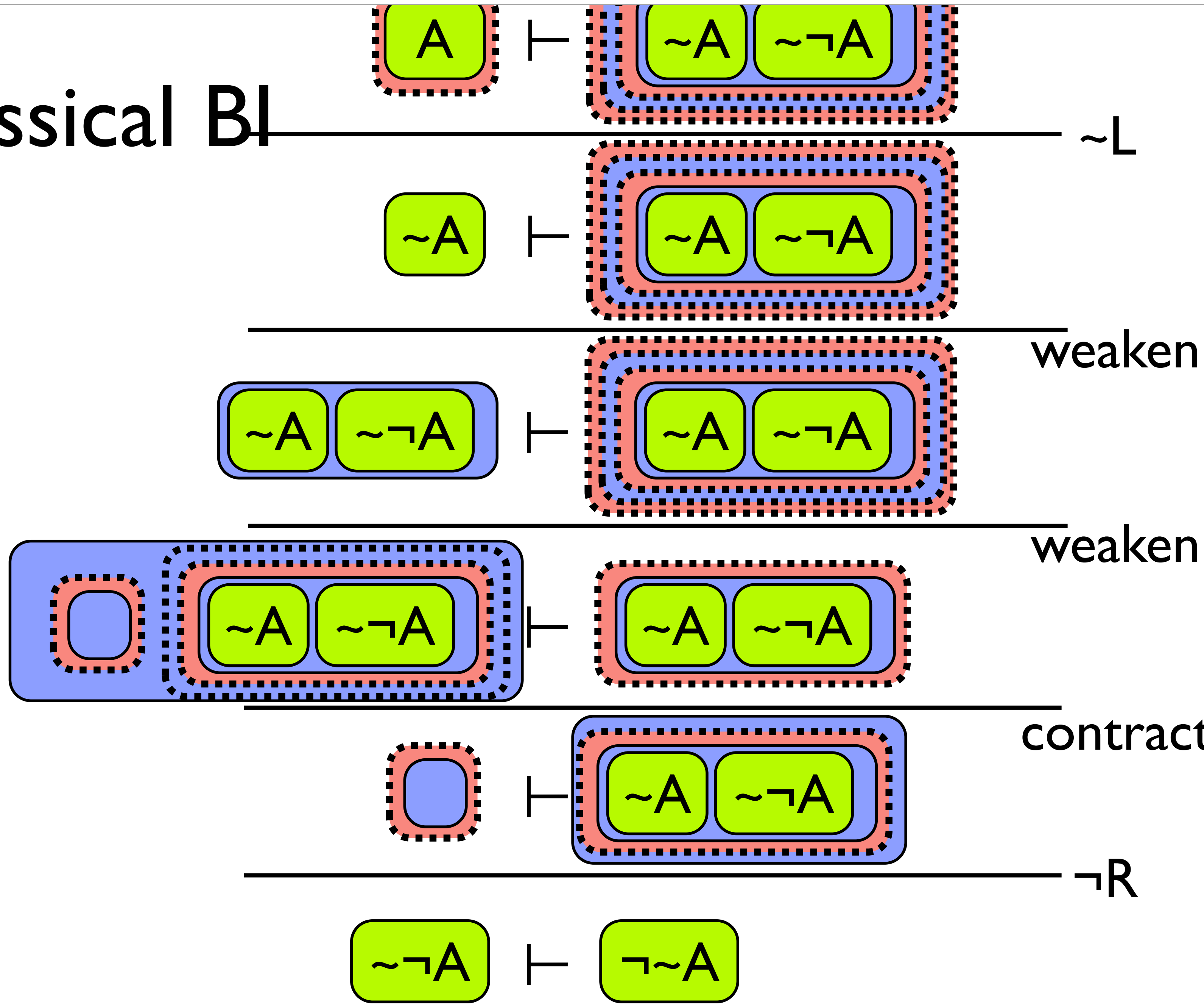


Classical BI



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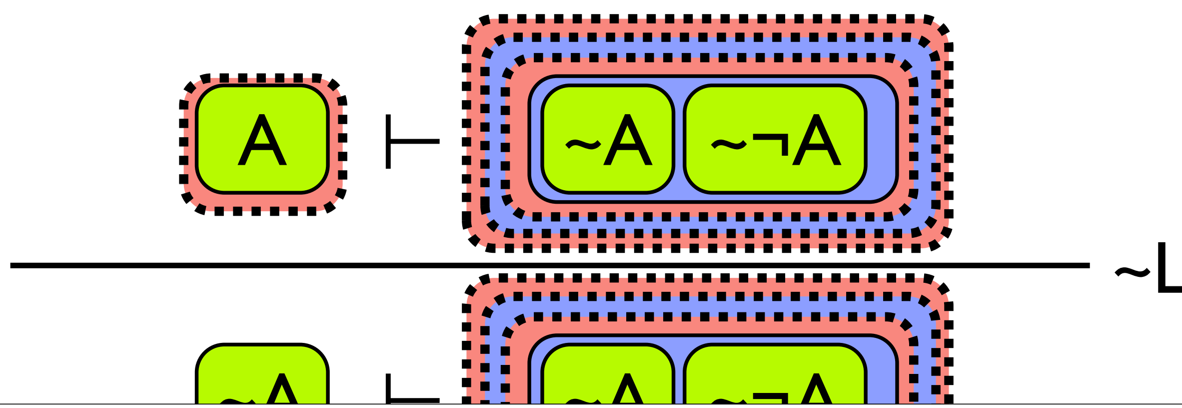
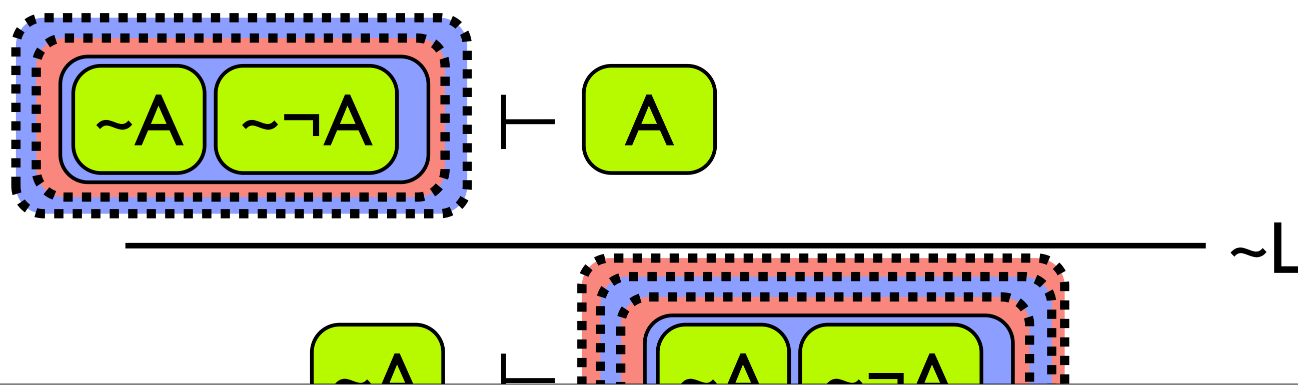


Classical BI



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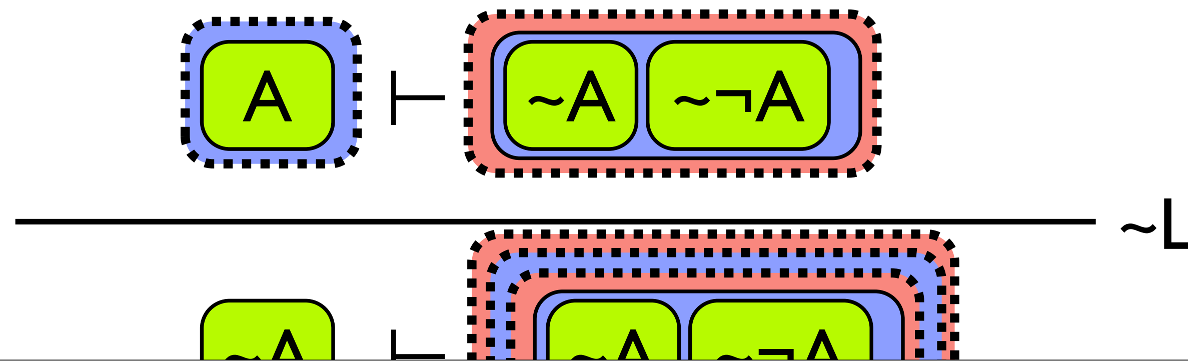

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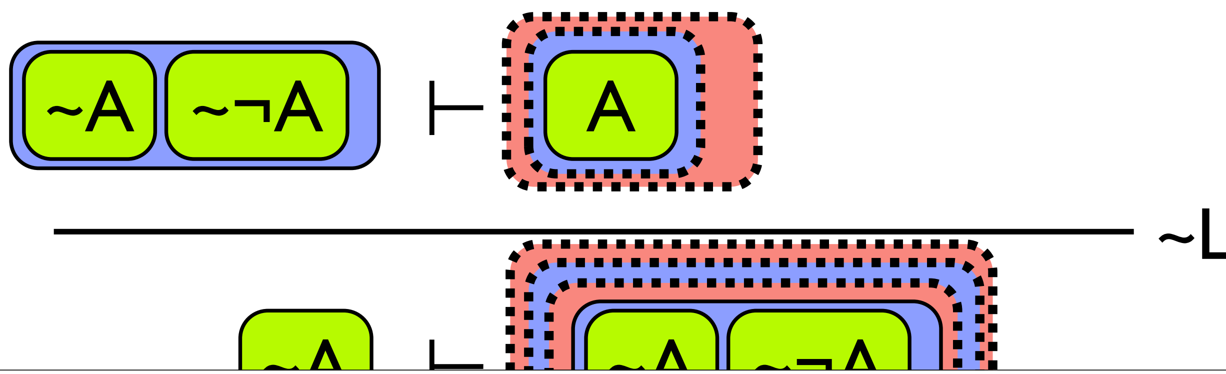

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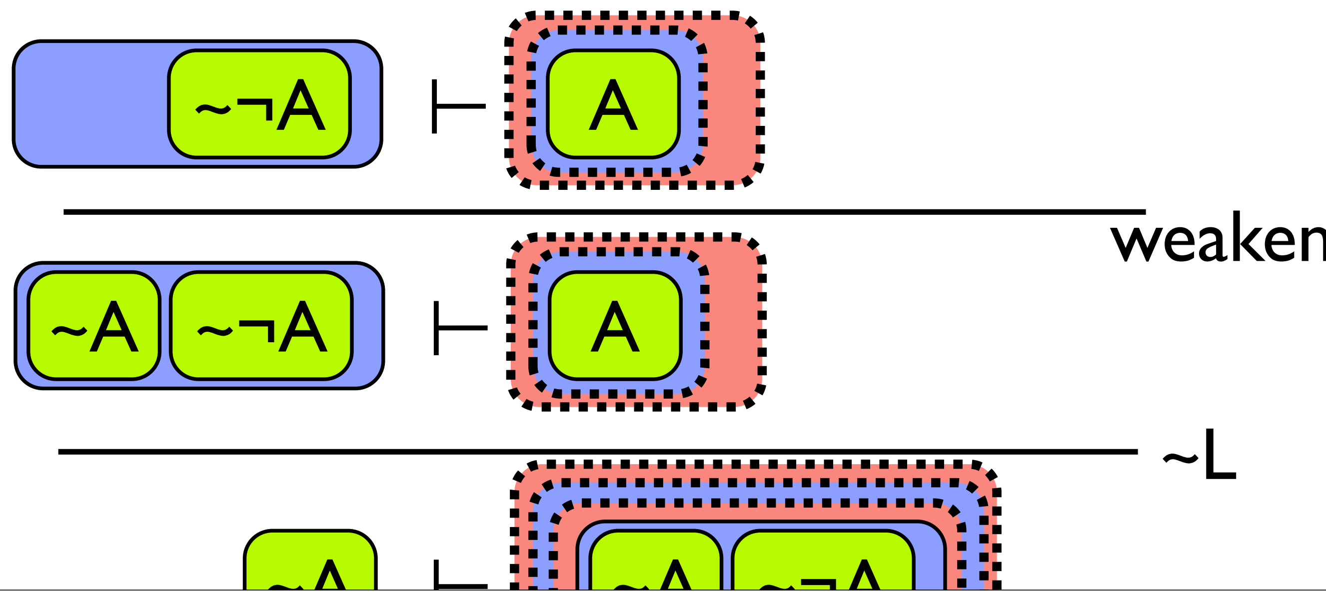

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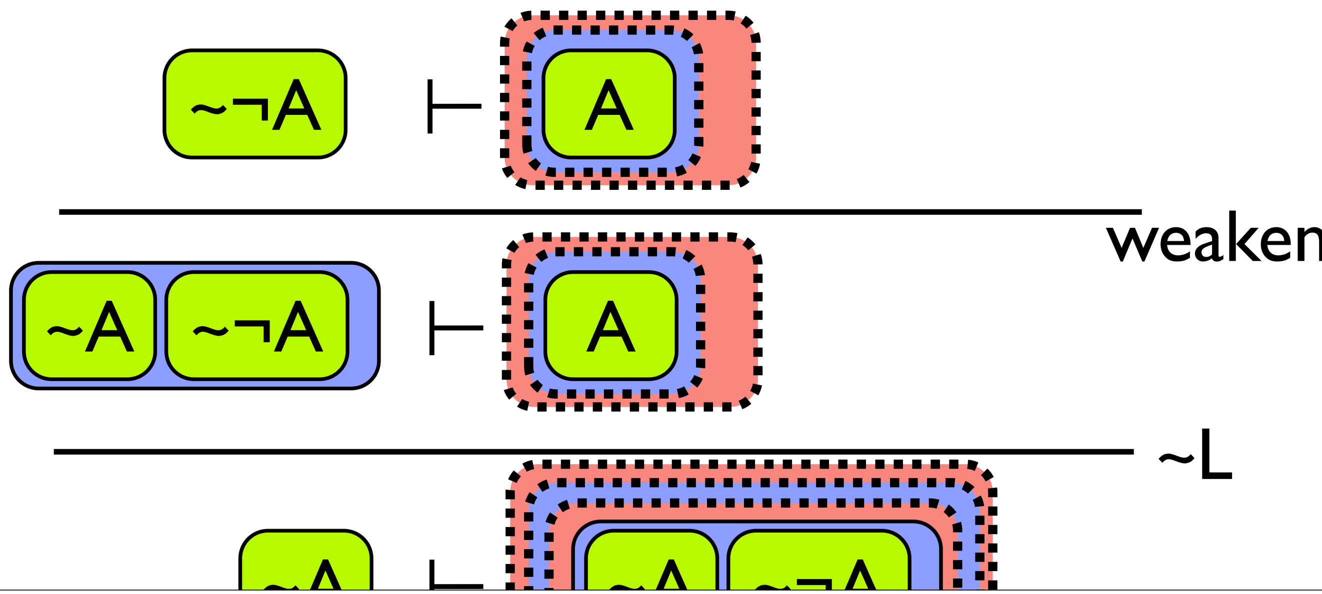

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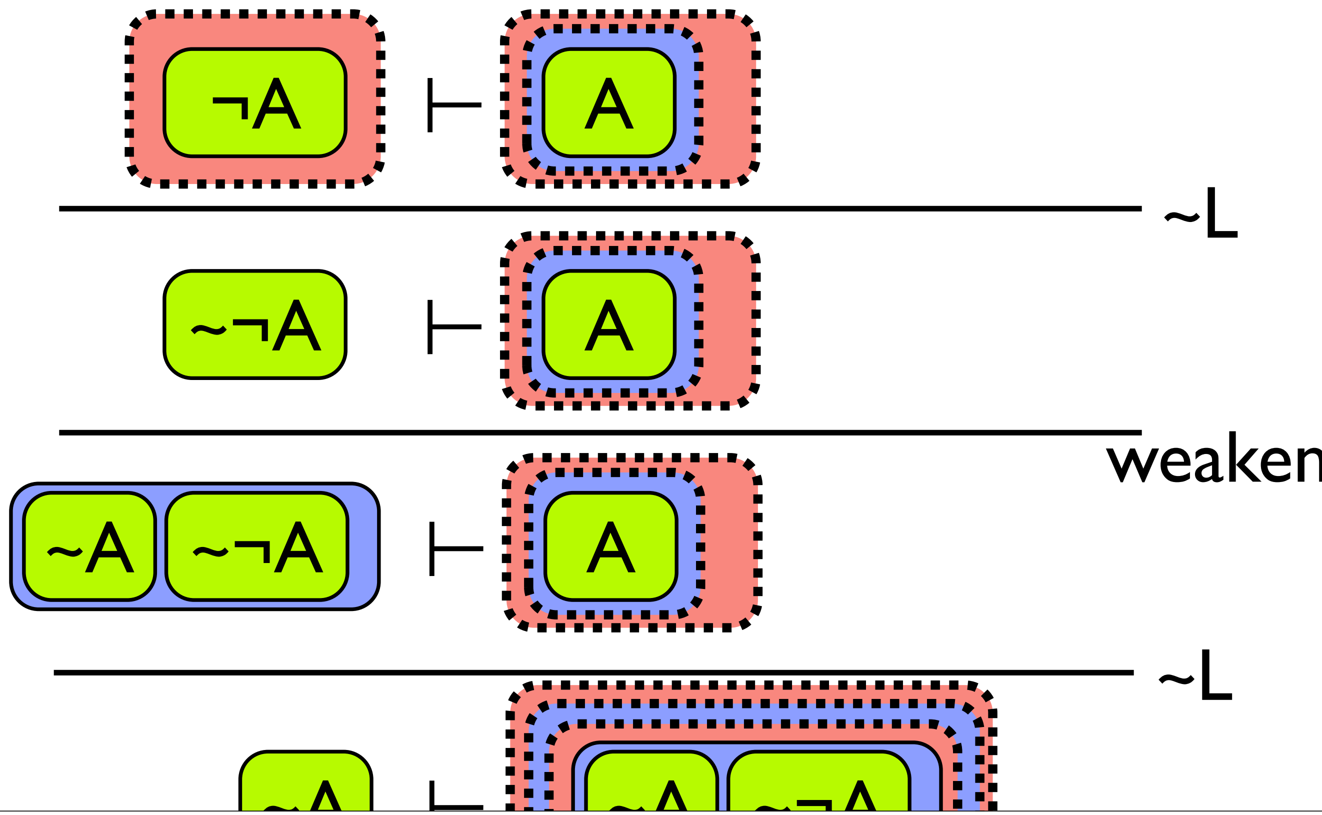

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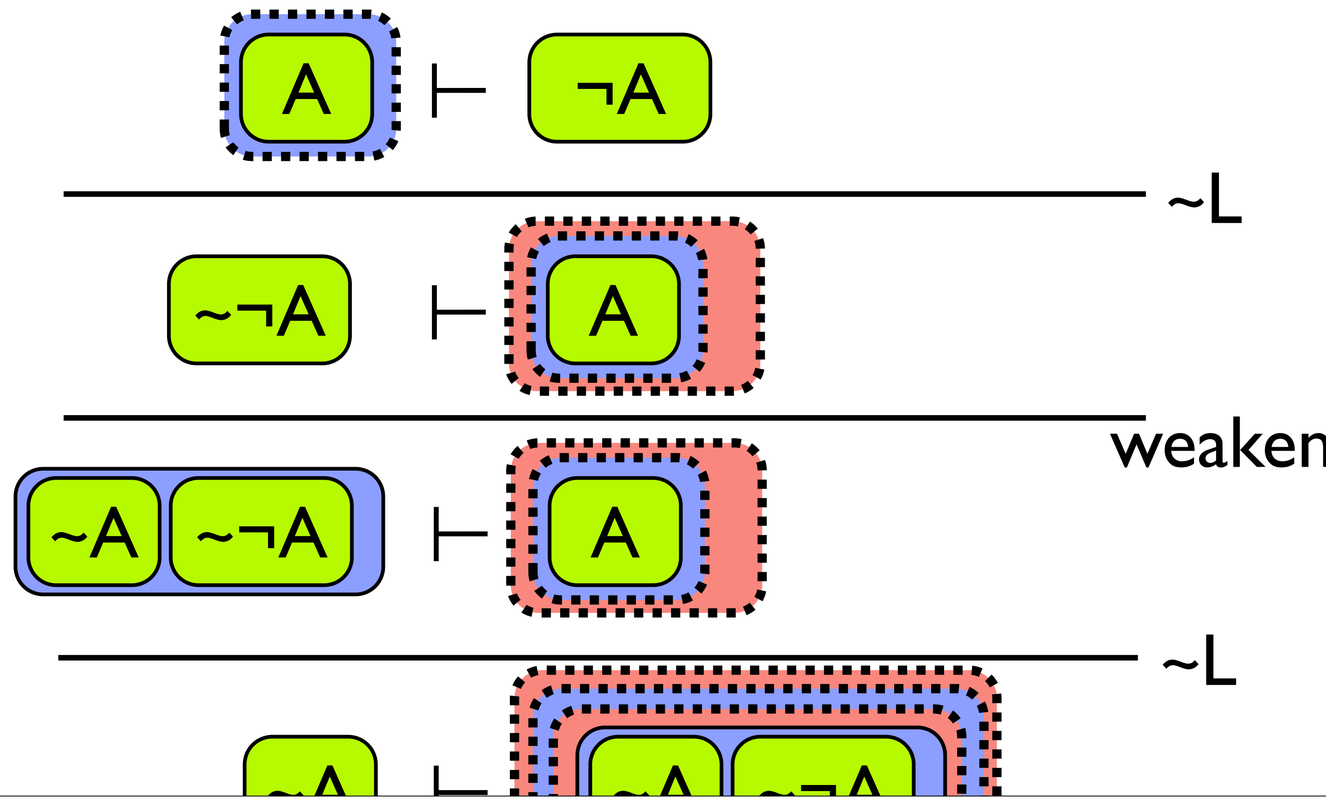

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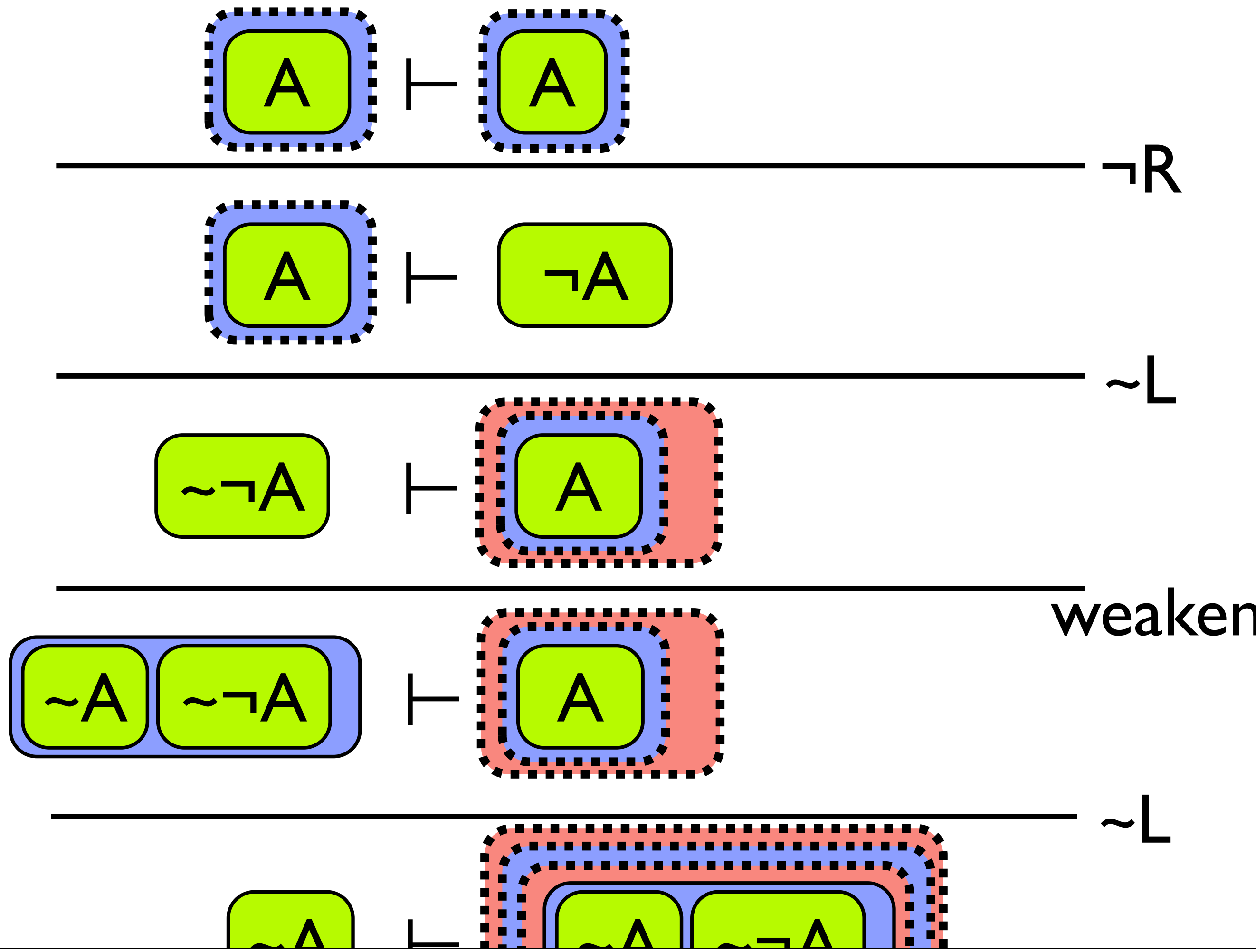
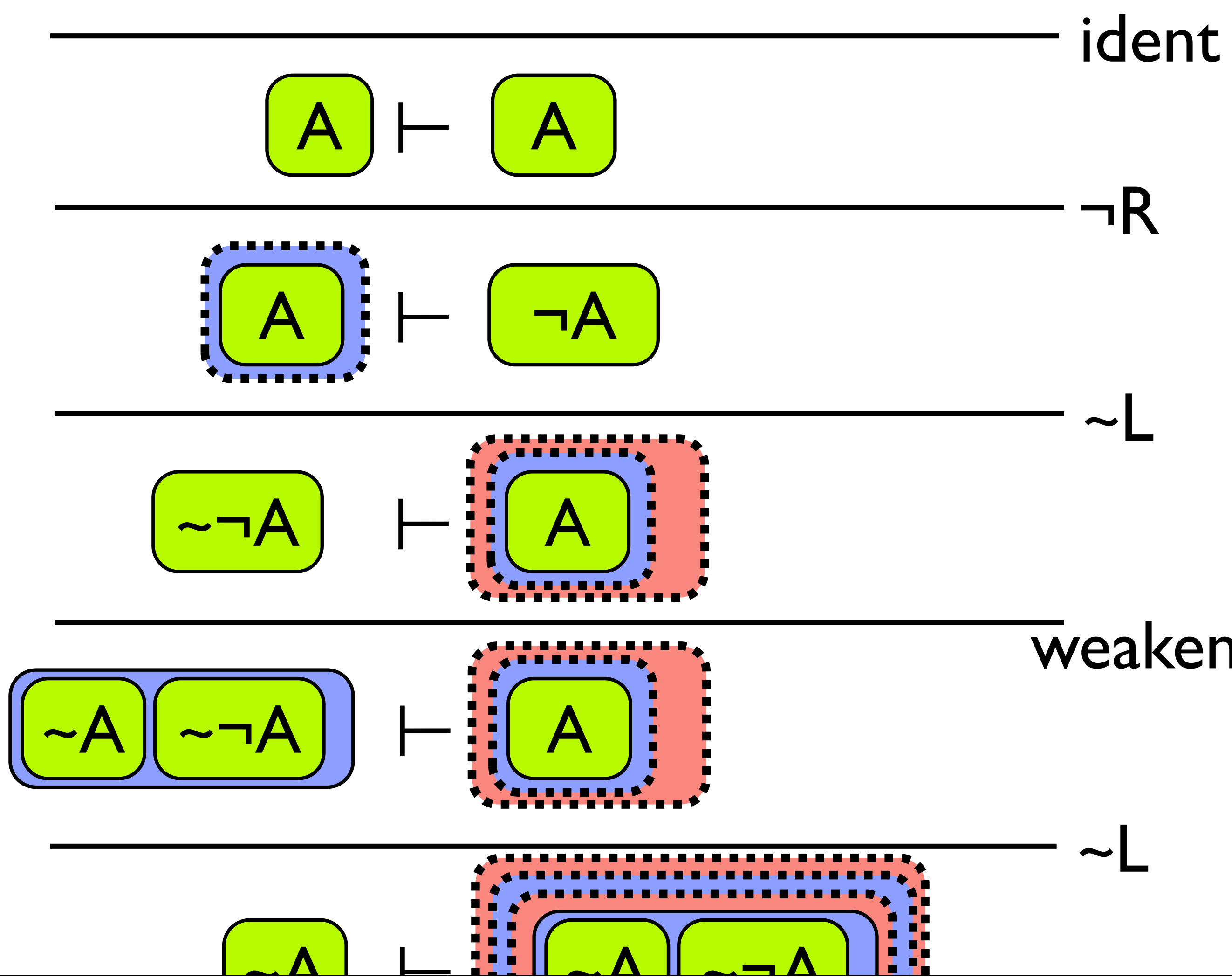


Classical BI



Classical BI to Boolean BI

All of these assumptions $\vdash$

$\vdash$ One of these conclusions

All of these resources $\vdash$

$\vdash$ ...

Classical BI to Boolean BI

All of these assumptions $\vdash$

All of these resources $\vdash$

$$\Gamma = A \mid \varnothing_a \mid \Gamma ; \Gamma \mid \varnothing_m \mid \Gamma , \Gamma$$

Classical BI to Boolean BI

All of these assumptions $\vdash$

$\vdash$ One of these conclusions

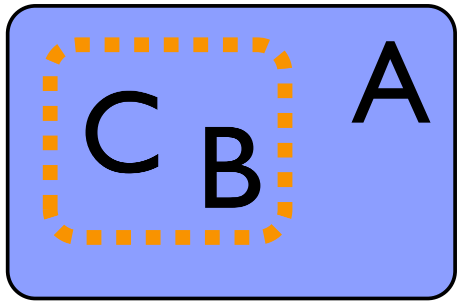
All of these resources $\vdash$

$$\Gamma = A \mid \emptyset_a \mid \Gamma ; \Gamma \mid \emptyset_m \mid \Gamma , \Gamma$$

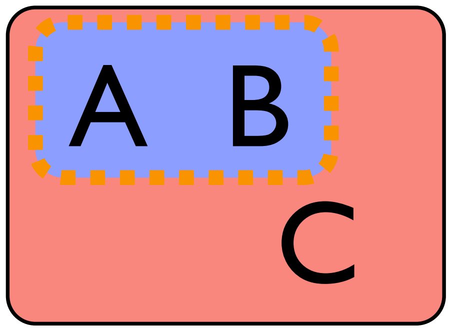
$$\Delta = A \mid \emptyset_a \mid \Delta ; \Delta$$

Classical BI to Boolean BI

$A; \#(B; C)$



$\#(A; B), C$

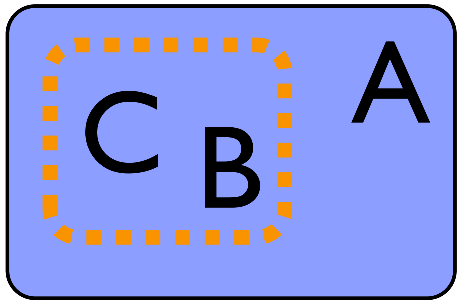


$$\Gamma = A \mid \varnothing_a \mid \Gamma ; \Gamma \mid \varnothing_m \mid \Gamma , \Gamma \mid \#\Delta$$

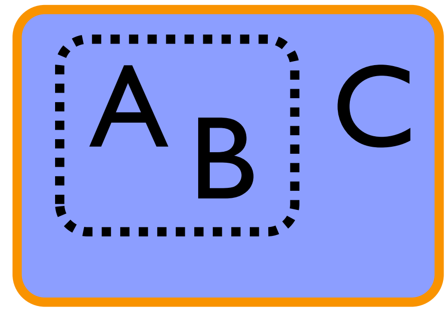
$$\Delta = A \mid \varnothing_a \mid \Delta ; \Delta$$

Classical BI to Boolean BI

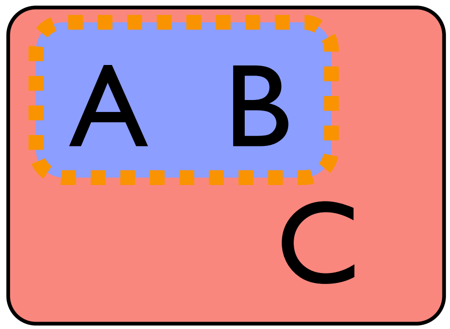
A; #(B; C)



#(A; B); C



#(A; B), C

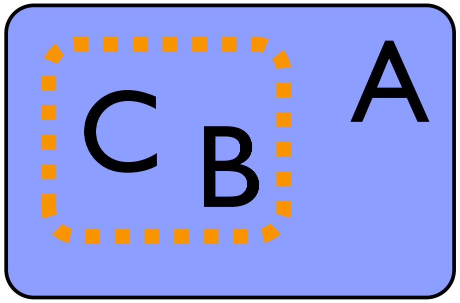


$$\Gamma = A \mid \varnothing_a \mid \Gamma ; \Gamma \mid \varnothing_m \mid \Gamma , \Gamma \mid \#\Delta$$

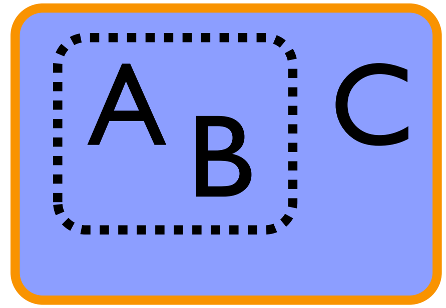
$$\Delta = A \mid \varnothing_a \mid \Delta ; \Delta \mid \#\Gamma$$

Classical BI to Boolean BI

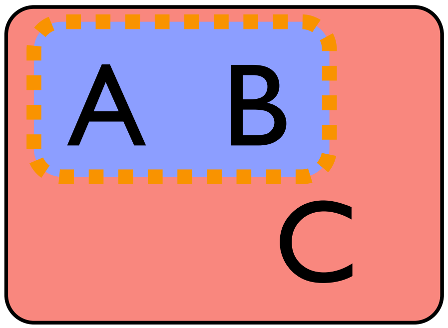
$A; \#(B; C)$



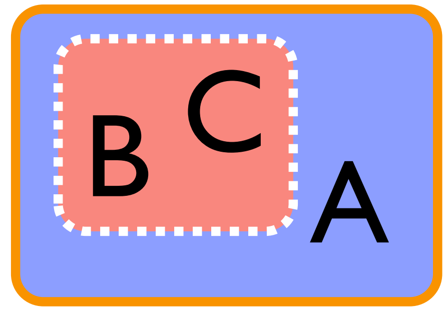
$\#(A; B); C$



$\#(A; B), C$



$(B, C) \multimap A$

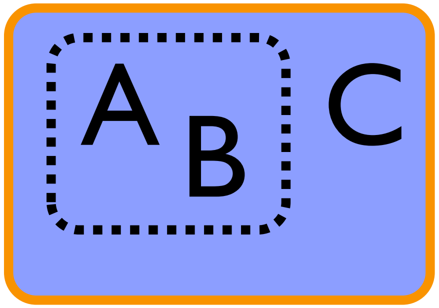


$$\Gamma = A \mid \varnothing_a \mid \Gamma ; \Gamma \mid \varnothing_m \mid \Gamma , \Gamma \mid \#\Delta$$

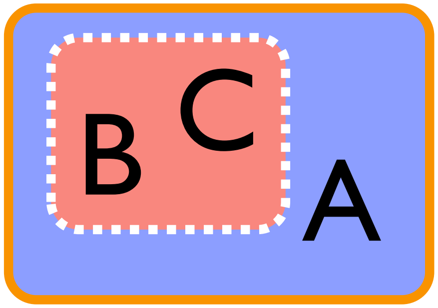
$$\Delta = A \mid \varnothing_a \mid \Delta ; \Delta \mid \#\Gamma \mid \Gamma \multimap \Delta$$

Classical BI to Boolean BI

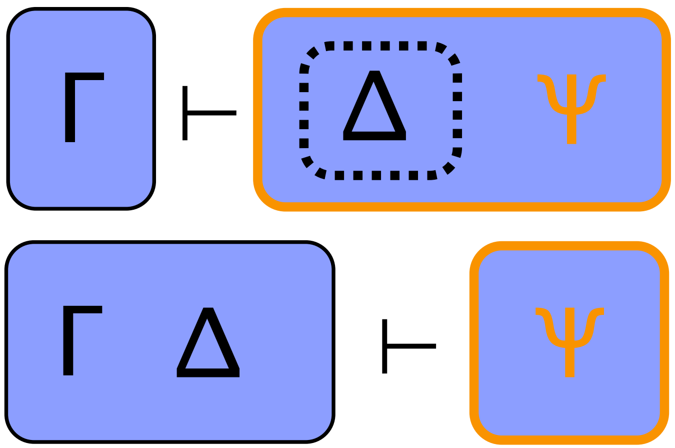
“Anti-contexts” can be weakened away
(same as for weakening on the left...)



“Holes” *can’t* be weakened away



Classical BI to Boolean BI

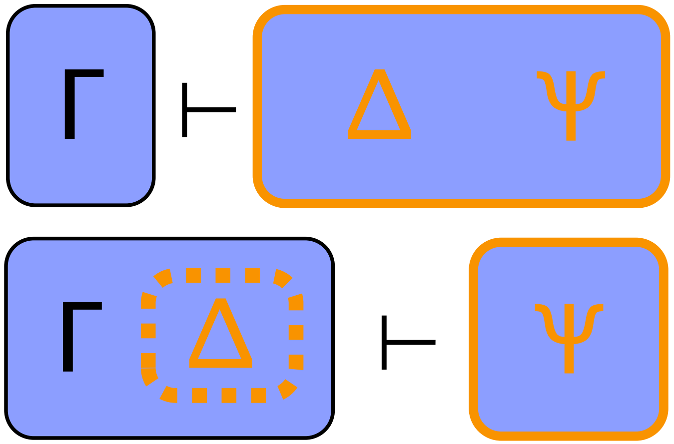


$\Gamma ; \Delta \vdash \Psi$

$\Gamma \vdash \#\Delta ; \Psi$

$\Delta ; \Gamma \vdash \Psi$

Classical BI to Boolean BI

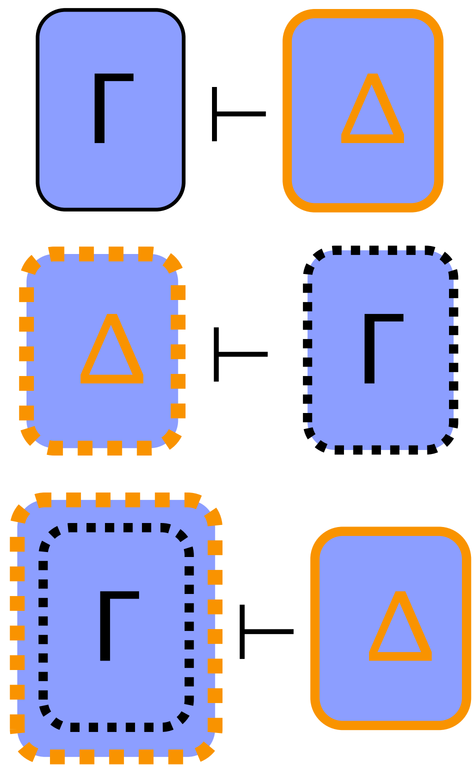


$\Gamma; \Delta \vdash \Psi$ $\qquad$ $\Gamma \vdash \Delta; \Psi$

$\Gamma \vdash \#\Delta; \Psi$ $\qquad$ $\Gamma; \#\Delta \vdash \Psi$

$\Delta; \Gamma \vdash \Psi$ $\qquad$ $\Gamma \vdash \Psi; \Delta$

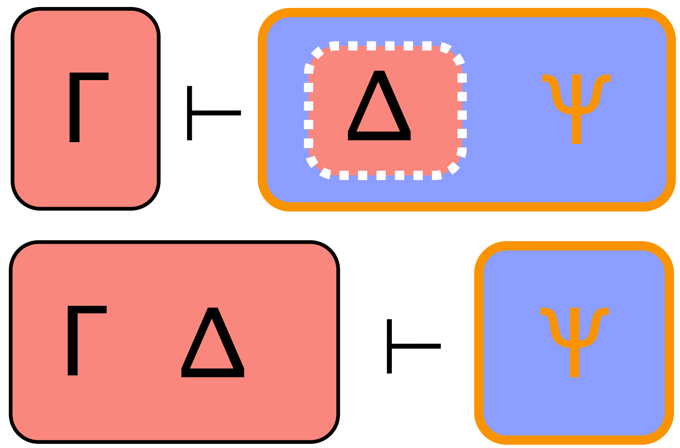
Classical BI to Boolean BI



$\Gamma; \Delta \vdash \Psi$	$\Gamma \vdash \Delta; \Psi$	$\Gamma \vdash \Delta$
$\Gamma \vdash \#\Delta; \Psi$	$\Gamma; \#\Delta \vdash \Psi$	$\#\Delta \vdash \#\Gamma$
$\Delta; \Gamma \vdash \Psi$	$\Gamma \vdash \Psi; \Delta$	$\#\#\Gamma \vdash \Delta$

Classical BI to Boolean BI

$$\Gamma, \Delta \vdash \Psi$$
$$\Gamma \vdash \Delta \multimap \Psi$$
$$\Delta, \Gamma \vdash \Psi$$



$$\Gamma ; \Delta \vdash \Psi$$
$$\Gamma \vdash \#\Delta ; \Psi$$
$$\Delta ; \Gamma \vdash \Psi$$

$$\Gamma \vdash \Delta ; \Psi$$
$$\Gamma ; \#\Delta \vdash \Psi$$
$$\Gamma \vdash \Psi ; \Delta$$

$$\Gamma \vdash \Delta$$
$$\#\Delta \vdash \#\Gamma$$
$$\#\#\Gamma \vdash \Delta$$

Classical BI to Boolean BI

$$\frac{\sharp A \vdash \Gamma}{\neg A \vdash \Gamma} \qquad \frac{\Gamma \vdash \sharp A}{\Gamma \vdash \neg A} \qquad \frac{A; B \vdash \Gamma}{A \wedge B \vdash \Gamma} \qquad \frac{\Delta \vdash A \quad \Gamma \vdash B}{\Delta; \Gamma \vdash A \wedge B}$$

$$\frac{\Gamma \vdash A; B}{\Gamma \vdash A \vee B} \qquad \frac{A \vdash \Delta \quad B \vdash \Gamma}{A \vee B \vdash \Delta; \Gamma} \qquad \frac{\Gamma; A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Gamma \vdash A \quad B \vdash \Delta}{A \Rightarrow B \vdash \#\Gamma; \Delta}$$

Boolean BI

$$\frac{\sharp A \vdash \Gamma}{\neg A \vdash \Gamma} \qquad \frac{\Gamma \vdash \sharp A}{\Gamma \vdash \neg A} \qquad \frac{A; B \vdash \Gamma}{A \wedge B \vdash \Gamma} \qquad \frac{\Delta \vdash A \quad \Gamma \vdash B}{\Delta; \Gamma \vdash A \wedge B}$$

$$\frac{A, B \vdash \Gamma}{A * B \vdash \Gamma} \qquad \frac{\Delta \vdash A \quad \Gamma \vdash B}{\Delta, \Gamma \vdash A * B}$$

$$\frac{\Gamma \vdash A; B}{\Gamma \vdash A \vee B} \qquad \frac{A \vdash \Delta \quad B \vdash \Gamma}{A \vee B \vdash \Delta; \Gamma} \qquad \frac{\Gamma; A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Gamma \vdash A \quad B \vdash \Delta}{A \Rightarrow B \vdash \#\Gamma; \Delta}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \mathrel{-\!\!*} B} \qquad \frac{\Gamma \vdash A \quad B \vdash \Delta}{A \mathrel{-\!\!*} B \vdash \Gamma \multimap \Delta}$$

Display Logic

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  year = {1982},
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  ee = {http://www.springerlink.com/content/b765t2nk68n0602p}
}
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Classical Bunched Implications

```
@inproceedings{1480923,
 author = {Brotherston,, James and Calcagno,, Cristiano},
 title = {Classical BI: a logic for reasoning about dualising resources},
 booktitle = {POPL '09: Proceedings of the 36th annual ACM SIGPLAN-SIGACT symposium on Principles of programming languages},
 year = {2009},
 isbn = {978-1-60558-379-2},
 pages = {328--339},
 location = {Savannah, GA, USA},
 doi = {http://doi.acm.org/10.1145/1480881.1480923},
 publisher = {ACM},
 address = {New York, NY, USA},
 }
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Boolean Bunched Implications

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@inproceedings{375719,
 author = {Ishtiaq,, Samin S. and O'Hearn,, Peter W.},
 title = {BI as an assertion language for mutable data structures},
 booktitle = {POPL '01: Proceedings of the 28th ACM SIGPLAN-SIGACT symposium on Principles of programming languages},
 year = {2001},
 isbn = {1-58113-336-7},
 pages = {14--26},
 location = {London, United Kingdom},
 doi = {http://doi.acm.org/10.1145/360204.375719},
 publisher = {ACM},
 address = {New York, NY, USA},
 }

@misc{brotherston-bbi,
  author = {James Brotherston},
  title = {A Cut-Free Proof Theory for Boolean BI},
  note = {In preparation},
}
```