The Logic of Bunched Implications

Presentation by Robert J. Simmons
Separation Logic, April 6, 2009
Big Picture

\[ P \Rightarrow P' \quad \{P'\} \quad C \quad \{R'\} \quad R' \Rightarrow R \]

- Valid formula of separation logic

\[ \forall s, h, \quad s, h \not\models P \Rightarrow P' \]

- A valid implication in BI is valid in sep. logic

- Not complete (example: pure propositions)
Proofs and provability

• Separation logic has a *model*
  • Valid props. true in all heaps/stores

• Both BI and Boolean BI have *axioms*
  • Valid props. are provable from the axioms

• BI has a *proof theory*
  • Valid props. have closed derivations with no hypotheses
  • Allows *hypothetical reasoning*
    (to prove $A \Rightarrow B$ assume $A$, prove $B$)
Outline: Today

• The logic of bunched implications
  • Substructural logics and bunchy contexts
• Natural deduction for BI
• Sequent calculus (& logic prog.) for BI
Outline: Wednesday

• (Maybe) a little more logic programming

• Other logics of bunched implication
  • Boolean BI
    An assertion language for separation logic
  • Classical BI
    Not compatible with separation logic!
Linear Logic (multiplicative)

\[ \Delta_1 \vdash A \]
\[ \Delta_2 \vdash B \]
\[ \Delta_1, \Delta_2 \vdash A \otimes B \]

Constructive Logic (additive)

\[ \Gamma \vdash A \]
\[ \Gamma \vdash B \]
\[ \Gamma \vdash A \land B \]
Linear Logic (multiplicative)

\[ \Delta_1 \vdash A \\
\Delta_2 \vdash B \]

\[ \Delta_1, \Delta_2 \vdash A \otimes B \]

Constructive Logic (additive)

\[ \Gamma \vdash A \\
\Gamma \vdash B \]

\[ \Gamma \vdash A \land B \]

\[ A \vdash A \]

\[ B \vdash B \]

\[ A, B \vdash A \otimes B \]

\[ B \vdash B \]

\[ A, B, B \vdash (A \otimes B) \otimes B \]

\[ A; B; B \vdash A \land B \]

\[ A; B; B \vdash (A \land B) \land B \]

\[ A; B; B \vdash B \]

\[ A; B; B \vdash A \land B \]

\[ A; B; B \vdash (A \land B) \land B \]
Contraction

\[ \Delta_1 \vdash A \]
\[ \Delta_2 \vdash B \]
\[ \Delta_1, \Delta_2 \vdash A \otimes B \]

\[ \Gamma \vdash A \]
\[ \Gamma \vdash B \]
\[ \Gamma \vdash A \land B \]

\[ A; B; B \not\vdash (A \otimes B) \otimes B \]

\[ A; B; B \vdash A \]
\[ A; B; B \vdash B \]
\[ A; B; B \vdash A \land B \]

\[ A; B; B \vdash (A \land B) \land B \]

\[ A; B; B \vdash A \]
\[ A; B; B \vdash B \]
\[ A; B; B \vdash A \land B \]

\[ A; B; B \vdash (A \land B) \land B \]
Weakening

\[
\Delta_1 \vdash A \\
\Delta_2 \vdash B
\]
\[
\Delta_1, \Delta_2 \vdash A \otimes B
\]

\[
\Gamma \vdash A \\
\Gamma \vdash B
\]
\[
\Gamma \vdash A \land B
\]

A, B, B, C \not\vdash (A \otimes B) \otimes B

A; B; C \vdash A

A; B; C \vdash B

A; B; C \vdash A \land B

A; B; C \vdash (A \land B) \land B
Exchange

\[
\Delta_1 \vdash A \\
\Delta_2 \vdash B \\
\Delta_1, \Delta_2 \vdash A \otimes B
\]

\[
\Gamma \vdash A \\
\Gamma \vdash B \\
\Gamma \vdash A \land B
\]

\[
B, B, A \vdash (A \otimes B) \otimes B \\
B, A, B \vdash (A \otimes B) \otimes B \\
A, B, B \vdash (A \otimes B) \otimes B
\]

\[
B; B; A \vdash (A \land B) \land B \\
B; A; B \vdash (A \land B) \land B \\
A; B; B \vdash (A \land B) \land B
\]
Linear Logic:
Additive and Multiplicative Zones

\[
\Gamma; \Delta, A \vdash B \rightarrow C \\
\Gamma; \Delta \vdash A \multimap B \rightarrow C
\]
Linear Logic:
Additive and Multiplicative Zones

Additive zone

Multiplicative zone

Conclusion

\[
\Gamma, B; \Delta, A \vdash C
\]

\[
\Gamma; \Delta, A \vdash B \rightarrow C
\]

\[
\Gamma; \Delta \vdash A \rightarrow B \rightarrow C
\]
Bunched implication:
Additive and Multiplicative Bunches

Additive combination

Multiplicative combination

Conclusion

\[ \Gamma; A \vdash B \rightarrow C \]

\[ \Gamma; A \vdash B \rightarrow C \]

\[ \Gamma \vdash A \rightarrow B \rightarrow C \]
Additive and Multiplicative Bunches

\[(A; (B, C)), (D, (E; (F, (G, H)))) \vdash Z\]
\[D, (((C, B); A), (E; ((H, F), G))) \vdash Z\]
Additive and Multiplicative Bunches

\[(A; (B, C)), (D, (E; (F, (G, H))))) \vdash Z\]
Additive and Multiplicative Bunches
Weakening
Additive and Multiplicative Bunches

Contraction

\[ \vdash Z \]

\[ \vdash Z \]
Additive and Multiplicative Bunches
“Commutative monoid equations”
Additive and Multiplicative Bunches
Formulas stand alone

\[ A = A = A \]

\[ A = A; \emptyset_a = A, \emptyset_m \]
Additive and Multiplicative Bunches
Contexts with a missing piece
Additive and Multiplicative Bunches
Contexts with a missing piece

Γ(−)
Additive and Multiplicative Bunches
Contexts with a missing piece

Γ(K)
Additive and Multiplicative Bunches Contexts with a missing piece

Γ(Δ)
Additive and Multiplicative Bunches

Empty additive context doesn’t disappear

Weaken
Natural Deduction for BI

\[ \emptyset_m \vdash I \]

\[ \emptyset_a \vdash \top \]

\[ A \vdash A \]

\[ \Gamma \vdash A \]
\[ \Gamma \vdash B \]
\[ \Gamma \vdash A \land B \]

\[ \Delta \vdash A \land B \]
\[ \Gamma(A; B) \vdash C \]
\[ \Gamma(\Delta) \vdash C \]

\[ \Gamma \vdash A \]
\[ \Delta \vdash B \]
\[ \Gamma, \Delta \vdash A \ast B \]

\[ \Delta \vdash A \ast B \]
\[ \Gamma(A, B) \vdash C \]
\[ \Gamma(\Delta) \vdash C \]

\[ \ast E \]
Natural Deduction for BI

\[ \vdash (P_1 \ast Q) \land (P_2 \ast Q) \]
Natural Deduction for BI

\[(P_1; P_2), Q \vdash (P_1 \ast Q) \land (P_2 \ast Q)\]
Natural Deduction for BI

\[ (P_1; P_2), Q \vdash (P_1 * Q) \]

\[ (P_1; P_2), Q \vdash (P_2 * Q) \]

\[ (P_1; P_2), Q \vdash (P_1 * Q) \land (P_2 * Q) \]
Natural Deduction for BI

\[
\begin{align*}
\frac{P_1 \vdash P_1}{(P_1; P_2) \vdash P_1} & \quad \frac{Q \vdash Q}{(P_1; P_2), Q \vdash (P_1 \ast Q)} & \quad \frac{P_1 \quad P_2 \quad Q}{\vdash (P_2 \ast Q)} \\
\hline
(P_1; P_2), Q \vdash (P_1 \ast Q) \quad & \quad (P_1; P_2), Q \vdash (P_1 \ast Q) \land (P_2 \ast Q)
\end{align*}
\]
Natural Deduction for Bi

\[
\begin{align*}
\frac{P_1 \vdash P_1}{(P_1; P_2) \vdash P_1} & \quad \frac{Q \vdash Q}{(P_1; P_2), Q \vdash (P_1 \ast Q)} \\
\frac{Q \vdash Q}{(P_1; P_2), Q \vdash (P_1 \ast Q)} & \quad \frac{Q \vdash Q}{(P_1; P_2), Q \vdash (P_2 \ast Q)} \\
\frac{Q \vdash Q}{(P_1; P_2), Q \vdash (P_1 \ast Q) \land (P_2 \ast Q)} &
\end{align*}
\]
Natural Deduction for BI

\[ \Delta \vdash A \ast B \]
\[ \Gamma(A, B) \vdash C \]
\[ \Gamma(\Delta) \vdash C \]
Natural Deduction for BI

\[ \Delta \vdash A \ast B \]
\[ \Gamma(A, B) \vdash C \]
\[ \Gamma(\Delta) \vdash C \]

\[ (P_1 \land P_2) \ast Q \vdash (P_1 \land P_2) \ast Q \]
\[ (P_1 \land P_2) \vdash (P_1 \ast Q) \]
\[ (P_2 \ast Q) \]

\[ (P_1 \land P_2) \ast Q \vdash (P_1 \ast Q) \land (P_2 \ast Q) \]
Natural Deduction for BI

\[
\begin{align*}
\Gamma; A \vdash B & \quad \Gamma \vdash A \Rightarrow B \\
\Gamma \vdash A \Rightarrow B & \\
\Gamma, A \vdash B & \quad \Gamma \vdash A \Leftarrow B \\
\Gamma \vdash A \Leftarrow B & \\
\end{align*}
\]
Natural Deduction for BI

\[
\begin{align*}
\emptyset_a \vdash P_0 \ast P_1 \Rightarrow P_2 \\
\emptyset_a ; P_0 \vdash P_1 \ast P_2 \\
\emptyset_a \vdash P_0 \Rightarrow P_1 \ast P_2
\end{align*}
\]
Natural Deduction for BI

We want these to be valid props., but they are “stuck in a bunch”

\[
\begin{align*}
P_1 &\vdash P_2 & Q_1 &\vdash Q_2 \\
P_1 \land Q_1 &\vdash P_2 \land Q_2 \\
(P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1, Q_1) &\vdash (P_2, Q_2) \\
\end{align*}
\]
Sequent Calculus for BI

$P_1 \ast Q_1$

$\vdash P_1 \ast Q_1 \quad (P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1, Q_1) \vdash (P_2 \ast Q_2)$

$(P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1 \ast Q_1) \vdash (P_2 \ast Q_2)$
Sequent Calculus for BI

\[ \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \ast B} \quad ^*R \quad \frac{\Gamma(A, B) \vdash C}{\Gamma(A \ast B) \vdash C} \quad ^*L \]

\[ (P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1, Q_1) \vdash (P_2 \ast Q_2) \]

\[ (P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1 \ast Q_1) \vdash (P_2 \ast Q_2) \]
Sequent Calculus for BI

\[
\begin{align*}
\Gamma & \vdash A & \Delta & \vdash B \\
\Gamma, \Delta & \vdash A \ast B & \Gamma(A, B) & \vdash C \\
\Gamma & \vdash A & \Gamma & \vdash B \\
\Gamma & \vdash A \land B & \Gamma(A; B) & \vdash C \\
\Gamma & \vdash A_i \\
\Gamma & \vdash A_1 \lor A_2 \\
\Gamma(A) & \vdash C & \Gamma(B) & \vdash C
\end{align*}
\]

\[
\begin{align*}
\ast R & \quad \ast L \\
\land R & \quad \land L \\
\lor R_i & \quad \lor L
\end{align*}
\]
Sequent Calculus for BI

\[
\begin{align*}
(P_0 \lor P_1) & \rightarrow Q \\
(P_0 \land Q) & \rightarrow (P_0 \lor P_1) \land Q \\
(P_0 \land Q) & \rightarrow (P_0 \lor P_1) \land Q \\
(P_0 \lor P_1) & \land Q \\
(P_0 \lor P_1) & \lor Q
\end{align*}
\]
Sequent Calculus for BI

\[
\begin{align*}
\frac{\Gamma; A \vdash B}{\Gamma \vdash A \implies B} & \quad \Rightarrow R \\
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \ast B} & \quad \ast R \\
\frac{\Gamma \vdash [t/x]A}{\Gamma \vdash \exists x. A} & \quad \exists R \\
\frac{\Delta \vdash A}{\Gamma(\Delta; A \implies B) \vdash C} & \quad \Rightarrow L \\
\frac{\Gamma(B) \vdash C}{\Gamma(\Delta, A \ast B) \vdash C} & \quad \ast L \\
\frac{\Gamma(\exists x. A) \vdash C}{\Gamma(\forall x. A) \vdash C} & \quad \forall R
\end{align*}
\]
Office Assignments

3 research groups (POP, Plaid, Sec)

SCS wants the 2 students in each office in Gates to be from different research groups
Office Assignments

- p(rjsimmon)
- p(cvarming)
- p(mtschantz)

- p(neelk)
- p(nbeckman)
- p(jfranklin)
∀x,y. p(x) * p(y) * ⊨ room(x,y)

⊢ ∃x,y. room(x,y)
Office Assignments

∀x,y. p(x) * p(y) * ⊨ room(x,y)

⊨ room(neelk, jfranklin)
Office Assignments

\[
p(nk) \land p(jf) \land \top \rightarrow room(nk,jf)
\]

\[\vdash room(neelk, jfranklin)\]
Office Assignments

\[
\text{room(neelk,jfranklin)} \\
\vdash \text{p(neelk)} \\
\text{p(cvarming)} \\
\text{p(mtschantz)} \\
\vdash \text{p(jfranklin)} \\
\text{p(nbeckman)} \\
\text{p(jfranklin)}
\]
Office Assignments

\[ \vdash p(\text{neelk}) \ast p(\text{jfranklin}) \ast \]

- p(rjsimmon)
- p(cvarming)
- p(ntschantz)
- p(neelk)
- p(nbeckman)
- p(jfranklin)
Office Assignments

\[ \vdash p(\text{neelk}) \]

\[ \vdash p(\text{jfranklin}) \]

\[ \vdash \top \]
Office Assignments

\[ \vdash p(\text{neelk}) \]

\[ \vdash p(\text{jfranklin}) \]

\[ \vdash \top \]
Office Assignments

\[ \vdash p(\text{neelk}) \]

\[ \vdash p(\text{jfranklin}) \]

\[ \vdash \top \]
Office Assignments

\[ p(\text{neelk}) \vdash p(\text{neelk}) \]

\[ p(\text{jfranklin}) \vdash p(\text{jfranklin}) \]

\[ \vdash \top \]
Office Assignments: Success!
Wrap-up

• BI: Logic sound for sep. logic
  • Axioms of BI have a proof theory

• Not covered: research on a proof theory for separation logic by Neel K.
  • Permits \((A \land I \vdash A \ast A)\), unlike BI

• Sequent calculus/logic programming!

• Next time: Classical BI, Boolean BI
The Logic of Bunched Implications

@article{DBLP:journals/bsl/OHearnP99,
  author = {Peter W. O'Hearn and David J. Pym},
  title = {The logic of bunched implications},
  journal = {Bulletin of Symbolic Logic},
  volume = {5},
  number = {2},
  year = {1999},
  pages = {215-244},
  ee = {http://www.math.ucla.edu/$\sim$asl/bsl/0502/0502-003.ps},
  bibsource = {DBLP, http://dblp.uni-trier.de}
}

Bunched Logic Programming

@inproceedings{DBLP:conf/cade/ArmelinP01,
  author = {Pablo A. Armelín and David J. Pym},
  title = {Bunched Logic Programming},
  booktitle = {IJCAR},
  year = {2001},
  pages = {289-304},
  ee = {http://link.springer.de/link/service/series/0558/bibs/2083/20830289.htm},
  crossref = {DBLP:conf/cade/2001},
  bibsource = {DBLP, http://dblp.uni-trier.de}
}

@proceedings{DBLP:conf/cade/2001,
  editor = {Rajeev Goré and Alexander Leitsch and Tobias Nipkow},
  booktitle = {IJCAR},
  publisher = {Springer},
  series = {Lecture Notes in Computer Science},
  volume = {2083},
  year = {2001},
  isbn = {3-540-42254-4},
  bibsource = {DBLP, http://dblp.uni-trier.de}
}

The Semantics of BI

@article{DBLP:journals/tcs/PymOY04,
  author = {David J. Pym and Peter W. O'Hearn and Hongseok Yang},
  title = {Possible worlds and resources: the semantics of BI},
  journal = {Theor. Comput. Sci.},
  volume = {315},
  number = {1},
  year = {2004},
  pages = {257-305},
  ee = {http://dx.doi.org/10.1016/j.tcs.2003.11.020},
  bibsource = {DBLP, http://dblp.uni-trier.de}
}