Cut admissibility for focused multiplicative, exponential linear logic

Robert J. Simmons, April 21, 2012

Our goal is to prove cut admissibility for the following logic:

$$
\begin{array}{lll}
A^+ & ::= p^+ \mid {\downarrow}A^- \mid {!}A^- \mid \mathbf{1} \mid A \otimes B & \\
A^- & ::= p^- \mid {\uparrow}A^+ \mid A \multimap B & \\
\Gamma & ::= \cdot \mid \Gamma, A^- & \textit{(multiset)} \\
\Delta & ::= \cdot \mid \Delta, A^+ \mid \Delta, A^- \mid \Delta, [A^-] \mid \Delta, \langle A^+ \rangle & \textit{(multiset)} \\
U & ::= A^- \mid A^+ \mid [A^+] \mid \langle A^- \rangle &
\end{array}
$$

$$
\frac{\Gamma; \Delta \vdash [A^+]}{\Gamma; \Delta \vdash A^+}\,\mathit{focus}_R^* \qquad
\frac{\Gamma; \Delta, [A^-] \vdash U}{\Gamma; \Delta, A^- \vdash U}\,\mathit{focus}_L^* \qquad
\frac{\Gamma, A^-; \Delta, [A^-] \vdash U}{\Gamma, A^-; \Delta \vdash U}\,\mathit{copy}^*
$$

$$
\frac{\Gamma; \Delta, \langle p^+ \rangle \vdash U}{\Gamma; \Delta, p^+ \vdash U}\,\eta^+ \qquad
\frac{}{\Gamma; \langle A^+ \rangle \vdash [A^+]}\,id^+ \qquad
\frac{\Gamma; \Delta \vdash \langle p^- \rangle}{\Gamma; \Delta \vdash p^-}\,\eta^- \qquad
\frac{}{\Gamma; [A^-] \vdash \langle A^- \rangle}\,id^-
$$

$$
\frac{\Gamma; \Delta \vdash A^+}{\Gamma; \Delta \vdash {\uparrow}A^+}\,{\uparrow}_R \qquad
\frac{\Gamma; \Delta, A^+ \vdash U}{\Gamma; \Delta, [{\uparrow}A^+] \vdash U}\,{\uparrow}_L \qquad
\frac{\Gamma; \Delta \vdash A^-}{\Gamma; \Delta \vdash [{\downarrow}A^-]}\,{\downarrow}_R \qquad
\frac{\Gamma; \Delta, A^- \vdash U}{\Gamma; \Delta, {\downarrow}A^- \vdash U}\,{\downarrow}_L
$$

$$
\frac{\Gamma; \cdot \vdash A^-}{\Gamma; \cdot \vdash [{!}A^-]}\,{!}_R \qquad
\frac{\Gamma, A^-; \Delta \vdash U}{\Gamma; \Delta, {!}A^- \vdash U}\,{!}_L \qquad
\frac{}{\Gamma; \cdot \vdash [\mathbf{1}]}\,\mathbf{1}_R \qquad
\frac{\Gamma; \Delta \vdash U}{\Gamma; \Delta, \mathbf{1} \vdash U}\,\mathbf{1}_L
$$

$$
\frac{\Gamma; \Delta_1 \vdash [A^+] \quad \Gamma; \Delta_2 \vdash [B^+]}{\Gamma; \Delta_1, \Delta_2 \vdash [A^+ \otimes B^+]}\,\otimes_R \qquad
\frac{\Gamma; \Delta, A^+, B^+ \vdash U}{\Gamma; \Delta, A^+ \otimes B^+ \vdash U}\,\otimes_L
$$

$$
\frac{\Gamma; \Delta, A^+ \vdash B^-}{\Gamma; \Delta \vdash A^+ \multimap B^-}\,{\multimap}_R \qquad
\frac{\Gamma; \Delta_1 \vdash [A^+] \quad \Gamma; \Delta_2, [B^-] \vdash U}{\Gamma; \Delta_1, \Delta_2, [A^+ \multimap B^-] \vdash U}\,{\multimap}_L
$$

We say Δ is stable if it includes no focus $[A^-]$ or positive proposition A^+, and say we write U is stable if it is not a focus $[A^+]$ or a negative proposition A^-. The rules focus_R, focus_L, and copy have a side condition that Δ and U are stable. This allows us to impose a global restriction: sequents with a focus are otherwise stable, and there is always at most one focus.

In the remainder of this note, when we write Δ or U it indicates a stable context/succeedant. When we write $\underline{\Delta}$ or $\underline{U}$, the context/succeedant is unconstrained aside from the global restriction, and when we write $\utilde{\Delta}$ or $\utilde{U}$, the context/succeedant is focus-free but may not be stable.

Theorem 1 (Cut admissibility)**.** *For all Γ, A^+, A^-, Δ, $\underline{\Delta}$, Δ', $\utilde{\Delta}'$, $\underline{\Delta}'$, U, $\utilde{U}$, and $\underline{U}$ containing no non-atomic suspensions $\langle A^+ \rangle$ or $\langle A^- \rangle$,*

1. *If $\Gamma; \Delta \vdash [A^+]$ and $\Gamma; \utilde{\Delta}', A^+ \vdash \utilde{U}$, then $\Gamma; \utilde{\Delta}', \Delta \vdash \utilde{U}$.* $\quad (\mathit{cut}[]A)$

2. *If $\Gamma; \Delta \vdash A^-$ and $\Gamma; \Delta', [A^-] \vdash U$, then $\Gamma; \Delta', \Delta \vdash U$.* $\quad (\mathit{cut}A[])$

3. *If $\Gamma; \underline{\Delta} \vdash A^+$ and $\Gamma; \Delta', A^+ \vdash U$, then $\Gamma; \Delta', \Delta \vdash U$.* $\quad (\mathit{cut}A^+)$

4. *If $\Gamma; \Delta \vdash A^-$ and $\Gamma; \underline{\Delta}', A^- \vdash \underline{U}$, then $\Gamma; \underline{\Delta}', \Delta \vdash \underline{U}$.* $\quad (\mathit{cut}A^-)$

5. *If $\Gamma; \cdot \vdash A^-$ and $\Gamma, A^-; \underline{\Delta}' \vdash \underline{U}$, then $\Gamma; \underline{\Delta}' \vdash \underline{U}$.* $\quad (\mathit{cut}!)$

Part 1:

$$\dfrac{\Gamma; \Delta \vdash [A^+] \qquad \Gamma; \underline{\Delta'}, A^+ \vdash \underline{U}}{\Gamma; \underline{\Delta'}, \Delta \vdash \underline{U}}$$

PRINCIPAL CUTS

$$\dfrac{\dfrac{}{\Gamma; \langle p^+\rangle \vdash [p^+]} \qquad \dfrac{\Gamma; \underline{\Delta'}, \langle p^+\rangle \vdash \underline{U}}{\Gamma; \underline{\Delta'}, p^+ \vdash \underline{U}}}{\Gamma; \underline{\Delta'}, \langle p^+\rangle \vdash \underline{U}} \quad \rightsquigarrow \quad \Gamma; \underline{\Delta'}, \langle p^+\rangle \vdash \underline{U}$$

$$\dfrac{\dfrac{\Gamma; \Delta \vdash A^-}{\Gamma; \Delta \vdash [\downarrow A^-]} \qquad \dfrac{\Gamma; \underline{\Delta'}, A^- \vdash \underline{U}}{\Gamma; \underline{\Delta'}, \downarrow A^- \vdash \underline{U}}}{\Gamma; \underline{\Delta'}, \Delta \vdash \underline{U}} \quad \rightsquigarrow \quad \dfrac{\Gamma; \Delta \vdash A^- \qquad \Gamma; \underline{\Delta'}, A^- \vdash \underline{U}}{\Gamma; \underline{\Delta'}, \Delta \vdash \underline{U}} \ \boxed{4}$$

$$\dfrac{\dfrac{\cancel{\Gamma; \Delta \vdash}\ \Gamma; \cdot \vdash A^-}{\Gamma; \cdot \vdash [!A^-]} \qquad \dfrac{\Gamma, A^-; \underline{\Delta'} \vdash \underline{U}}{\Gamma; \underline{\Delta'}, !A^- \vdash \underline{U}}}{\Gamma; \underline{\Delta'} \vdash \underline{U}} \quad \rightsquigarrow \quad \dfrac{\Gamma; \cdot \vdash A^- \qquad \Gamma, A^-; \underline{\Delta} \vdash \underline{U}}{\Gamma; \underline{\Delta'} \vdash \underline{U}} \ \boxed{5}$$

$$\dfrac{\dfrac{}{\Gamma; \cdot \vdash [1]} \qquad \dfrac{\Gamma; \underline{\Delta'} \vdash \underline{U}}{\Gamma; \underline{\Delta'}, 1 \vdash \underline{U}}}{\Gamma; \underline{\Delta'} \vdash \underline{U}} \quad \rightsquigarrow \quad \Gamma; \underline{\Delta} \vdash \underline{U}$$

$$\dfrac{\dfrac{\Gamma; \Delta_1 \vdash [A_1] \qquad \Gamma; \Delta_2 \vdash [A_2]}{\Gamma; \Delta_1, \Delta_2 \vdash [A_1 \otimes A_2]} \qquad \dfrac{\Gamma; \underline{\Delta'}, A_1, A_2 \vdash \underline{U}}{\Gamma; \underline{\Delta'}, A_1 \otimes A_2 \vdash \underline{U}}}{\Gamma; \underline{\Delta'}, \Delta_1, \Delta_2 \vdash \underline{U}} \quad \rightsquigarrow \quad \dfrac{\Gamma; \Delta_2 \vdash [A_2] \qquad \dfrac{\Gamma; \Delta_1 \vdash [A_1] \qquad \Gamma; \underline{\Delta'}, A_1, A_2 \vdash \underline{U}}{\Gamma; \underline{\Delta'}, \Delta_1, A_2 \vdash \underline{U}}}{\Gamma; \underline{\Delta'}, \Delta_1, \Delta_2 \vdash \underline{U}}$$

$$\text{Part 1:} \quad \frac{\Gamma;\Delta\vdash[A^+] \qquad \Gamma;\underline{\Delta'},A^+\vdash\underline{U}}{\Gamma;\underline{\Delta'},\Delta\vdash\underline{U}}$$

In order to handle the ~~cases~~ principal cuts for $\multimap$ and $\otimes$, we need to allow the context and succeedent for Part 1 to be unstable. This means we need to duplicate some right commutative cases. ~~that will also appear in~~

RIGHT COMMUTATIVE CUTS

$$\frac{\Gamma;\Delta\vdash[A^+] \qquad \dfrac{\Gamma;\underline{\Delta'},\langle p^+\rangle,A^+\vdash\underline{U}}{\Gamma;\underline{\Delta'},p^+,A^+\vdash\underline{U}}}{\Gamma;\underline{\Delta'},p^+,\Delta\vdash\underline{U}} \quad \leadsto \quad \frac{\dfrac{\Gamma;\Delta\vdash[A^+] \qquad \Gamma;\underline{\Delta'},\langle p^+\rangle,A^+\vdash\underline{U}}{\Gamma;\underline{\Delta'},\langle p^+\rangle,\Delta\vdash\underline{U}}}{\Gamma;\underline{\Delta'},p^+,\Delta\vdash\underline{U}}$$

$$\frac{\Gamma;\Delta\vdash[A^+] \qquad \dfrac{\Gamma;\underline{\Delta'},A^+\vdash\langle p^-\rangle}{\Gamma;\underline{\Delta'},A^+\vdash p^-}}{\Gamma;\underline{\Delta'},\Delta\vdash p^-} \quad \leadsto \quad \frac{\dfrac{\Gamma;\Delta\vdash[A^+] \qquad \Gamma;\underline{\Delta'},A^+\vdash\langle p^-\rangle}{\Gamma;\underline{\Delta'},\Delta\vdash\langle p^-\rangle}}{\Gamma;\underline{\Delta'},\Delta\vdash p^-}$$

$$\frac{\Gamma;\Delta\vdash[A^+] \qquad \dfrac{\Gamma;\underline{\Delta'},A^+\vdash B^+}{\Gamma;\underline{\Delta'},A^+\vdash{\uparrow}B^+}}{\Gamma;\underline{\Delta'},\Delta\vdash{\uparrow}B^+} \quad \leadsto \quad \frac{\dfrac{\Gamma;\Delta\vdash[A^+] \qquad \Gamma;\underline{\Delta'},A^+\vdash B^+}{\Gamma;\underline{\Delta'},\Delta\vdash B^+}}{\Gamma;\underline{\Delta'},\Delta\vdash{\uparrow}B^+}$$

$$\frac{\Gamma;\Delta\vdash[A^+] \qquad \dfrac{\Gamma;\underline{\Delta'},B^-,A^+\vdash\underline{U}}{\Gamma;\underline{\Delta'},{\downarrow}B^-,A^+\vdash\underline{U}}}{\Gamma;\underline{\Delta'},{\downarrow}B^-,\Delta\vdash\underline{U}} \quad \leadsto \quad \frac{\dfrac{\Gamma;\Delta\vdash[A^+] \qquad \Gamma;\underline{\Delta'},B^-,A^+\vdash\underline{U}}{\Gamma;\underline{\Delta'},B^-,\Delta\vdash\underline{U}}}{\Gamma;\underline{\Delta'},{\downarrow}B^-,\Delta\vdash\underline{U}}$$

$$\frac{\Gamma;\Delta\vdash[A^+] \qquad \dfrac{\Gamma,B^-;\underline{\Delta'},A^+\vdash\underline{U}}{\Gamma;\underline{\Delta'},!B^-,A^+\vdash\underline{U}}}{\Gamma;\underline{\Delta'},!B^-,\Delta\vdash\underline{U}} \quad \leadsto \quad \frac{\dfrac{\dfrac{\Gamma;\Delta\vdash[A^+]}{\Gamma,B^-;\Delta\vdash[A^+]}\,W \qquad \Gamma,B^-;\underline{\Delta'},A^+\vdash\underline{U}}{\Gamma,B^-;\underline{\Delta'},\Delta\vdash\underline{U}}}{\Gamma;\underline{\Delta'},!B^-,\Delta\vdash\underline{U}}$$

~~$\Gamma;\underline{\Delta'},!B^-,\Delta$~~

$$\begin{array}{c} \Gamma ; \Delta \vdash [A^+] \quad \dfrac{\Gamma ; \underline{\Delta}', A^+ \vdash \underline{U}}{\Gamma ; \underline{\Delta}', 1, A^+ \vdash \underline{U}} \\ \hdashline \Gamma ; \underline{\Delta}', 1, \Delta \vdash \underline{U} \end{array} \quad \Rrightarrow \quad \dfrac{\begin{array}{c} \Gamma ; \Delta \vdash [A^+] \quad \Gamma ; \underline{\Delta}', A^+ \vdash \underline{U} \\ \hdashline \Gamma ; \underline{\Delta}', \Delta \vdash \underline{U} \end{array}}{\Gamma ; \underline{\Delta}', 1, \Delta \vdash \underline{U}}$$

$$\begin{array}{c} \Gamma ; \Delta \vdash [A^+] \quad \dfrac{\Gamma ; \underline{\Delta}', B_1, B_2, A^+ \vdash \underline{U}}{\Gamma ; \underline{\Delta}', B_1 \otimes B_2, A^+ \vdash \underline{U}} \\ \hdashline \Gamma ; \underline{\Delta}', B_1 \otimes B_2, \Delta \vdash \underline{U} \end{array} \quad \Rrightarrow \quad \dfrac{\begin{array}{c} \Gamma ; \Delta \vdash [A^+] \quad \Gamma ; \underline{\Delta}', B_1, B_2, A^+ \vdash \underline{U} \\ \hdashline \Gamma ; \underline{\Delta}', B_1, B_2, \Delta \vdash U \end{array}}{\Gamma ; \underline{\Delta}', B_1 \otimes B_2, \Delta \vdash \underline{U}}$$

$$\begin{array}{c} \Gamma ; \Delta \vdash [A^+] \quad \dfrac{\Gamma ; \underline{\Delta}', A^+, B_1 \vdash B_2}{\Gamma ; \underline{\Delta}', A^+ \vdash B_1 \multimap B_2} \\ \hdashline \Gamma ; \underline{\Delta}', \Delta \vdash B_1 \multimap B_2 \end{array} \quad \Rrightarrow \quad \dfrac{\begin{array}{c} \Gamma ; \Delta \vdash [A^+] \quad \Gamma ; \underline{\Delta}', A^+, B_1 \vdash B_2 \\ \hdashline \Gamma ; \underline{\Delta}', \Delta, B_1 \vdash B_2 \end{array}}{\Gamma ; \underline{\Delta}', \Delta \vdash B_1 \multimap B_2}$$

Part 2: $$\frac{\Gamma; \Delta \vdash A^- \qquad \Gamma; \Delta', [A^-] \vdash U}{\Gamma; \Delta', \Delta \vdash U}$$

PRINCIPAL CUTS

$$\frac{\dfrac{\Gamma; \Delta \vdash \langle p^- \rangle}{\Gamma; \Delta \vdash p^-} \qquad \dfrac{}{\Gamma; [p^-] \vdash \langle p^- \rangle}}{\Gamma; \Delta \vdash \langle p^- \rangle} \rightsquigarrow \Gamma; \Delta \vdash \langle p^- \rangle$$

$$\frac{\dfrac{\Gamma; \Delta \vdash A^+}{\Gamma; \Delta \vdash \uparrow A^+} \qquad \dfrac{\Gamma; \Delta', A^+ \vdash U}{\Gamma; \Delta', [\uparrow A^+] \vdash U}}{\Gamma; \Delta', \Delta \vdash U} \rightsquigarrow \frac{\Gamma; \Delta \vdash A^+ \qquad \Gamma; \Delta', A^+ \vdash U}{\Gamma; \Delta', \Delta \vdash U} \; \boxed{3}$$

$$\frac{\dfrac{\Gamma; \Delta, A^+ \vdash B^-}{\Gamma; \Delta \vdash A^+ \multimap B^-} \qquad \dfrac{\Gamma; \Delta'_A \vdash [A^+] \qquad \Gamma; \Delta', [B^-] \vdash U}{\Gamma; \Delta', \Delta'_A, [A^+ \multimap B^-] \vdash U}}{\Gamma; \Delta', \Delta'_A, \Delta \vdash U}$$

$$\not\rightsquigarrow \; \cancel{\frac{\Gamma; \Delta', \Delta_A \vdash B^- \qquad \Gamma; \Delta', [B^-] \vdash U}{\Gamma; \Delta', \Delta'_A, \Delta \vdash U}}$$

$$\rightsquigarrow \frac{\dfrac{\Gamma; \Delta'_A \vdash A^+ \qquad \Gamma; \Delta, A^+ \vdash B^-}{\Gamma; \Delta, \Delta'_A \vdash B^-}\;\boxed{1} \qquad \Gamma; \Delta', [B^-] \vdash U}{\Gamma; \Delta', \Delta'_A, \Delta \vdash U}$$

We could have the first derivation be $\Gamma; \underline{\Delta} \vdash A^-$, giving more left commutative cases, if we wanted, ~~but the unstabl~~ which would align with Part 1's right commutative cases.

This is unnecessary, however: the instability in 1 is needed only to deal with principal cuts for $A \otimes B$ and $A \multimap B$.

Part 3: $$\frac{\Gamma ; \underline{\Delta} \vdash A^+ \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \underline{\Delta} \vdash U}$$

LEFT COMMUTATIVE CUTS

$$\frac{\dfrac{\Gamma ; \Delta \vdash [A^+]}{\Gamma ; \Delta \vdash A^+} \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \Delta \vdash U} \quad \rightsquigarrow \quad \frac{\Gamma ; \Delta \vdash [A^+] \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \Delta \vdash U}\ \boxed{1}$$

$$\frac{\dfrac{\Gamma ; \Delta , [B^-] \vdash A^+}{\Gamma ; \Delta , B^- \vdash A^+} \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \Delta , B^- \vdash U} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma ; \Delta , [B^-] \vdash A^+ \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \Delta , [B^-] \vdash U}}{\Gamma ; \Delta' , \Delta , B^- \vdash U}$$

$$\frac{\dfrac{\Gamma , B^- ; \Delta , [B^-] \vdash A^+}{\Gamma , B^- ; \Delta \vdash A^+} \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma , B^- ; \Delta' , \Delta \vdash A^+} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma , B^- ; \Delta , [B^-] \vdash A^+ \qquad \Gamma , B^- ; \Delta' , A^+ \vdash U}{\Gamma , B^- ; \Delta' , \Delta , [B^-] \vdash A^+}}{\Gamma , B^- ; \Delta' , \Delta \vdash A^+}$$

$$\frac{\dfrac{\Gamma ; \underline{\Delta} , \langle p^+ \rangle \vdash A^+}{\Gamma ; \underline{\Delta} , p^+ \vdash A^+} \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \underline{\Delta} , p^+ \vdash U} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma ; \underline{\Delta} , \langle p^+ \rangle \vdash A^+ \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \underline{\Delta} , \langle p^+ \rangle \vdash U}}{\Gamma ; \Delta' , \underline{\Delta} , p^+ \vdash U}$$

$$\frac{\dfrac{\Gamma ; \Delta , B^+ \vdash A^+}{\Gamma ; \Delta , [\uparrow B^+] \vdash A^+} \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \Delta , [\uparrow B^+] \vdash U} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma ; \Delta , B^+ \vdash A^+ \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \Delta , B^+ \vdash U}}{\Gamma ; \Delta' , \Delta , [\uparrow B^+] \vdash U}$$

$$\frac{\dfrac{\Gamma ; \underline{\Delta} , B^- \vdash A^+}{\Gamma ; \underline{\Delta} , \downarrow B^- \vdash A^+} \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \underline{\Delta} , \downarrow B^- \vdash U} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma ; \underline{\Delta} , B^- \vdash A^+ \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \underline{\Delta} , B^- \vdash U}}{\Gamma ; \Delta' , \underline{\Delta} , \downarrow B^- \vdash U}$$

$$\frac{\dfrac{\Gamma , B^- ; \underline{\Delta} \vdash A^+}{\Gamma ; \underline{\Delta} , !B^- \vdash A^+} \qquad \Gamma ; \Delta' , A^+ \vdash U}{\Gamma ; \Delta' , \underline{\Delta} , !B^- \vdash U} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma , B^- ; \underline{\Delta} \vdash A^+ \qquad \dfrac{\Gamma ; \Delta' , A^+ \vdash U}{\Gamma , B^- ; \Delta' , A^+ \vdash U}\ W}{\Gamma , B^- ; \Delta' , \underline{\Delta} \vdash U}}{\Gamma ; \Delta' , \underline{\Delta} , !B^- \vdash U}$$

$$\dfrac{\dfrac{\Gamma ; \underline{\Delta} \vdash A^+}{\Gamma ; \underline{\Delta}, 1 \vdash A^+} \qquad \Gamma ; \Delta', A^+ \vdash U}{\Gamma ; \Delta', \underline{\Delta}, 1 \vdash U} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma ; \underline{\Delta} \vdash A^+ \qquad \Gamma ; \Delta', A^+ \vdash U}{\Gamma ; \Delta', \underline{\Delta} \vdash U}}{\Gamma ; \Delta', \underline{\Delta}, 1 \vdash U}$$

$$\dfrac{\dfrac{\Gamma ; \underline{\Delta}, B_1, B_2 \vdash A^+}{\Gamma ; \underline{\Delta}, B_1 \otimes B_2 \vdash A^+} \qquad \Gamma ; \Delta ; A^+ \vdash U}{\Gamma ; \Delta, \underline{\Delta}, B_1 \otimes B_2 \vdash U} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma ; \underline{\Delta}, B_1, B_2 \vdash A^+ \qquad \Gamma ; \Delta ; A^+ \vdash U}{\Gamma ; \Delta, \underline{\Delta}, B_1, B_2 \vdash U}}{\Gamma ; \Delta, \underline{\Delta}, B_1 \otimes B_2 \vdash U}$$

Part 4: $\dfrac{\Gamma; \Delta \vdash A^- \qquad \Gamma; \underline{\Delta'}, A^- \vdash \underline{U}}{\Gamma; \underline{\Delta'}, \Delta \vdash \underline{U}}$

RIGHT COMMUTATIVE CUTS

$$\dfrac{\Gamma; \Delta \vdash A^- \qquad \dfrac{\Gamma; \Delta', A^- \vdash [B^+]}{\Gamma; \Delta', A^- \vdash B^+}}{\Gamma; \Delta', \Delta \vdash B^+} \leadsto \dfrac{\dfrac{\Gamma; \Delta \vdash A^- \qquad \Gamma; \Delta', A^- \vdash [B^+]}{\Gamma; \Delta', \Delta \vdash [B^+]}}{\Gamma; \Delta', \Delta \vdash B^+}$$

$$\dfrac{\Gamma; \Delta \vdash A^- \qquad \dfrac{\Gamma; \Delta', [A^-] \vdash U}{\Gamma; \Delta', A^- \vdash U}}{\cancel{\Gamma; \Delta \vdash}\ \Gamma; \Delta', \Delta \vdash U} \leadsto \dfrac{\Gamma; \Delta \vdash A^- \qquad \Gamma; \Delta', [A^-] \vdash U}{\Gamma; \Delta', \Delta \vdash U} \quad \boxed{2}$$

$$\dfrac{\Gamma; \Delta \vdash A^- \qquad \dfrac{\Gamma; \Delta', [B^-], A^- \vdash U}{\Gamma; \Delta', B^-, A^- \vdash U}}{\Gamma; \Delta', B^-, \Delta \vdash U} \leadsto \dfrac{\dfrac{\Gamma; \Delta \vdash A^- \qquad \Gamma; \Delta', [B^-], \Delta \vdash U}{\Gamma; \Delta', [B^-], \Delta \vdash U}}{\Gamma; \Delta', B^-, \Delta \vdash U}$$

$$\dfrac{\Gamma, B^-; \Delta \vdash A^- \qquad \dfrac{\Gamma, B^-; \Delta', A^-, [B^-] \vdash U}{\Gamma, B^-; \Delta', A^- \vdash U}}{\Gamma, B^-; \Delta', \Delta \vdash U} \leadsto \dfrac{\dfrac{\Gamma, B^-; \Delta \vdash A^- \qquad \Gamma, B^-; \Delta', A^-, [B^-] \vdash U}{\Gamma, B^-; \Delta', \Delta, [B^-] \vdash U}}{\Gamma, B^-; \Delta', \Delta \vdash U}$$

$$\dfrac{\Gamma; \Delta \vdash A^- \qquad \dfrac{\Gamma; \underline{\Delta'}, \langle p^+ \rangle, A^- \vdash \underline{U}}{\Gamma; \underline{\Delta'}, p^+, A^- \vdash \underline{U}}}{\Gamma; \underline{\Delta'}, p^+, \Delta \vdash \underline{U}} \leadsto \dfrac{\dfrac{\Gamma; \Delta \vdash A^- \qquad \Gamma; \underline{\Delta'}, \langle p^+ \rangle, A^- \vdash \underline{U}}{\Gamma; \underline{\Delta'}, \langle p^+ \rangle, \Delta \vdash \underline{U}}}{\Gamma; \underline{\Delta'}, p^+, \Delta \vdash \underline{U}}$$

$$\dfrac{\Gamma; \Delta \vdash A^- \qquad \cancel{\dfrac{\Gamma}{\Gamma; \Delta \vdash p^-}} \quad \dfrac{\Gamma; \underline{\Delta'}, A^- \vdash \langle p^- \rangle}{\Gamma; \underline{\Delta'}, A^- \vdash p^-}}{\Gamma; \underline{\Delta'}, \Delta \vdash p^-} \leadsto \dfrac{\dfrac{\Gamma; \Delta \vdash A^- \qquad \Gamma; \underline{\Delta'}, A^- \vdash \langle p^- \rangle}{\Gamma; \underline{\Delta'}, \Delta \vdash \langle p^- \rangle}}{\Gamma; \underline{\Delta'}, \Delta \vdash p^-}$$

$$\dfrac{\Gamma; \Delta \vdash A^- \qquad \dfrac{\Gamma; \underline{\Delta'}, A^- \vdash B^+}{\Gamma; \underline{\Delta'}, A^- \vdash {\uparrow} B^+}}{\Gamma; \underline{\Delta'}, \Delta \vdash {\uparrow} B^+} \leadsto \dfrac{\dfrac{\Gamma; \Delta \vdash A^- \qquad \Gamma; \underline{\Delta'}, A^- \vdash B^+}{\Gamma; \underline{\Delta'}, \Delta \vdash B^+}}{\Gamma; \underline{\Delta'}, \Delta \vdash {\uparrow} B^+}$$

$$\dfrac{\Gamma;\Delta\vdash A^- \qquad \dfrac{\Gamma;\Delta',A^-,B^+\vdash U}{\Gamma;\Delta',A^-,[\uparrow B^+]\vdash U}}{\Gamma;\Delta',\Delta,[\uparrow B^+]\vdash U} \;\approx\!> \; \dfrac{\dfrac{\Gamma;\Delta\vdash A^- \qquad \Gamma;\Delta',A^-,B^+\vdash U}{\Gamma;\Delta',\Delta,B^+\vdash U}}{\Gamma;\Delta',\Delta,[\uparrow B^+]\vdash U}$$

$$\dfrac{\Gamma;\Delta\vdash A^- \qquad \dfrac{\Gamma;\Delta',A^-\vdash B^-}{\Gamma;\Delta',A^-\vdash[\downarrow B^-]}}{\Gamma;\Delta',\Delta\vdash[\downarrow B^-]} \;\approx\!> \; \dfrac{\dfrac{\Gamma;\Delta\vdash A^- \qquad \Gamma;\Delta',A^-\vdash B^-}{\Gamma;\Delta',\Delta\vdash B^-}}{\Gamma;\Delta',\Delta\vdash[\downarrow B^-]}$$

$$\dfrac{\Gamma;\Delta\vdash A^- \qquad \dfrac{\Gamma;\underline{\Delta}',A^-,B^-\vdash\underline{U}}{\Gamma;\underline{\Delta}',A^-,\downarrow B^-\vdash\underline{U}}}{\Gamma;\underline{\Delta}',\Delta,\downarrow B^-\vdash\underline{U}} \;\approx\!> \; \dfrac{\dfrac{\Gamma;\Delta\vdash A^- \qquad \Gamma;\underline{\Delta}',A^-,B^-\vdash\underline{U}}{\Gamma;\underline{\Delta}',\Delta,B^-\vdash\underline{U}}}{\Gamma;\underline{\Delta}',\Delta,\downarrow B^-\vdash\underline{U}}$$

$$\cancel{\Gamma;\Delta\vdash A^- \qquad \Gamma;\cdot\vdash[!A^-]} \quad \text{no } A^-$$

$$\dfrac{\Gamma;\Delta\vdash A^-}{\qquad} \quad W$$

$$\dfrac{\Gamma;\Delta\vdash A^- \qquad \dfrac{\cancel{\Gamma;\Delta}\,\Gamma,B^-;\underline{\Delta}',A^-\vdash\underline{U}}{\Gamma;\underline{\Delta}',!B^-,A^-\vdash\underline{U}}}{\Gamma;\underline{\Delta}',!B^-,\Delta\vdash\underline{U}} \;\approx\!> \; \dfrac{\dfrac{\cancel{\Gamma;\Delta}\,\Gamma,B^-;\Delta\vdash A^- \qquad \Gamma,B^-;\underline{\Delta}',A^-\vdash\underline{U}}{\Gamma,B^-;\underline{\Delta}',\Delta\vdash\underline{U}}}{\Gamma;\underline{\Delta}',!B^-,\Delta\vdash\underline{U}}$$

$$\dfrac{\Gamma;\Delta\vdash A^- \qquad \dfrac{\Gamma;\underline{\Delta}',A^-\vdash\underline{U}}{\Gamma;\underline{\Delta}',1,A^-\vdash\underline{U}}}{\Gamma;\underline{\Delta}',1,\Delta\vdash\underline{U}} \;\approx\!> \; \dfrac{\dfrac{\Gamma;\Delta\vdash A^- \qquad \Gamma;\underline{\Delta}',A^-\vdash\underline{U}}{\Gamma;\underline{\Delta}',\Delta\vdash U}}{\Gamma;\underline{\Delta}',1,\Delta\vdash U}$$

$$\dfrac{\Gamma;\Delta\vdash A^- \qquad \dfrac{\Gamma;\Delta_1,A^-\vdash[B_1] \qquad \Gamma;\Delta_2\vdash[B_2]}{\Gamma;\Delta_1,A^-,\Delta_2\vdash[B_1\otimes B_2]}}{\Gamma;\Delta_1,\Delta,\Delta_2\vdash[B_1\otimes B_2]} \;\approx\!> \; \dfrac{\dfrac{\Gamma;\Delta\vdash A^- \qquad \Gamma;\Delta_1,A^-\vdash[B_1]}{\Gamma;\Delta_1,\Delta\vdash[B_1]} \qquad \Gamma;\Delta_2\vdash[B_2]}{\Gamma;\Delta_1,\Delta,\Delta_2\vdash[B_1\otimes B_2]}$$

$$\dfrac{\Gamma;\Delta\vdash A^- \qquad \dfrac{\Gamma;\Delta_1\vdash[B_1] \qquad \Gamma;\Delta_2,A^-\vdash[B_2]}{\Gamma;\Delta_1,\Delta_2,A^-\vdash[B_1\otimes B_2]}}{\Gamma;\Delta_1,\Delta_2,\Delta\vdash[B_1\otimes B_2]} \;\approx\!> \; \dfrac{\Gamma;\Delta_1\vdash[B_1] \qquad \dfrac{\Gamma;\Delta\vdash A^- \qquad \Gamma;\Delta_2,A^-\vdash[B_2]}{\Gamma;\Delta_2,\Delta\vdash[B_2]}}{\Gamma;\Delta_1,\Delta_2,\Delta\vdash[B_1\otimes B_2]}$$

$$\dfrac{\Gamma;\Delta \vdash A^- \qquad \dfrac{\Gamma;\underline{\Delta}',B_1,B_2,A^- \vdash \underline{U}}{\Gamma;\underline{\Delta}',B_1\otimes B_2,A^- \vdash \underline{U}}}{\Gamma;\underline{\Delta}',B_1\otimes B_2,\Delta \vdash \underline{U}} \;\rightsquigarrow\; \dfrac{\dfrac{\Gamma;\Delta\vdash A^- \qquad \Gamma;\underline{\Delta}',B_1,B_2,A^-\vdash \underline{U}}{\Gamma;\underline{\Delta}',B_1,B_2,\Delta\vdash\underline{U}}}{\Gamma;\underline{\Delta}',B_1\otimes B_2,\Delta\vdash\underline{U}}$$

$$\cancel{\Gamma;\Delta\vdash A^-} \qquad \cancel{\dfrac{\Gamma;}{\Gamma;\underline{\Delta}'\vdash A^+ \multimap B^-}}$$

$$\dfrac{\Gamma;\Delta\vdash A^- \qquad \dfrac{\Gamma;\underline{\Delta}',A^-,B_1\vdash B_2}{\Gamma;\underline{\Delta}',A^-\vdash B_1\multimap B_2}}{\Gamma;\underline{\Delta}',A^-\vdash B_1\multimap B_2} \;\rightsquigarrow\; \dfrac{\dfrac{\Gamma;\Delta\vdash A^- \qquad \Gamma;\underline{\Delta}',\Delta,B_1\vdash B_2}{\Gamma;\underline{\Delta}',A^-,B_1\vdash B_2}}{\Gamma;\underline{\Delta}',A^-\vdash B_1\multimap B_2}$$

$$\cancel{\Gamma;\Delta\vdash A^- \qquad \Gamma;\Delta_1,A}$$

$$\dfrac{\Gamma;\Delta\vdash A^- \qquad \dfrac{\Gamma;\Delta_1,A^-\vdash[B_1] \qquad \Gamma;\Delta_2,[B_2]\vdash U}{\Gamma;\Delta_1,A^-,\Delta_2,[B_1\multimap B_2]\vdash U}}{\Gamma;\Delta_1,\Delta,\Delta_2,[B_1\multimap B_2]\vdash U} \;\rightsquigarrow\; \dfrac{\dfrac{\Gamma;\Delta\vdash A^- \qquad \Gamma;\Delta_1,A^-\vdash[B_1]}{\Gamma;\Delta_1,\Delta\vdash[B_1]} \qquad \Gamma;\Delta_2,[B_2]\vdash U}{\Gamma;\Delta_1,\Delta_2,\Delta,[B_1\multimap B_2]\vdash U}$$

$$\dfrac{\Gamma;\Delta\vdash A^- \qquad \dfrac{\Gamma;\Delta_1\vdash[B_1] \qquad \Gamma;\Delta_2,A^-,[B_2]\vdash U}{\Gamma;\Delta_1,\Delta_2,A^-,[B_1\multimap B_2]\vdash U}}{\Gamma;\Delta_1,\Delta_2,\Delta[B_1\multimap B_2]\vdash U} \;\rightsquigarrow\; \dfrac{\Gamma;\Delta_1\vdash[B_1] \qquad \dfrac{\Gamma;\Delta\vdash A^- \qquad \Gamma;\Delta_2,A^-,[B_2]\vdash U}{\Gamma;\Delta_2,\Delta,[B_2]\vdash U}}{\Gamma;\Delta_1,\Delta_2,\Delta,[B_1\multimap B_2]\vdash U}$$

Part 5: $\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma, A; \underline{\Delta}\vdash\underline{U}}{\Gamma;\underline{\Delta}\vdash\underline{U}}$

RIGHT COMMUTATIVE CUTS

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta\vdash[B^+]}{\Gamma,A^-;\Delta\vdash B^+}}{\Gamma;\Delta\vdash B^+} \approx\!> \dfrac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta\vdash[B^-]}{\Gamma;\Delta\vdash[B^+]}}{\Gamma;\Delta\vdash B^+}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta,[B^-]\vdash U}{\Gamma,A^-;\Delta,B^-\vdash U}}{\Gamma;\Delta,B^-\vdash U} \approx\!> \dfrac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta,[B^-]\vdash U}{\Gamma;\Delta,[B^-]\vdash U}}{\Gamma;\Delta,B^-\vdash U}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta,[A^-]\vdash U}{\Gamma,A^-;\Delta\vdash U}}{\Gamma;\Delta\vdash U} \qquad \dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta,[A^-]\vdash U}{\Gamma;\Delta,[A^-]\vdash U}}{\Gamma;\Delta\vdash U}\ \boxed{2}$$

$$\dfrac{\Gamma,B^-\cancel{\Gamma};\cdot\vdash A^- \qquad \dfrac{\Gamma,B^-,A^-;\Delta,[B^-]\vdash U}{\Gamma,B^-,A^-;\Delta\vdash U}}{\cancel{\Gamma;\Delta}\,\Gamma,B^-;\Delta\vdash U} \approx\!> \dfrac{\dfrac{\Gamma,B^-;\cdot\vdash A^- \qquad \Gamma,B^-,A^-;\Delta,[B^-]\vdash U}{\Gamma,B^-;\Delta,[B^-]\vdash U}}{\Gamma,B^-;\Delta\vdash U}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\underline{\Delta},\langle p^+\rangle\vdash\underline{U}}{\Gamma,A^-;\underline{\Delta},p^+\vdash\underline{U}}}{\Gamma;\underline{\Delta},p^+\vdash\underline{U}} \approx\!> \dfrac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\underline{\Delta},\langle p^+\rangle\vdash\underline{U}}{\Gamma;\underline{\Delta},\langle p^+\rangle\vdash\underline{U}}}{\Gamma;\underline{\Delta},p^+\vdash\underline{U}}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{}{\Gamma,A^-;\langle B^+\rangle\vdash[B^+]}}{\Gamma;\langle B^+\rangle\vdash[B^+]} \approx\!> \dfrac{}{\Gamma;\langle B^+\rangle\vdash[B^+]}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \cancel{\dfrac{\Gamma;\underline{\Delta}}{\Gamma;\underline{\Delta}\vdash p^-}}\ \dfrac{\Gamma,A^-;\underline{\Delta}\vdash\langle p^-\rangle}{\Gamma,A^-;\underline{\Delta}\vdash p^-}}{\Gamma;\underline{\Delta}\vdash p^-} \approx\!> \dfrac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\underline{\Delta}\vdash\langle p^-\rangle}{\Gamma;\underline{\Delta}\vdash\langle p^+\rangle}}{\Gamma;\underline{\Delta}\vdash p^-}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{}{\Gamma,A^-;[B^-]\vdash\langle B^-\rangle}}{\Gamma;[B^-]\vdash\langle B^-\rangle} \rightsquigarrow \dfrac{}{\Gamma;[B^-]\vdash\langle B^-\rangle}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta\vdash B^+}{\Gamma,A^-;\Delta\vdash{\uparrow}B^+}}{\Gamma;\Delta\vdash{\uparrow}B^+} \rightsquigarrow \dfrac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta\vdash B^+}{\Gamma;\Delta\vdash B^+}}{\Gamma;\Delta\vdash{\uparrow}B^+}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta,B^+\vdash U}{\Gamma,A^-;\Delta,[{\uparrow}B^+]\vdash U}}{\Gamma;\Delta,[{\uparrow}B^+]\vdash U} \rightsquigarrow \dfrac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta,B^+\vdash U}{\Gamma;\Delta,B^+\vdash U}}{\Gamma;\Delta,[{\uparrow}B^+]\vdash U}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta\vdash B^-}{\Gamma,A^-;\Delta\vdash[{\downarrow}B^-]}}{\Gamma;\Delta\vdash[{\downarrow}B^-]} \rightsquigarrow \dfrac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta\vdash B^-}{\Gamma;\Delta\vdash B^-}}{\Gamma;\Delta\vdash[{\downarrow}B^-]}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta,B^-\vdash U}{\Gamma,A^-;\Delta,{\downarrow}B^-\vdash U}}{\Gamma;\Delta,{\downarrow}B^-\vdash U} \rightsquigarrow \dfrac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta,B^-\vdash U}{\Gamma;\Delta,B^-\vdash U}}{\Gamma;\Delta,{\downarrow}B^-\vdash U}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\cdot\vdash B^-}{\Gamma,A^-;\cdot\vdash[!B^-]}}{\Gamma;\cdot\vdash[!B^-]} \rightsquigarrow \dfrac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\cdot\vdash B^-}{\Gamma;\cdot\vdash B^-}}{\Gamma;\cdot\vdash[!B^-]}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-,B^-;\Delta\vdash U}{\Gamma,A^-;\Delta,!B^-\vdash U}}{\Gamma;\Delta,!B^-\vdash U} \rightsquigarrow \dfrac{\dfrac{\dfrac{\Gamma;\cdot\vdash A^-}{\Gamma,B^-;\cdot\vdash A^-}\,W \qquad \Gamma,A^-,B^-;\Delta\vdash U}{\cancel{\Gamma;\Delta}\ \Gamma,B^-;\Delta\vdash U}}{\Gamma;\Delta,!B^-\vdash U}$$

$$\dfrac{\Gamma;\cdot\vdash A^- \qquad \dfrac{}{\Gamma,A^-;\cdot\vdash[1]}}{\cancel{\Gamma,A^-;}\ \Gamma;\cdot\vdash[1]} \rightsquigarrow \dfrac{}{\Gamma;\cdot\vdash[1]}$$

$$\frac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta\vdash U}{\Gamma,A^-;\Delta,1\vdash U}}{\Gamma;\Delta,1\vdash U} \Rightarrow \frac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta\vdash U}{\Gamma;\Delta\vdash U}}{\Gamma;\Delta,1\vdash U}$$

$$\frac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta_1\vdash[B_1] \qquad \Gamma,A^-;\Delta_2\vdash[B_2]}{\Gamma,A^-;\Delta_1,\Delta_2\vdash[B_1\otimes B_2]}}{\cancel{\Gamma,A^-;\Delta}\ \Gamma;\Delta_1,\Delta_2\vdash[B_1\otimes B_2]}$$

$$\Rightarrow \frac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta_1\vdash[B_1]}{\Gamma;\Delta_1\vdash[B_1]} \qquad \dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta_2\vdash[B_2]}{\Gamma;\Delta_2\vdash[B_2]}}{\Gamma;\Delta_1,\Delta_2\vdash[B_1\otimes B_2]}$$

$$\frac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta,B_1,B_2\vdash U}{\Gamma,A^-;\Delta,B_1\otimes B_2\vdash U}}{\Gamma;\Delta,B_1\otimes B_2\vdash U} \Rightarrow \frac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta,B_1,B_2\vdash U}{\Gamma;\Delta,B_1,B_2\vdash U}}{\Gamma;\Delta,B_1\otimes B_2\vdash U}$$

$$\frac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta,B_1\vdash B_2}{\Gamma,A^-;\Delta\vdash B_1\multimap B_2}}{\Gamma;\Delta\vdash B_1\multimap B_2} \Rightarrow \frac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta,B_1\vdash B_2}{\Gamma;\Delta,B_1\vdash B_2}}{\Gamma;\Delta\vdash B_1\multimap B_2}$$

$$\frac{\Gamma;\cdot\vdash A^- \qquad \dfrac{\Gamma,A^-;\Delta_1\vdash[B_1] \qquad \Gamma,A^-;\Delta_2,[B_2]\vdash U}{\Gamma,A^-;\Delta_1,\Delta_2,[B_1\multimap B_2]\vdash U}}{\Gamma;\Delta_1,\Delta_2,[B_1\multimap B_2]\vdash U}$$

$$\Rightarrow \frac{\dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta_1\vdash[B_1]}{\Gamma;\Delta_1\vdash[B_1]} \qquad \dfrac{\Gamma;\cdot\vdash A^- \qquad \Gamma,A^-;\Delta_2,[B_2]\vdash U}{\Gamma;\Delta_2,[B_2]\vdash U}}{\Gamma;\Delta_1,\Delta_2,[B_1\multimap B_2]\vdash U}$$