

Structural focalization

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Focusing, introduced by Jean-Marc Andreoli in the context of classical linear logic [Andreoli 1992], defines a normal form for sequent calculus derivations that cuts down on the number of possible derivations by eagerly applying invertible rules and grouping sequences of non-invertible rules. A focused sequent calculus is defined relative to some non-focused sequent calculus; *focalization* is the property that every non-focused derivation can be transformed into a focused derivation.

In this paper, we present a focused sequent calculus for propositional intuitionistic logic and prove the focalization property relative to a standard presentation of propositional intuitionistic logic. Compared to existing approaches, the proof is quite concise, depending only on the internal soundness and completeness of the focused logic. In turn, both of these properties can be established (and mechanically verified) by structural induction in the style of Pfenning’s structural cut elimination without the need for any tedious and repetitious invertibility lemmas. The proof of cut admissibility for the focused system, which establishes internal soundness, is not particularly novel. The proof of identity expansion, which establishes internal completeness, is a major contribution of this work.

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1. INTRODUCTION

The propositions of intuitionistic propositional logic are easily recognizable and standard: we will consider a logic with atomic propositions, falsehood, disjunction, truth, conjunction, and implication.

$$P, Q ::= p \mid \perp \mid P_1 \vee P_2 \mid \top \mid P_1 \wedge P_2 \mid P_1 \supset P_2$$

The sequent calculus presentation for intuitionistic logic is also standard; the system in Figure 1 is precisely the propositional fragment of Kleene’s sequent system G_3 as presented in [Pfenning 2000]. Contexts Γ are, as usual, considered to be unordered multisets of propositions P , and the structural properties of exchange, weakening, and contraction are admissible (each left rule implicitly contains a contraction).

Sequent calculi are a nice way of presenting logics, and a logic’s sequent calculus presentation is a convenient setting in which to establish the logic’s metatheory in a way that is straightforwardly mechanizable in proof assistants (like Twelf or Agda) that are organized around the idea of structural induction. There are two key metatheoretic properties that we are interested in. The first, cut admissibility, justifies the use of lemmas: if we know P (if we have a derivation of the sequent $\Gamma \longrightarrow P$) and we know that Q follows from assuming P (if we have a derivation of the sequent $\Gamma, P \longrightarrow Q$), then we can come to know Q without the additional assumption of P (we can obtain a derivation of the sequent $\Gamma \longrightarrow Q$).¹ A proof of the cut admissibility property establishes the *internal soundness* of a logic

¹In common practice, the words *proof* and *derivation* are used interchangeably. In this article, we will be careful to refer to the formal objects constructed using sequent calculus rules (such as those in Figure 1) as *derivations*. Except when discussing natural deduction, the words *proof* and *theorem* will refer to theorems proved about these formal objects; these are frequently called *metatheorems* in the literature.

$$\boxed{\Gamma \longrightarrow P}$$

$$\begin{array}{c}
\overline{\Gamma, p \longrightarrow p} \textit{ init} \quad (\textit{no rule } \perp_R) \quad \overline{\Gamma, \perp \longrightarrow Q} \perp_L \\
\frac{\Gamma \longrightarrow P_1}{\Gamma \longrightarrow P_1 \vee P_2} \vee_{R1} \quad \frac{\Gamma \longrightarrow P_2}{\Gamma \longrightarrow P_1 \vee P_2} \vee_{R2} \\
\frac{\Gamma, P_1 \vee P_2, P_1 \longrightarrow Q \quad \Gamma, P_1 \vee P_2, P_2 \longrightarrow Q}{\Gamma, P_1 \vee P_2 \longrightarrow Q} \vee_L \\
\overline{\Gamma \longrightarrow \top} \top_R \quad (\textit{no rule } \top_L) \quad \frac{\Gamma \longrightarrow P_1 \quad \Gamma \longrightarrow P_2}{\Gamma \longrightarrow P_1 \wedge P_2} \wedge_R \\
\frac{\Gamma, P_1 \wedge P_2, P_1 \longrightarrow Q}{\Gamma, P_1 \wedge P_2 \longrightarrow Q} \wedge_{L1} \quad \frac{\Gamma, P_1 \wedge P_2, P_2 \longrightarrow Q}{\Gamma, P_1 \wedge P_2 \longrightarrow Q} \wedge_{L2} \\
\frac{\Gamma, P_1 \longrightarrow P_2}{\Gamma \longrightarrow P_1 \supset P_2} \supset_R \quad \frac{\Gamma, P_1 \supset P_2 \longrightarrow P_1 \quad \Gamma, P_1 \supset P_2, P_2 \longrightarrow Q}{\Gamma, P_1 \supset P_2 \longrightarrow Q} \supset_L
\end{array}$$

Fig. 1. Sequent calculus for intuitionistic logic.

– it implies that there are no closed proofs of contradiction, even by circuitous reasoning using lemmas. The identity property asserts that assuming P is always sufficient to conclude P , that is, that the sequent $\Gamma, P \longrightarrow P$ is always derivable. A proof of the identity property establishes the *internal completeness* of a logic. We call these properties *internal*, following Pfenning [2010], to emphasize that these are properties of the deductive system itself and not a comment on the system’s relationship to any external semantics.

There is a tradition in logic, dating back to Gentzen [1935], that views the sequent calculus as a convenient formalism for proving a logic’s metatheoretic properties while viewing natural deduction proofs as the “true proof objects.”² One reason for this bias towards natural deduction is that natural deduction proofs have nice normalization properties. A natural deduction proof is *normal* if there are no instances of an introduction rule immediately followed by an elimination rule of the same connective; such detours give rise to *local reductions* which eliminate the detour, such as this one:

$$\frac{\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{P_1 \textit{ true} \quad P_2 \textit{ true}} \wedge_I}{\frac{P_1 \wedge P_2 \textit{ true}}{P_1 \textit{ true}} \wedge_{E1}} \Longrightarrow_R \quad \mathcal{D}_1 \quad P_1 \textit{ true}$$

The *normalization* property says that every natural deduction proof can be transformed into a normal natural deduction proof.

We are frequently interested in the set of normal natural deduction proofs of a given proposition. As an example, there is exactly one normal natural deduction proof for $(p \wedge q) \supset (r \wedge s) \supset (p \wedge r)$. Presented as a derivation, that natural deduction

²This discussion assumes a basic familiarity with natural deduction. We refer the interested reader to Girard, Taylor, and Lafont’s *Proofs and Types* [Girard et al. 1989]; the aforementioned quote comes from Section 5.4 of that work.

proof looks like this:

$$\frac{\frac{\frac{}{p \wedge q \text{ true}}{p \text{ true}} \text{ hyp}_u \quad \frac{\frac{}{r \wedge s \text{ true}}{r \text{ true}} \text{ hyp}_v}{\wedge_{E1}}}{\wedge_{E1}} \quad \frac{}{p \wedge r \text{ true}} \wedge_I}{\frac{}{(r \wedge s) \supset (p \wedge r) \text{ true}} \supset_I^v} \supset_I^u \quad \frac{}{(p \wedge q) \supset (r \wedge s) \supset (p \wedge r) \text{ true}} \supset_I^u$$

Under the standard proof term assignment for natural deduction, this (normal) natural deduction proof corresponds to the (irreducible) proof term $\lambda x.\lambda y.\langle \pi_1 x, \pi_1 y \rangle$.

In contrast, there are many sequent calculus derivations of the same proposition. Here's one of them:

$$\frac{\frac{\frac{}{p \wedge q, r \wedge s, p \longrightarrow p} \text{ init}}{\wedge_{L1}} \quad \frac{\frac{}{p \wedge q, r \wedge s, r \longrightarrow r} \text{ init}}{\wedge_{L1}}}{\wedge_R} \quad \frac{}{p \wedge q, r \wedge s \longrightarrow p \wedge r} \wedge_R}{\supset_R} \quad \frac{}{p \wedge q \longrightarrow (r \wedge s) \supset (p \wedge r)} \supset_R}{\cdot \longrightarrow (p \wedge q) \supset (r \wedge s) \supset (p \wedge r)} \supset_R$$

Reading from bottom to top, this derivation decomposes $(p \wedge q) \supset (r \wedge s) \supset (p \wedge r)$ on the right, then decomposes $(r \wedge s) \supset (p \wedge r)$ on the right, then decomposes $p \wedge r$ on the right, and then (in one branch) decomposes $p \wedge q$ on the left while (in the other branch) decomposing $r \wedge s$ on the left. Other possibilities include decomposing $p \wedge q$ on the left before decomposing $(r \wedge s) \supset (p \wedge r)$ on the right and decomposing $r \wedge s$ on the left before $p \wedge r$ on the right; there are at least six different derivations even if you don't count derivations that do useless decompositions on the left.

These different derivations are particularly problematic if our goal is to do proof search for sequent calculus derivations, as inessential differences between derivations correspond to unnecessary choice points that a proof search procedure will need to backtrack over. It was in this context that Andreoli originally introduced the idea of focusing. Some connectives, such as implication $A \supset B$, are called *asynchronous* because their right rules can always be applied eagerly, without backtracking, during bottom-up proof search. Other connectives, such as disjunction $A \vee B$, are called *synchronous* because their right rules cannot be applied eagerly. For instance, the \vee_{R1} rule cannot be applied eagerly if we are looking for a proof of $p \longrightarrow \perp \vee p$. This asynchronous or synchronous character is the connective's *polarity*.³

³Andreoli dealt with a one-sided classical sequent calculus; in intuitionistic logic, it is common to call asynchronous connectives *right-asynchronous* and *left-synchronous*. Similarly, it is common to call synchronous connectives *right-synchronous* and *left-asynchronous*.

Synchronicity or polarity, a property of connectives, is closely connected to (and sometimes conflated with) a property of rules called *invertibility*; a rule is invertible if the conclusion of the rule implies each of the premises. So \supset_R is invertible ($\Gamma \longrightarrow P_1 \supset P_2$ implies $\Gamma, P_1 \longrightarrow P_2$) but \supset_L is not ($\Gamma, P_1 \supset P_2 \longrightarrow C$ does not imply $\Gamma, P_1 \supset P_2 \longrightarrow P_1$). Rules that can be applied eagerly need to be invertible, so asynchronous connectives have invertible right rules and synchronous connectives have invertible left rules. Therefore, another synonym for asynchronous is *right-invertible*, and another synonym for synchronous is *left-invertible*. This terminology would be misleading in our case, as both the left and right rules for conjunction in Figure 1 are invertible.

Andreoli’s key observation was that proof search only needs to consider derivations that have two alternating phases. In *inversion* phases, we eagerly apply (invertible) right rules to asynchronous connectives and (invertible) left rules to synchronous ones. When this is no longer possible, we begin a *focusing* phase by putting a single remaining proposition *in focus*, repeatedly decomposing it (and only it) by applying right rules to synchronous connectives and left rules to asynchronous ones. Andreoli described this restricted form of proof search as a regular proof search procedure in a restricted sequent calculus; such sequent calculi, and derivations in them, are called *focused* as opposed to *unfocused* [Andreoli 1992].

In order to adopt such a proof search strategy, it is important to know that the strategy is both sound (i.e., the proof search strategy will only say “the sequent has a derivation” if that is the case) and complete (i.e., the proof search strategy is capable of finding a proof if one exists). Soundness proofs for focusing are usually easy: focused derivations are essentially a syntactic refinement of the unfocused derivations. Completeness, the nontrivial direction, involves turning unfocused derivations into focused ones. This process is *focalization*.⁴ Thus, an effective procedure for focalization is a constructive witness to the completeness of focusing.

The techniques described in this article are general and can be straightforwardly transferred to other modal and substructural logics, as explored in the author’s dissertation [Simmons 2012]. Our approach has three key components, which we will now discuss in turn.

Focalization via cut and identity. Existing focalization proofs almost all fall prey to the need to prove multiple tedious invertibility lemmas describing the interaction of each rule with every other rule; this results in proofs that are unrealistic to write out, difficult to check, and exhausting to contemplate mechanizing. The way forward was first suggested by Chaudhuri [2006]. In his dissertation, he established the focalization property for linear logic as the consequence of the focused logic’s internal soundness (the cut admissibility property) and completeness (the identity property). Stating and proving the identity property for a focused sequent calculus has remained a challenge, however. A primary contribution of this work is *identity expansion*, a generalization of the identity property that is amenable to mechanized proof by structural induction on propositions. This identity property is, in turn, part of our larger development, a proof of the focalization property that entirely avoids the tedious invertibility lemmas that plague existing approaches. (We review existing techniques used to prove the focalization property in Section 6.)

Refining the focused calculus. The focused logic presented in this article is essentially equivalent to the presentation of LJF given by Liang and Miller [2009], a point we will return to in Section 2.1. A reader familiar with LJF will note three non-cosmetic differences. The first two, our use of a *polarized* variant of intuition-

⁴The usage of *focus*, *focusing*, *focussing*, and *focalization* is not standard in the literature. We use the words *focus* and *focusing* to describe a logic (e.g. the focused sequent calculus) and aspects of that logic (e.g. focused derivations, propositions in focus, left- or right-focused sequents, and focusing phases). *Focalization*, derived from the French *focalisation*, is reserved exclusively for the act of producing a focused derivation given an unfocused derivation; the focalization *property* establishes that focalization is always possible.

istic logic and our novel treatment of atomic and suspended propositions, will be discussed further in Section 2. A third change is that LJF does not force any particular ordering for the application of rules during an inversion phase. A seemingly inevitable consequence of this choice is that the proof of focalization must establish the equivalence of all permutations of these invertible rules; this is one of the aforementioned tedious invertibility lemmas that plague proofs of the focalization property. Our focused logic, like many others (including Andreoli’s original system) fixes a particular inversion order.

We introduce a new calculus rather than reusing an existing one in order to present a focused logic and focalization proof that is computationally clean and straightforward to mechanize. Our desire to mechanize proofs of the focalization property also informed our decision to use propositional intuitionistic logic. All proofs in this article are mechanized in both Twelf [Pfenning and Schürmann 1999] and Agda [Norell 2007], though we will only mention the Twelf development. The concrete basis for our claim of computational cleanliness is that our mechanizations are complete artifacts capturing the constructive content of the proofs we present, and the size of this artifact scales linearly relative to the number of connectives; the approaches we call “tedious” tend to scale quadratically.

Proof terms. Since Andreoli’s original work, focused sequent calculus derivations have been shown to be isomorphic to normal natural deduction proofs for restricted fragments of logic [Cervesato and Pfenning 2003] and variations on the usual focusing discipline [Howe 2001]. Such results challenge the position that natural deduction proofs are somehow more fundamental than sequent calculus derivations and also indicate that focalization is a fundamental property of logic. In Section 2.4, we present a proof term language for polarized intuitionistic logic that directly captures the branching and binding structure of focused derivations. The result is a term language generalizing the *spine form* of Cervesato and Pfenning [2003].

By presenting our cut admissibility and identity expansion proofs at the level of proof terms, we emphasize that the constructive content of a cut admissibility proof is a substitution function on proof terms generalizing the *hereditary substitution* of Watkins et al. [2002] in a spine form setting. Similarly, the constructive content of our identity expansion proof is a novel η -expansion property on proof terms.

1.1 Outline

This article is dealing with three “soundness” properties and three “completeness” properties, so it is important to carefully explain what we’re doing and when; the following discussion is represented graphically in Figure 2.

We present a new proof of the completeness of *focusing* (the focalization property, Theorem 4) for intuitionistic logic; the proof of the focalization property follows from the *internal* soundness and completeness of the focused sequent calculus (cut admissibility, Theorem 2, and identity expansion, Theorem 3). We will start, in Section 2, by motivating a polarized presentation of logic that syntactically differentiates the synchronous and asynchronous propositions. We then present a focused sequent calculus for polarized propositional intuitionistic logic and formally state the soundness and completeness of focusing. We also prove the soundness of focusing (the de-focalization property, Theorem 1) in this section, but it’s pretty

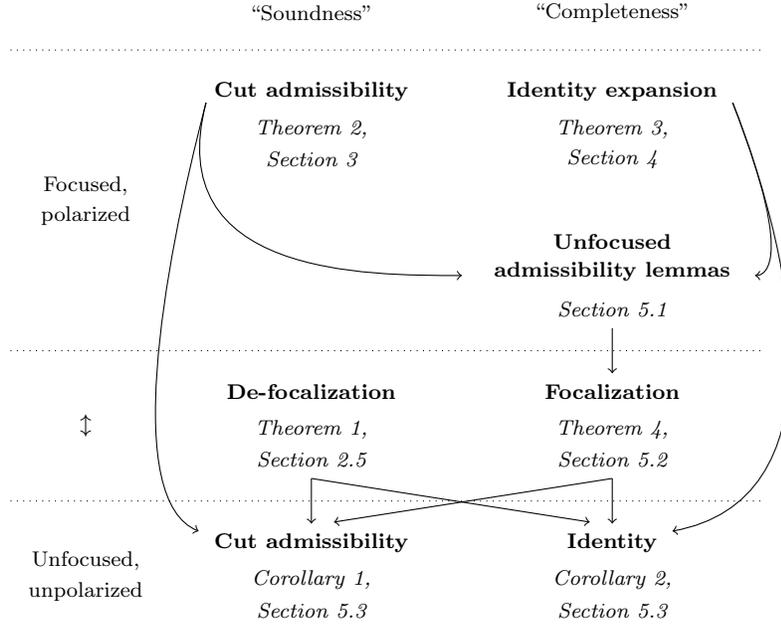


Fig. 2. Theorems and their dependencies.

boring and independent of the proofs of cut admissibility, identity expansion, and the completeness of focusing.

Internal soundness for the focused sequent calculus is established by the cut admissibility theorem in Section 3, and internal completeness for the focused sequent calculus is established in Section 4 using a generalization of the identity expansion theorem first developed in [Simmons and Pfenning 2011b]. In Section 5 we prove the focalization property by showing *unfocused admissibility*, a group of lemmas establishing that the focused sequent calculus can act like an unfocused sequent calculus. Finally, rather than proving the internal soundness and completeness (cut and identity) for the unfocused system directly, we show that these properties can be established as corollaries of the first four theorems. In Section 6 we conclude with an overview of existing proofs of the focalization property.

We will henceforth avoid using the words *soundness* and *completeness* as much as possible. Instead, we will refer to the cut admissibility and identity theorems for the focused and unfocused sequent calculi by name, and will refer to the soundness and completeness of focusing as de-focalization and focalization, respectively.

2. POLARIZED LOGIC

There is a significant line of work on polarity in logic dating back to Andreoli [1992] and Girard [1993]. That line of work holds that polarity is a refinement property of propositions, that the asynchronous and synchronous propositions are syntactic refinements of the set of all propositions. We can determine the synchronous or

$$\begin{array}{ll}
(p^+)^\bullet = p^+ & (p^-)^\bullet = p^- \\
(\downarrow A^-)^\bullet = (A^-)^\bullet & (\uparrow A^+)^\bullet = (A^+)^\bullet \\
(\perp)^\bullet = \perp & \\
(A^+ \vee B^+)^\bullet = (A^+)^\bullet \vee (B^+)^\bullet & (A^+ \supset B^-)^\bullet = (A^+)^\bullet \supset (B^-)^\bullet \\
(\top^+)^\bullet = \top & (\top^-)^\bullet = \top \\
(A^+ \wedge^+ B^+)^\bullet = (A^+)^\bullet \wedge (B^+)^\bullet & (A^- \wedge^- B^-)^\bullet = (A^-)^\bullet \wedge (B^-)^\bullet
\end{array}$$

Fig. 3. Erasure of polarized propositions.

asynchronous character of a proposition by inspecting its outermost connective.⁵

In a 1991 note published to the LINEAR list [Girard 1991], Girard introduced the idea of syntactically differentiating the *positive* propositions (those Andreoli called synchronous) from the *negative* propositions (those Andreoli called asynchronous) while mediating between the two with *shifts*: the upshift $\uparrow A^+$ includes positive propositions in the negative ones, and the downshift $\downarrow A^-$ includes negative propositions in the positive ones. This *polarization*⁶ of logic was developed further by Girard in Ludics [Girard 2001] and treated extensively in Laurent and Zeilberger’s Ph.D. theses [Laurent 2002; Zeilberger 2009b]. These are the propositions, positive and negative, for polarized intuitionistic logic:

$$\begin{array}{l}
A^+, B^+, C^+ ::= p^+ \mid \downarrow A^- \mid \perp \mid A^+ \vee B^+ \mid \top^+ \mid A^+ \wedge^+ B^+ \\
A^-, B^-, C^- ::= p^- \mid \uparrow A^+ \mid A^+ \supset B^- \mid \top^- \mid A^- \wedge^- B^-
\end{array}$$

Linear logic is able to unambiguously assign all connectives to one category or the other, but in intuitionistic logic, truth \top and conjunction $P_1 \wedge P_2$ can be understood as having either a positive character (corresponding to $\mathbf{1}$ and $A^+ \otimes B^+$ in linear logic) or a negative character (corresponding to \top and $A^- \& B^-$ in linear logic). We take the maximally general approach and allow both versions of truth and conjunction, which are decorated to emphasize their polarity.

The shifts introduced by Girard were modalities that might change the provability of a proposition. We adopt the later stance of Zeilberger [2009b], McLaughlin and Pfenning [2009], and others: shifts influence the structure of proofs, but not the provability of propositions. Therefore, we expect there to be a focused proof of A^+ or A^- if and only if there is an unfocused proof of $(A^+)^\bullet$ or $(A^-)^\bullet$, where $(-)^{\bullet}$ is the *erasure* function given in Figure 3. The polarity of an atomic proposition can be arbitrary as long as it is consistent, as if each individual atomic proposition p is really intrinsically positive or negative, but the calculus in Figure 1 didn’t notice.

Shifts and polarization are computationally interesting phenomena. Our view of polarization lines up with the *call-by-push-value* system independently developed

⁵Linear logic naturally has two polarities. Other systems, like Girard’s LU and Liang and Miller’s LKU, use more than these two polarities [Girard 1993; Liang and Miller 2011]. Furthermore, in LU the polarity of a proposition is determined by more than just the outermost connective.

⁶For the purposes of this article we are making somewhat artificial distinction between *polarity*, Andreoli’s classification of propositions as asynchronous and synchronous, and *polarization*, the segregation of these two classes as positive or negative propositions by using shifts. This distinction is not a standard one: “synchronous” and “positive” are elsewhere used interchangeably, “polarity” is used to describe why we call “polarization,” and so on.

by Levy: positive propositions correspond to *value types* and negative propositions correspond to *computation types* [Levy 2004]. Shifts are also useful in theorem proving. By employing different *polarization strategies*, the name for partial inverses of erasure, the Imogen theorem prover can simulate fully-focused LJF proof search, proof search in an unfocused logic like Kleene’s G_3 , and proof search in many other partially-focused systems in between [McLaughlin and Pfenning 2009]. Furthermore, forward-chaining versus backward-chaining logic programming can be seen as arising from particular polarization strategies in uniform proof search [Chaudhuri et al. 2008]. However, in this article polarization is primarily a technical device. Shifts makes it clearer that certain proofs are structurally inductive over propositions, especially identity expansion (Theorem 3). Additionally, the focus-interrupting nature of the shift is a critical part of our unfocused admissibility lemmas.⁷

2.1 Sequent calculus

We will develop our focused sequent calculus in two stages; in the first stage we do not consider atomic propositions. We can present sequents for our polarized logic in two equivalent ways. In the *one-sequent* view, we say that all sequents have the form $\Gamma; L \vdash U$. The components of a sequent are defined by the following (currently incomplete) grammar:

<i>Hypothetical contexts</i>	$\Gamma ::= \cdot \mid \Gamma, A^- \mid \dots$
<i>Inversion contexts</i>	$\Omega ::= \cdot \mid A^+, \Omega$
<i>Antecedents</i>	$L ::= \Omega \mid [A^-]$
<i>Succedents</i>	$U ::= [A^+] \mid A^+ \mid A^- \mid \dots$

This first view requires us to further restrict the form of sequents for two reasons. First, we only want to focus on one proposition at a time, so only one right focus $[A^+]$ or left focus $[A^-]$ should be present in a sequent. Second, focus and inversion phases should not overlap, so it must be the case that $L = \cdot$ when $U = [A^+]$ and that $U \neq A^-$ when $L = [A^-]$. Given this restriction, the first view of sequents is equivalent to a *three-sequent* view in which there are three different sequent forms:

- Right focus: $\Gamma \vdash [A^+]$, where $L = \cdot$ and is therefore omitted,
- Inversion: $\Gamma; \Omega \vdash U$, where $U \neq [A^+]$, and
- Left focus: $\Gamma; [A^-] \vdash U$, where U is *stable* (more about this shortly).

The sequent calculus for polarized intuitionistic logic in Figure 4 is presented in terms of this three-sequent view. The *right focus* sequent $\Gamma \vdash [A^+]$ describes a state in which non-invertible right rules are being applied to positive propositions, the *left focus* sequent $\Gamma; [A^-] \vdash U$ describes a state in which non-invertible left rules

⁷Something like a shift is necessary for Liang and Miller’s focalization proofs as well. Because their logic has polarity but no shifts, they construct *delays* $\delta^+(A) = \top^+ \wedge^+ A$ and $\delta^-(A) = \top^+ \supset A$ to force asynchronous or synchronous connectives to behave (respectively) like synchronous or asynchronous ones [Liang and Miller 2009]. This gets in the way of defining logical fragments by adding or removing connectives, as the presence of the connective \top^+ is fundamental to their completeness proof.

$\Gamma \vdash [A^+] - \textit{right focus}$

$$\begin{array}{c}
 \frac{\Gamma; \cdot \vdash A^-}{\Gamma \vdash [\downarrow A^-]} \downarrow_R \\
 \\
 \text{(no rule } \perp_R) \quad \frac{\Gamma \vdash [A^+]}{\Gamma \vdash [A^+ \vee B^+]} \vee_{R1} \quad \frac{\Gamma \vdash [B^+]}{\Gamma \vdash [A^+ \vee B^+]} \vee_{R2} \\
 \\
 \frac{}{\Gamma \vdash [\top^+]} \top_R^+ \quad \frac{\Gamma \vdash [A^+] \quad \Gamma \vdash [B^+]}{\Gamma \vdash [A^+ \wedge^+ B^+]} \wedge_R^+
 \end{array}$$

$\Gamma; \Omega \vdash U - \textit{inversion}, U \neq [A^+]$

$$\begin{array}{c}
 \frac{\Gamma \vdash [A^+]}{\Gamma; \cdot \vdash A^+} \textit{foc}_R \quad \frac{U \textit{ stable} \quad \Gamma, A^-; [A^-] \vdash U}{\Gamma, A^-; \cdot \vdash U} \textit{foc}_L \quad \frac{}{A^+ \textit{ stable}} \\
 \\
 \frac{\Gamma, A^-; \Omega \vdash U}{\Gamma; \downarrow A^-, \Omega \vdash U} \downarrow_L \\
 \\
 \frac{}{\Gamma; \perp, \Omega \vdash U} \perp_L \quad \frac{\Gamma; A^+, \Omega \vdash U \quad \Gamma; B^+, \Omega \vdash U}{\Gamma; A^+ \vee B^+, \Omega \vdash U} \vee_L \\
 \\
 \frac{\Gamma; \Omega \vdash U}{\Gamma; \top^+, \Omega \vdash U} \top_L^+ \quad \frac{\Gamma; A^+, B^+, \Omega \vdash U}{\Gamma; A^+ \wedge^+ B^+, \Omega \vdash U} \wedge_L^+ \\
 \\
 \frac{\Gamma; \cdot \vdash A^+}{\Gamma; \cdot \vdash \uparrow A^+} \uparrow_R \quad \frac{\Gamma; A^+ \vdash B^-}{\Gamma; \cdot \vdash A^+ \supset B^-} \supset_R \\
 \\
 \frac{}{\Gamma; \cdot \vdash \top^-} \top_R^- \quad \frac{\Gamma; \cdot \vdash A^- \quad \Gamma; \cdot \vdash B^-}{\Gamma; \cdot \vdash A^- \wedge^- B^-} \wedge_R^-
 \end{array}$$

$\Gamma; [A^-] \vdash U - \textit{left focus}, U \textit{ must be stable}$

$$\begin{array}{c}
 \frac{\Gamma; A^+ \vdash U}{\Gamma; [\uparrow A^+] \vdash U} \uparrow_L \quad \frac{\Gamma \vdash [A^+] \quad \Gamma; [B^-] \vdash U}{\Gamma; [A^+ \supset B^-] \vdash U} \supset_L \\
 \\
 \text{(no rule } \top_L^-) \quad \frac{\Gamma; [A^-] \vdash U}{\Gamma; [A^- \wedge^- B^-] \vdash U} \wedge_{L1}^- \quad \frac{\Gamma; [B^-] \vdash U}{\Gamma; [A^- \wedge^- B^-] \vdash U} \wedge_{L2}^-
 \end{array}$$

Fig. 4. Focused sequent calculus for polarized intuitionistic logic (sans suspended propositions).

are being applied to negative propositions, and the *inversion* sequent $\Gamma; \Omega \vdash U$ describes everything else. While we treat the hypothetical context Γ informally as a multiset, the *inversion context* Ω is *not* a multiset. Instead, it should be thought of as an ordered sequence of positive propositions: the empty sequence is written as “.” and “,” is an associative append operator. Whenever the context Ω is non-empty, the *only* applicable rule is the one that decomposes the left-most positive connective in the context.

The picture in Figure 4 is quite uniform: every rule except for \textit{foc}_R and \textit{foc}_L breaks down a single connective, while the shift rules regulate the focus and inversion phases. Read from bottom up, they put an end to the process of breaking

down a proposition under focus (\downarrow_R, \uparrow_L) and to the process of breaking down a proposition with inversion (\downarrow_L, \uparrow_R). All other rules maintain focus or inversion on the subformulas of a proposition.

The conclusions of foc_R and foc_L are inversion sequents with empty inversion contexts and succedents U that are stable, meaning that there is no possibility of applying an invertible rule. These *stable sequents* (sometimes called *neutral*) have an important place in focused sequent calculi. In the introduction, we claimed that there was an essential equivalence between LJF and our focused presentation, but this is only true if we ignore the internal structure of focus and inversion phases and work with *synthetic rules*, the derivation fragments comprised of one or more inversion phases stacked on top of a single focused phase. In the synthetic view of focusing, we abstract away from the internal structure of focusing phases to emphasize the stable sequents that lie between them [Andreoli 2001]. LJF and our presentation of polarized intuitionistic logic have different internal structure, but we claim that the systems give rise to the same synthetic rules.⁸ Aside from our use of shifts, we depart from Liang and Miller in ways that are technically relevant but that are invisible at the level of synthetic connectives.

The stability requirement for U in left focus sequents $\Gamma; [A^-] \vdash U$ can equivalently be stated as an extra premise U *stable* for the rule \uparrow_L , which is done in the accompanying Twelf development.

2.2 Suspended propositions

The pleasant picture of focusing given above is complicated when we consider atomic propositions. Atomic propositions are best understood as stand-ins for arbitrary propositions, and so our polarized logic has both positive atomic propositions (stand-ins for arbitrary positive propositions) and negative atomic propositions (stand-ins for arbitrary negative propositions).

When we are performing inversion and we reach an atomic proposition, we do not have enough information to break down that proposition any further, but we have not reached a shift. We have to do something different. What we do is *suspend* that atomic proposition, either in the hypothetical context or in the succedent. We represent a suspended atomic proposition as $\langle p^+ \rangle$ or $\langle p^- \rangle$. If we wanted to closely follow existing focused sequent calculi, we would introduce two more rules for proving atomic propositions in focus using a suspended atomic proposition. The resulting extension of Figure 4 would look something like this:

$$\frac{\Gamma, \langle p^+ \rangle; \Omega \vdash U}{\Gamma; p^+, \Omega \vdash U} \quad \frac{\Gamma; \cdot \vdash \langle p^- \rangle}{\Gamma; \cdot \vdash p^-} \quad \frac{}{\Gamma, \langle p^+ \rangle \vdash [p^+]} \quad \frac{}{\Gamma; [p^-] \vdash \langle p^- \rangle} \quad \frac{}{\langle p^- \rangle \text{ stable}}$$

This treatment is not incorrect and is obviously analogous to the *init* rule from the unfocused system in Figure 1. Nevertheless, we contend that this is a design error and a large part of why it has historically been difficult to prove the identity theorem for focused systems. We generalize these rules by allowing the hypothetical context to contain arbitrary suspended positive propositions (not just atomic

⁸Our presentation does not have the first-order quantifiers present in LJF, and LJF lacks a negative unit \top^- , but quantifiers can be added to our system easily, and the same is true for \top^- in LJF.

$$\begin{array}{c}
\overline{\Gamma, \langle A^+ \rangle \vdash [A^+]} \quad id^+ \quad \overline{\Gamma; [A^-] \vdash \langle A^- \rangle} \quad id^- \\
\overline{\langle A^- \rangle \text{ stable}} \quad \frac{\Gamma, \langle p^+ \rangle; \Omega \vdash U}{\Gamma; p^+, \Omega \vdash U} \eta^+ \quad \frac{\Gamma; \cdot \vdash \langle p^- \rangle}{\Gamma; \cdot \vdash p^-} \eta^-
\end{array}$$

Fig. 5. Focused sequent calculus, extended with suspended propositions

positive propositions) and allowing the succedent to contain arbitrary suspended negative propositions (not just atomic negative propositions).

This generalization allows us to finally give the complete grammar of hypothetical contexts and succedents:

$$\begin{array}{ll}
\textit{Hypothetical contexts} & \Gamma ::= \cdot \mid \Gamma, A^- \mid \Gamma, \langle A^+ \rangle \\
\textit{Succedents} & U ::= [A^+] \mid A^+ \mid A^- \mid \langle A^- \rangle
\end{array}$$

The rules for atomic propositions, extending Figure 4, are given in Figure 5. The η^+ and η^- rules are the same as the ones we discussed above, reflecting the fact that the inversion process *must* suspend itself at an atomic proposition and *should not* suspend itself any earlier. The id^+ and id^- rules directly describe an identity or hypothesis principle, but only for suspended propositions.

These more general id^+ and id^- rules allow us to define two substitution principles, which are critical for the proof of identity expansion in Section 4. The derivation of a right-focused sequent $[A^+]$ can discharge $\langle A^+ \rangle$ in the hypothetical context, and the derivation of a left-focused sequent $[A^-]$ can discharge $\langle A^- \rangle$ in the succedent. Written as admissible rules, these two *focal substitution* principles are as follows:

$$\frac{\Gamma \vdash [A^+] \quad \Gamma, \langle A^+ \rangle; L \vdash U}{\Gamma; L \vdash U} \textit{subst}^+ \quad \frac{\Gamma; L \vdash \langle A^- \rangle \quad \Gamma; [A^-] \vdash U}{\Gamma; L \vdash U} \textit{subst}^-$$

It is straightforward to establish the positive focal substitution principle by induction over the derivation of $\Gamma, \langle A^+ \rangle; L \vdash U$, and it is likewise straightforward to establish the negative substitution principle by induction over the derivation of $\Gamma; L \vdash \langle A^- \rangle$. When the last rule in the derivation we're inducting over is id^+ or id^- , we return the derivation we're not inducting over, and in every other case we apply the induction hypothesis directly.⁹

The admissible rules \textit{subst}^+ and \textit{subst}^- are *uniform* substitution principles. This means that, in the accompanying Twelf development, it is possible to get them both for free from the LF function space, the same way we get weakening of the hypothetical context for free and generally take it for granted. This is natural in the case of positive focal substitution: we interpret the suspended atomic proposition $\langle A^+ \rangle$ as a uniform assumption that A^+ is provable in right focus. It is more counterintuitive to get negative focal substitution for free in LF; we refer the reader to the accompanying Twelf development for details.

The logic extended with these more general id^+ and id^- rules conservatively extends the logic with the more traditional rules we initially proposed. Reading

⁹In the case of the rules η^+ and \downarrow_L , we also apply an admissible weakening principle to the derivation we're not inducting over.

$$\begin{array}{l}
(\Omega)^\bullet \quad (\cdot)^\bullet = \cdot \quad (A^+, \Omega)^\bullet = (A^+)^\bullet, (\Omega)^\bullet \\
(\Gamma)^\otimes \quad (\cdot)^\otimes = \cdot \quad (\Gamma, A^-)^\otimes = (\Gamma)^\otimes, (A^-)^\bullet \quad (\Gamma, \langle p^+ \rangle)^\otimes = (\Gamma)^\otimes, p^+ \\
(L)^\otimes \quad ([A^-])^\otimes = (A^-)^\bullet \quad (\Omega)^\otimes = (\Omega)^\bullet \\
(U)^\otimes \quad ([A^+])^\otimes = (A^+)^\bullet \quad (A^+)^\otimes = (A^+)^\bullet \quad (A^-)^\otimes = (A^-)^\bullet \quad (\langle p^- \rangle)^\otimes = p^-
\end{array}$$

Fig. 6. Erasure of contexts and succedents.

rules from bottom to top, the rules η^+ and η^- are the only ones that introduce suspended propositions. Therefore, given the derivation of a sequent where every suspended proposition is atomic, we know that every instance of id^+ and id^- in that derivation acts on an atomic proposition. We call sequents where every suspended proposition is atomic *suspension-normal* sequents. Certain operations, in particular erasure and cut admissibility, are only defined on suspension-normal sequents and derivations of these sequents.

2.3 Erasure and focalization

We presented the erasure of propositions in Figure 3, and Figure 6 describes the erasure of a polarized contexts and sequents. Note that erasure is only defined on hypothetical contexts Γ and succedents U that are suspension-normal.

Erasure is a pretty boring operation, important mainly because it allows us to state soundness and completeness of focusing. We want to understand completeness in terms of stable, suspension-normal sequents, so the correctness of focusing states that, if Γ and U are stable and suspension-normal, $\Gamma; \cdot \vdash U$ if and only if $(\Gamma)^\otimes \longrightarrow (U)^\otimes$. The backward (completeness) direction is focalization and the forward (soundness) direction is de-focalization.

Many different polarized propositions will typically erase to the same unpolarized proposition. The proposition used in the example from the introduction, $(p \wedge q) \supset (r \wedge s) \supset (p \wedge r)$, is the erasure of each of the following:

$$\downarrow(p \wedge^- q) \supset \downarrow(r \wedge^- s) \supset (p \wedge^- r) \quad (1)$$

$$(p \wedge^+ q) \supset (r \wedge^+ s) \supset \uparrow(p \wedge^+ r) \quad (2)$$

$$\downarrow(\uparrow p \wedge^- \uparrow q) \supset \uparrow \downarrow(\downarrow(\uparrow r \wedge^- \uparrow s) \supset \uparrow \downarrow(\uparrow p \wedge^- \uparrow r)) \quad (3)$$

Note that the first proposition implies a negative polarity for all atomic propositions and the last two propositions imply a positive polarity.

The first and second propositions each have *exactly one* focused derivation, just as the unpolarized propositions had exactly one normal natural deduction proof. The unique derivation of (1) is structurally similar to the unfocused derivation from

$$\boxed{\Gamma \vdash V : [A^+] - \text{values } V}$$

$$\frac{}{\Gamma, z : \langle A^+ \rangle \vdash z : [A^+]} id^+ \quad \frac{\Gamma; \cdot \vdash N : A^-}{\Gamma \vdash \text{thunk } N : [\downarrow A^-]} \downarrow_R$$

$$(\text{no rule } \perp_R) \quad \frac{\Gamma \vdash V : [A^+]}{\Gamma \vdash \text{inl } V : [A^+ \vee B^+]} \vee_{R1} \quad \frac{\Gamma \vdash V : [B^+]}{\Gamma \vdash \text{inr } V : [A^+ \vee B^+]} \vee_{R2}$$

$$\frac{}{\Gamma \vdash \langle \rangle^+ : [\top^+]} \top_R^+ \quad \frac{\Gamma \vdash V_1 : [A^+] \quad \Gamma \vdash V_2 : [B^+]}{\Gamma \vdash \langle V_1, V_2 \rangle^+ : [A^+ \wedge^+ B^+]} \wedge_R^+$$

$$\boxed{\Gamma; \Omega \vdash N : U - \text{terms } N, U \neq [A^+]}$$

$$\frac{\Gamma \vdash V : [A^+]}{\Gamma; \cdot \vdash \text{ret } V : A^+} foc_R \quad \frac{U \text{ stable} \quad \Gamma, x : A^-; [A^-] \vdash S : U}{\Gamma, x : A^-; \cdot \vdash x \circ S : U} foc_L \quad \frac{}{A^+ \text{ stable}} \quad \frac{}{\langle A^- \rangle \text{ stable}}$$

$$\frac{\Gamma, z : \langle p^+ \rangle; \Omega \vdash N : U}{\Gamma; p^+, \Omega \vdash \langle z \rangle.N : U} \eta^+ \quad \frac{\Gamma, x : A^-; \Omega \vdash N : U}{\Gamma; \downarrow A^-, \Omega \vdash x.N : U} \downarrow_L$$

$$\frac{}{\Gamma; \perp, \Omega \vdash \text{abort} : U} \perp_L \quad \frac{\Gamma; A^+, \Omega \vdash N_1 : U \quad \Gamma; B^+, \Omega \vdash N_2 : U}{\Gamma; A^+ \vee B^+, \Omega \vdash [N_1, N_2] : U} \vee_L$$

$$\frac{\Gamma; \Omega \vdash N : U}{\Gamma; \top^+, \Omega \vdash \langle \rangle.N : U} \top_L^+ \quad \frac{\Gamma; A^+, B^+, \Omega \vdash N : U}{\Gamma; A^+ \wedge^+ B^+, \Omega \vdash \times N : U} \wedge_L^+$$

$$\frac{\Gamma; \cdot \vdash N : \langle p^- \rangle}{\Gamma; \cdot \vdash \langle N \rangle : p^-} \eta^- \quad \frac{\Gamma; \cdot \vdash N : A^+}{\Gamma; \cdot \vdash \{N\} : \uparrow A^+} \uparrow_R \quad \frac{\Gamma; A^+ \vdash N : B^-}{\Gamma; \cdot \vdash \lambda N : A^+ \supset B^-} \supset_R$$

$$\frac{}{\Gamma; \cdot \vdash \langle \rangle^- : \top^-} \top_R^- \quad \frac{\Gamma; \cdot \vdash N_1 : A^- \quad \Gamma; \cdot \vdash N_2 : B^-}{\Gamma; \cdot \vdash \langle N_1, N_2 \rangle^- : A^- \wedge^- B^-} \wedge_R^-$$

$$\boxed{\Gamma; [A^-] \vdash S : U - \text{spines } S, U \text{ must be stable}}$$

$$\frac{}{\Gamma; [A^-] \vdash \text{NIL} : \langle A^- \rangle} id^- \quad \frac{\Gamma; A^+ \vdash N : U}{\Gamma; [\uparrow A^+] \vdash \text{pm } N : U} \uparrow_L \quad \frac{\Gamma \vdash V : [A^+] \quad \Gamma; [B^-] \vdash S : U}{\Gamma; [A^+ \supset B^-] \vdash V; S : U} \supset_L$$

$$(\text{no rule } \top_L^-) \quad \frac{\Gamma; [A^-] \vdash S : U}{\Gamma; [A^- \wedge^- B^-] \vdash \pi_1; S : U} \wedge_{L1}^- \quad \frac{\Gamma; [B^-] \vdash S : U}{\Gamma; [A^- \wedge^- B^-] \vdash \pi_2; S : U} \wedge_{L2}^-$$

Fig. 7. Proof terms for the focused sequent calculus.

clearly as an option in a logic with shifts; Andreoli's focused classical linear logic [1992], Chaudhuri's focused intuitionistic linear logic [2006], and Liang and Miller's LJF [2009] all approach focalization for a logic where there are no shifts and where polarity is derived from a proposition's topmost connective. From our polarized perspective, these approaches can all be seen as defining a particular polarization strategy $(\Gamma \longrightarrow P)^\circ$ that transforms unpolarized sequents into polarized ones. It is then possible to state and prove a strictly weaker focalization property: that $(\Gamma \longrightarrow P)^\circ$ is derivable if and only if $\Gamma \longrightarrow P$ is derivable.

2.4 Proof terms

While it is convenient and traditional to define a logic in terms of rules, we follow Herbelin [1995] in noting that it is sometimes easier to manipulate derivations using an appropriately-designed proof term presentation of the logic. Our proof term language is primarily a generalization of the *spine form* introduced by Cervesato and Pfenning [2003].¹⁰ Spine form is a proof term assignment for the so-called uniform proofs, the focused fragment of a logic that only includes the negative (or asynchronous) propositions. In spine form, *terms* and *spines* correspond to derivations of inversion sequents and left-focused sequents, respectively; we also consider *values* corresponding to derivations of right-focused sequents.

$$\begin{array}{ll}
\text{Values} & V ::= z \mid \mathbf{thunk} N \mid \mathbf{inl} V \mid \mathbf{inr} V \mid \langle \rangle^+ \mid \langle V_1, V_2 \rangle^+ \\
\text{Terms } N, M & ::= \mathbf{ret} V \mid x \circ S \mid \langle z \rangle.N \mid x.N \mid \mathbf{abort} \mid [N_1, N_2] \mid \langle \rangle.N \mid \times N \\
& \quad \mid \langle N \rangle \mid \{N\} \mid \lambda N \mid \langle \rangle^- \mid \langle N_1, N_2 \rangle^- \\
\text{Spines } S & ::= \mathbf{NIL} \mid \mathbf{pm} N \mid V; S \mid \pi_1; S \mid \pi_2; S
\end{array}$$

The separation of our syntax into three categories corresponds to the three-sequent view of our calculus. We will also refer to proof terms generically as *expressions* E when we want to invoke the one-sequent view of our system.

Only two terms bind new variables. The term $\langle z \rangle.N$, corresponding to the rule η^+ , binds a positive variable z (a variable corresponding to a suspended proposition). The term $x.N$, corresponding to the rule \uparrow_L , binds a new negative variable x (a variable corresponding to a negative proposition). We will freely span the Curry-Howard correspondence (or “propositions as types”), calling a value that corresponds to a derivation of the right-focused sequent $\Gamma \vdash [A^+]$ a value focused on A^+ and calling a spine that corresponds to a derivation of the left-focused sequent $\Gamma; [A^-] \vdash B^+$ or $\Gamma; [A^-] \vdash \langle B^- \rangle$ a spine of type B^+ or B^- (respectively) focused on A^- . Terms that correspond to stable sequents $\Gamma; \cdot \vdash A^+$ and $\Gamma; \cdot \vdash \langle A^- \rangle$ are (respectively) terms of type A^+ or A^- . Terms corresponding to derivations of more general inversion sequents like $\Gamma; \Omega \vdash A^+$ and $\Gamma; \Omega \vdash A^-$ are (respectively) terms of type A^+ introducing Ω and terms introducing Ω and A^- . We reserve the word *type* for stable succedents to emphasize that these are the important elements in the synthetic view of focusing.

It is possible to re-present the entire sequent calculus from Figures 4 and 5 annotating sequents with values, terms, and spines; the result is Figure 7. This “Curry-style” view, which sees types as *extrinsic* to the proof terms, is helpful as a reference, but it does not otherwise serve our purposes. Instead, we will proceed with a “Church-style” view of types as *intrinsic*. This necessitates thinking of proof terms as carrying some extra annotations; Pfenning writes these as superscripts [Pfenning 2008], but which we will follow Girard in leaving them implicit [Girard et al. 1989]. In particular, positive variables z must be annotated with positive propositions, negative variables x must be annotated with negative propositions, and \mathbf{inl} , \mathbf{inr} , π_1 , and π_2 must be annotated with the branch of the disjunction or conjunction that was not taken. This suffices to ensure that proof terms are in 1-to-1

¹⁰We also draw inspiration from the syntax of CLF [Watkins et al. 2002], call-by-push-value [Levy 2004], and Modernized Algol [Harper 2012] for our syntax.

Proof term:	$\lambda x_1. \lambda x_2. \langle \langle x_1 \circ (\pi_1; \text{NIL}) \rangle, \langle x_2 \circ (\pi_1; \text{NIL}) \rangle \rangle^-$
SML:	$\text{fn } x1 \Rightarrow \text{fn } x2 \Rightarrow (\#1 \ x1, \ #1 \ x2)$
Type:	$\downarrow(p \wedge^- q) \supset \downarrow(r \wedge^- s) \supset (p \wedge^- r)$
Proof term:	$\lambda \times z_1. z_2. \lambda \times z_3. z_4. \{\text{ret } \langle z_1, z_3 \rangle^+\}$
SML:	$\text{fn } (z1, z2) \Rightarrow \text{fn } (z3, z4) \Rightarrow (z1, z3)$
Type:	$(p \wedge^+ q) \supset (r \wedge^+ s) \supset \uparrow(p \wedge^+ r)$
Proof term:	$\lambda f. \lambda g. \lambda \langle z \rangle. \lambda [(\langle z_1 \rangle. \langle f \circ \langle z_1; z; \text{NIL} \rangle),$ $(\langle z_2 \rangle. \langle g \circ \langle z_2, z \rangle^+; \text{NIL} \rangle)]$
SML:	$\text{fn } f \Rightarrow \text{fn } g \Rightarrow \text{fn } z \Rightarrow (\text{fn Inl } z1 \Rightarrow f \ z1 \ z$ $\quad \mid \text{Inr } z2 \Rightarrow g \ (z2, z))$
Type:	$\downarrow(p^+ \supset s^+ \supset r^-) \supset \downarrow(q^+ \wedge^+ s^+ \supset r^-) \supset s^+ \supset (p^+ \vee q^+) \supset r^-$
Proof term:	$\lambda \langle z \rangle. \lambda f. \lambda g. \{f \circ \langle z; \text{pm } [(\langle z_1 \rangle. g \circ \langle z_1; \text{pm } \langle z_3 \rangle. \text{ret } z_3),$ $(\langle z_2 \rangle. \text{ret } z_2)] \rangle\}$
SML:	$\text{fn } z \Rightarrow \text{fn } f \Rightarrow \text{fn } g \Rightarrow (\text{case } (f \ z) \text{ of}$ $\quad \text{Inl } z1 \Rightarrow (\text{case } g \ z1 \ \text{of } z3 \Rightarrow z3)$ $\quad \mid \text{Inr } z2 \Rightarrow z2)$
Type:	$p^+ \supset \downarrow(p^+ \supset \uparrow(q^+ \vee r^+)) \supset \downarrow(q^+ \supset \uparrow r^+) \supset \uparrow r^+$

Fig. 8. Some proof terms and their rough translation Standard ML.

correspondence with sequent calculus derivations modulo the structural properties of exchange and weakening.¹¹

We will make a habit of presenting proof terms for admissible rules as well. The admissible focal substitution principles labeled subst^+ and subst^- above are respectively associated with the functions $[V/z]E$ and $[E]S$ that act on proof terms:

$$\frac{\Gamma \vdash V : [A^+] \quad \Gamma, z : \langle A^+ \rangle; L \vdash E : U}{\Gamma; L \vdash [V/z]E : U} \quad \frac{\Gamma; L \vdash E : \langle A^- \rangle \quad \Gamma; [A^-] \vdash S : U}{\Gamma; L \vdash [E]S : U}$$

Patterns. Our proof term calculus departs in one important way from most presentations of focused proof terms. In other work, the trend is to introduce the variables needed for an inversion phase all at once in a syntactic entity called a *pattern*; one significant example is Krishnaswami’s presentation of ML-style pattern matching and pattern compilation in the context of a focused sequent calculus [Krishnaswami 2009]. We do not use patterns because doing so would not be faithful to the LF encoding of Figure 4 used in the accompanying Twelf development; patterns cause the inductive structure of proof terms and sequents to deviate, even if they remain in 1-to-1 correspondence.

While a full discussion of patterns is beyond the scope of this article, we also want to suggest that our choice is the natural one from the perspective of the sequent calculus. Patterns are certainly relevant in the study of logic and programming

¹¹Our desire to adopt an intrinsic or Church-style view of proof terms and present cut admissibility and identity expansion using proof terms is one reason we use so many syntactic markers in our proof terms. These markers ($\times N$, $\langle N \rangle$, and $\{N\}$, etc.) may be omitted in a Curry-style presentation.

languages, but they seem more in line with natural deduction presentations of logic or with higher-order focused presentations, which can be seen as a synthesis of natural deduction and sequent calculus presentations [Zeilberger 2009b; Brock-Nannestad and Schürmann 2010].

Examples. Using Standard ML's syntax as an imperfect proxy for a natural-deduction system with pattern matching, we give, in Figure 8, some proof terms and our suggestion as to the corresponding natural deduction term. Note that if M is a term introducing B^- then the proof term corresponding to $\downarrow A^- \supset B^-$ is $\lambda x.M$, though in this case the familiar construct is comprised of two smaller constructs, the proof term corresponding to \supset_R and the proof term corresponding to \downarrow_L .

2.5 De-focalization

We conclude this section by presenting the de-focalization property, that $\Gamma; \cdot \vdash U$ implies $(\Gamma)^\circledast \longrightarrow (U)^\circledast$, which we can prove independently of any of the standard metatheoretic results for either system. In order to generalize the induction hypothesis, we define a new sequent form $\Gamma; \Psi \longrightarrow P$, where Ψ is an ordered sequence that mimics the inversion context Ω . The meaning of this sequent is defined by two rules which force the ordered Ψ context to introduce its contents into the hypothetical context Γ in a left-to-right order:

$$\frac{\Gamma, P; \Psi \longrightarrow Q}{\Gamma; P, \Psi \longrightarrow Q} \text{ cons} \quad \frac{\Gamma \longrightarrow Q}{\Gamma; \cdot \longrightarrow Q} \text{ nil}$$

With this definition, we can state the appropriate generalization of the induction hypothesis; our desired de-focalization property is a corollary.

THEOREM 1 DE-FOCALIZATION. *If $\Gamma; L \vdash U$, then $(\Gamma)^\circledast; (L)^\circledast \longrightarrow (U)^\circledast$*

We can also state Theorem 1 using the three-sequent view of our logic. This statement of the theorem has three parts:

- (1) If $\Gamma \vdash [A^+]$, then $(\Gamma)^\circledast; \cdot \longrightarrow (A^+)^\bullet$,
- (2) If $\Gamma; \Omega \vdash U$, then $(\Gamma)^\circledast; (\Omega)^\bullet \longrightarrow (U)^\circledast$, and
- (3) If $\Gamma; [A^-] \vdash U$, then $(\Gamma)^\circledast; (A^-)^\bullet \longrightarrow (U)^\circledast$.

PROOF. By induction and case analysis on the given derivation; $\mathcal{D} :: \Gamma \longrightarrow P$ denotes that \mathcal{D} is a derivation of $\Gamma \longrightarrow P$. Twenty-one of the twenty-four cases are blindingly straightforward, such as this one:

$$\text{Case. } \mathcal{D} = \frac{\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{\Gamma \vdash [A^+] \quad \Gamma \vdash [B^+]}}{\Gamma \vdash [A^+ \wedge B^+]} \wedge_R^+$$

$$\begin{array}{ll} \mathcal{E}_1 :: (\Gamma)^\circledast; \cdot \longrightarrow (A^+)^\bullet & \text{by the i.h. (part 1) on } \mathcal{D}_1 \\ \mathcal{E}'_1 :: (\Gamma)^\circledast \longrightarrow (A^+)^\bullet & \text{by inversion on } \mathcal{E}_1 \\ \mathcal{E}_2 :: (\Gamma)^\circledast; \cdot \longrightarrow (B^+)^\bullet & \text{by the i.h. (part 1) on } \mathcal{D}_2 \\ \mathcal{E}'_2 :: (\Gamma)^\circledast \longrightarrow (B^+)^\bullet & \text{by inversion on } \mathcal{E}_2 \\ \mathcal{E} :: (\Gamma)^\circledast \longrightarrow (A^+)^\bullet \wedge (B^+)^\bullet & \text{by rule } \wedge_R \text{ on } \mathcal{E}_1 \text{ and } \mathcal{E}_2 \\ \mathcal{E} :: (\Gamma)^\circledast \longrightarrow (A^+ \wedge^+ B^+)^\bullet & (A^+ \wedge^+ B^+)^\bullet = (A^+)^\bullet \wedge (B^+)^\bullet \\ \mathcal{E}' :: (\Gamma)^\circledast; \cdot \longrightarrow (A^+ \wedge^+ B^+)^\bullet & \text{by rule } \text{nil} \text{ on } \mathcal{E}. \end{array}$$

For three cases corresponding to the rules \perp_L , \vee_L , and \wedge_L^+ , a secondary induction is needed to show the admissibility, in the unfocused sequent calculus, of left rules that have a context Ψ .

$$\text{Case. } \mathcal{D} = \frac{\frac{\mathcal{D}_1}{\Gamma; A^+, B^+, \Omega \vdash U}}{\Gamma; A^+ \wedge B^+, \Omega \vdash U} \wedge_L^+$$

$$\begin{array}{ll} \mathcal{E}_1 :: (\Gamma)^\otimes; (A^+, B^+, \Omega)^\bullet \longrightarrow (U)^\otimes & \text{by i.h. (part 2) on } \mathcal{D}_1 \\ \mathcal{E}_1 :: (\Gamma)^\otimes; (A^+)^\bullet, (B^+)^\bullet, (\Omega)^\bullet \longrightarrow (U)^\otimes & (A^+, B^+, \Omega)^\bullet = (A^+)^\bullet, (B^+)^\bullet, (\Omega)^\bullet \\ \mathcal{E}'_1 :: (\Gamma)^\otimes, (A^+)^\bullet; (B^+)^\bullet, (\Omega)^\bullet \longrightarrow (U)^\otimes & \text{by inversion on } \mathcal{E}_1 \\ \mathcal{E}''_1 :: (\Gamma)^\otimes, (A^+)^\bullet, (B^+)^\bullet; (\Omega)^\bullet \longrightarrow (U)^\otimes & \text{by inversion on } \mathcal{E}'_1 \\ \mathcal{E} :: (\Gamma)^\otimes, (A^+)^\bullet \wedge (B^+)^\bullet; (\Omega)^\bullet \longrightarrow (U)^\otimes & \text{by lemma on } \mathcal{E}''_1 \\ \mathcal{E} :: (\Gamma)^\otimes, (A^+ \wedge^+ B^+)^\bullet; (\Omega)^\bullet \longrightarrow (U)^\otimes & (A^+ \wedge^+ B^+)^\bullet = (A^+)^\bullet \wedge (B^+)^\bullet \\ \mathcal{E}' :: (\Gamma)^\otimes; (A^+ \wedge^+ B^+)^\bullet, (\Omega)^\bullet \longrightarrow (U)^\otimes & \text{by rule } \mathit{cons} \text{ on } \mathcal{E}' \\ \mathcal{E}'' :: (\Gamma)^\otimes; (A^+ \wedge^+ B^+, \Omega)^\bullet \longrightarrow (U)^\otimes & (A^+ \wedge^+ B^+, \Omega)^\bullet = (A^+ \wedge^+ B^+)^\bullet, (\Omega)^\bullet \end{array}$$

The necessary lemma is that $\Gamma, P_1, P_2; \Psi \longrightarrow Q$, implies $\Gamma, P_1 \wedge P_2; \Psi \longrightarrow Q$. We proceed by induction on Ψ and by case analysis on the structure of the given derivation.

$$\text{Subcase. } \mathcal{D} = \frac{\frac{\mathcal{D}_1}{\Gamma, P_1, P_2, P; \Psi \longrightarrow Q}}{\Gamma, P_1, P_2; P, \Psi \longrightarrow Q} \mathit{cons}$$

$$\begin{array}{ll} \mathcal{D}'_1 :: \Gamma, P, P_1, P_2; \Psi \longrightarrow Q & \text{by exchange on } \mathcal{D}_1 \\ \mathcal{E}_1 :: \Gamma, P, P_1 \wedge P_2; \Psi \longrightarrow Q & \text{by i.h. on } \mathcal{D}'_1 \\ \mathcal{E}'_1 :: \Gamma, P_1 \wedge P_2, P; \Psi \longrightarrow Q & \text{by exchange on } \mathcal{E}_1 \\ \mathcal{E} :: \Gamma, P_1 \wedge P_2; P, \Psi \longrightarrow Q & \text{by rule } \mathit{cons} \text{ on } \mathcal{E}'_1 \end{array}$$

$$\text{Subcase. } \mathcal{D} = \frac{\frac{\mathcal{D}_1}{\Gamma, P_1, P_2 \longrightarrow Q}}{\Gamma, P_1, P_2; \cdot \longrightarrow Q} \mathit{nil}$$

$$\begin{array}{ll} \mathcal{D}'_1 :: \Gamma, P_1 \wedge P_2, P_1, P_2 \longrightarrow Q & \text{by weakening on } \mathcal{D}_1 \\ \mathcal{E}_1 :: \Gamma, P_1 \wedge P_2, P_1 \longrightarrow Q & \text{by rule } \wedge_{L2} \text{ on } \mathcal{D}'_1 \\ \mathcal{E}'_1 :: \Gamma, P_1 \wedge P_2 \longrightarrow Q & \text{by rule } \wedge_{L1} \text{ on } \mathcal{E}_1 \\ \mathcal{E} :: \Gamma, P_1 \wedge P_2; \cdot \longrightarrow Q & \text{by rule } \mathit{nil} \text{ on } \mathcal{E}'_1 \end{array}$$

The 22 other cases of the main theorem and the 2 other lemmas are similar. This theorem is named **sound** in the accompanying Twelf development. \square

The lemma for \wedge_L^+ and the two similar lemmas for \perp_L and \vee_L are as close as we will get to the tedious invertibility lemmas encountered by other proofs of the focalization property. Because of the way we have structured our system, each lemma only requires induction and case analysis over the definition of $\Gamma; \Psi \longrightarrow P$, which is defined by two rules, cons and nil . Therefore, our proof remains linear in the number of connectives and rules, rather than quadratic as in other approaches.

3. CUT ADMISSIBILITY

The statement of cut admissibility in an unpolarized logic is that $\Gamma \longrightarrow P$ and $\Gamma, P \longrightarrow Q$ imply $\Gamma \longrightarrow Q$. In polarized logic, we have positive and negative

propositions, so the cut admissibility theorem must, at minimum, have two parts: a negative cut, that $\Gamma; \cdot \vdash A^-$ and $\Gamma, A^-; \cdot \vdash U$ imply $\Gamma; \cdot \vdash U$, and a positive cut, that $\Gamma; \cdot \vdash A^+$ and $\Gamma, A^+ \vdash U$ imply $\Gamma; \cdot \vdash U$. The actual proof of cut admissibility will require further generalization, but these statements are corollaries. As in the statement of de-focalization, we state cut admissibility using the one-sequent view of sequents in order to cut down on the number of individual statements that we need to consider (4 parts instead of the 7 we would need otherwise).

THEOREM 2 CUT ADMISSIBILITY. *If Γ and U are suspension-normal, then*

- (1) *If $\Gamma \vdash [A^+]$ and $\Gamma, A^+, \Omega \vdash U$, then $\Gamma; \Omega \vdash U$,*
- (2) *If $\Gamma; \cdot \vdash A^-$, $\Gamma; [A^-] \vdash U$, and U stable, then $\Gamma; \cdot \vdash U$,*
- (3) *If $\Gamma; \cdot \vdash A^-$ and $\Gamma, A^-; L \vdash U$, then $\Gamma; L \vdash U$, and*
- (4) *If $\Gamma; L \vdash A^+$, $\Gamma, A^+ \vdash U$, and U stable, then $\Gamma; L \vdash U$.*

Beyond the additional cases needed to deal with shifts, the proof of focused cut admissibility mirrors structural cut admissibility proofs for unfocused sequent calculi. In fact, the organization strategy imposed by this four-part statement of cut admissibility makes explicit the informal organization strategy of principal, left commutative, and right commutative cuts that Pfenning used to present the many cases of structural cut admissibility proofs [Pfenning 2000].

Before discussing the proof of Theorem 2, we will show how we write the four parts of cut admissibility at the level of proof terms. By Curry-Howard, cut admissibility corresponds to a reduction operation on proof terms that was named *hereditary substitution* by Watkins et al. [2002].

Principal cuts (parts 1 and 2) are cases where the *principal formula* (that is, A^+ or A^-) is the proposition being decomposed in the last rule of both given derivations. In a focused sequent calculus, this naturally happens when the principal formula is in focus in one sequent and in inversion in the other. We will refer to the operation of principal cuts on proof terms as *principal substitution*:

$$\frac{\Gamma \vdash V : [A^+] \quad \Gamma, A^+, \Omega \vdash N : U}{\Gamma; \Omega \vdash (V \bullet N)^{A^+} : U} \text{ cut}^+$$

$$\frac{\Gamma; \cdot \vdash M : A^- \quad \Gamma; [A^-] \vdash S : U \quad U \text{ stable}}{\Gamma; \cdot \vdash (M \bullet S)^{A^-} : U} \text{ cut}^-$$

Right commutative cuts (part 3) deal with all cases where the second given derivation decomposes a proposition other than the principal formula. The action on proof terms is *rightist substitution*:

$$\frac{\Gamma; \cdot \vdash M : A^- \quad \Gamma, x:A^-; L \vdash E : U}{\Gamma; L \vdash \llbracket M/x \rrbracket^{A^-} E : U} \text{ rsubst}$$

Left commutative cuts (part 4) deal with all cases where the first given derivation ends in a left rule. The action on proof terms is a *leftist substitution*:

$$\frac{\Gamma; L \vdash E : A^+ \quad \Gamma, A^+ \vdash N : U \quad U \text{ stable}}{\Gamma; L \vdash \llbracket E \rrbracket^{A^+} N : U} \text{ lsubst}$$

PROOF. The proof of cut admissibility is by lexicographic induction. In each invocation of the induction hypothesis, either

- the principal formula A^+ or A^- gets smaller, or else it stays the same and
- the “part size” (as in parts 1-4) decreases, or else both the principal formula and part size stay the same and either
 - we are in part 3 and the second given derivation gets smaller, or
 - we are in part 4 and the first given derivation gets smaller.

This is actually a refinement of the standard structural induction metric presented by Pfenning [2000], which is itself a structural-induction-flavored reinterpretation of the metric used by Gentzen [1935] that forms the basis of most cut elimination proofs. The extra lexicographic ordering on “part size” is nonstandard, but is needed here to justify the appeals to principal substitution from rightist and leftist substitution. When we look at the computational content of cut admissibility, we can see that rightist substitutions only break apart the second given derivation and that leftist substitutions only break apart the first derivation, and that these substitutions do not call one another directly. Unlike the usual induction argument for cut admissibility, there is no commitment made to the first derivation staying the same or getting smaller while we are performing rightist substitution; the same is true for the second derivation in leftist substitution. While it is beyond the scope of this article, this alternate induction metric is helpful when formalizing structural focalization in Agda.

Due to the conciseness (certainly) and clarity (optimistically) of such a presentation, we present the cases of this proof using only proof terms. This critically relies on the fact that we understand all of our values, terms, and spines to be intrinsically typed (and therefore in 1-to-1 correspondence with focused sequent calculus derivations).

Principal substitution. This is where the action is; it’s where both terms are decomposed simultaneously in concert as the type gets smaller. Rightist and leftist substitutions, in comparison, are just looking around for places where principal substitution can happen.

$$\boxed{(V \bullet N)^{A^+} = N'} \text{ (part 1)}$$

$$\begin{aligned} (z \bullet \langle z' \rangle . N)^{p^+} &= [z/z']N \\ (\text{thunk } M \bullet x.N)^{\downarrow A^-} &= \llbracket M/x \rrbracket^{A^-} N \\ (\text{inl } V \bullet [N_1, N_2])^{A^+ \vee B^+} &= (V \bullet N_1)^{A^+} \\ (\text{inr } V \bullet [N_1, N_2])^{A^+ \vee B^+} &= (V \bullet N_2)^{B^+} \\ (\top^+ \bullet \langle \rangle . N)^{\top^+} &= N \\ ((V_1, V_2)^+ \bullet \times N)^{A^+ \wedge B^+} &= (V_2 \bullet (V_1 \bullet N)^{A^+})^{B^+} \end{aligned}$$

In the case where $A^+ = p^+$, we invoke focal substitution $[z/z']N$ to do variable-for-variable substitution. This can also be seen as a use of contraction.

$$\boxed{(M \bullet S)^{A^-} = N'} \text{ (part 2)}$$

$$\begin{aligned}
((x \circ S) \bullet \text{NIL})^{P^-} &= x \circ S \\
(\{M\} \bullet \text{pm } N)^{\uparrow A^+} &= \llbracket M \rrbracket^{A^+} N \\
(\lambda N \bullet V; S)^{A^+ \supset B^-} &= ((V \bullet N)^{A^+} \bullet S)^{B^-} \\
(\langle M_1, M_2 \rangle^- \bullet \pi_1; S)^{A^- \wedge^- B^-} &= (M_1 \bullet S)^{A^-} \\
(\langle M_1, M_2 \rangle^- \bullet \pi_2; S)^{A^- \wedge^- B^-} &= (M_2 \bullet S)^{B^-}
\end{aligned}$$

Rightist substitution. This is closest to the traditional form of substitution that we're used to from natural deduction: we churn through the second term to find all the places where x , the variable we're substituting M for, occurs (if, indeed, any exist). When we find an occurrence of this distinguished variable, which can only happen when the expression that we're substituting into is a term that has decided to focus on x , we call to negative principal substitutions (part 2). In traditional substitution we'd just plop M down at the places where x occurred, but to do that in this setting would introduce a cut!

$$\boxed{\llbracket M/x \rrbracket^{A^-} V = V'} \quad (\text{part 3, } E = V)$$

$$\begin{aligned}
\llbracket M/x \rrbracket^{A^-} z &= z \\
\llbracket M/x \rrbracket^{A^-} \{N\} &= \{\llbracket M/x \rrbracket^{A^-} N\} \\
\llbracket M/x \rrbracket^{A^-} \text{inl } V &= \text{inl } (\llbracket M/x \rrbracket^{A^-} V) \\
\llbracket M/x \rrbracket^{A^-} \text{inr } V &= \text{inr } (\llbracket M/x \rrbracket^{A^-} V) \\
\llbracket M/x \rrbracket^{A^-} \langle \rangle^+ &= \langle \rangle^+ \\
\llbracket M/x \rrbracket^{A^-} \langle V_1, V_2 \rangle^+ &= \langle (\llbracket M/x \rrbracket^{A^-} V_1), (\llbracket M/x \rrbracket^{A^-} V_2) \rangle^+
\end{aligned}$$

$$\boxed{\llbracket M/x \rrbracket^{A^-} N = N'} \quad (\text{part 3, } E = N)$$

$$\begin{aligned}
\llbracket M/x \rrbracket^{A^-} \text{ret } V &= \text{ret } (\llbracket M/x \rrbracket^{A^-} V) \\
\llbracket M/x \rrbracket^{A^-} (x \circ S) &= (M \bullet \llbracket M/x \rrbracket^{A^-} S)^{A^-} \\
\llbracket M/x \rrbracket^{A^-} (x' \circ S) &= x' \circ (\llbracket M/x \rrbracket^{A^-} S) \quad (\text{if } x \neq x') \\
\llbracket M/x \rrbracket^{A^-} \langle z \rangle.N &= \langle z \rangle.(\llbracket M/x \rrbracket^{A^-} N) \\
\llbracket M/x \rrbracket^{A^-} x'.N &= x'.(\llbracket M/x \rrbracket^{A^-} N) \\
\llbracket M/x \rrbracket^{A^-} \text{abort} &= \text{abort} \\
\llbracket M/x \rrbracket^{A^-} [N_1, N_2] &= [(\llbracket M/x \rrbracket^{A^-} N_1), (\llbracket M/x \rrbracket^{A^-} N_2)] \\
\llbracket M/x \rrbracket^{A^-} \langle \rangle.N &= \langle \rangle.(\llbracket M/x \rrbracket^{A^-} N) \\
\llbracket M/x \rrbracket^{A^-} \times N &= \times (\llbracket M/x \rrbracket^{A^-} N) \\
\llbracket M/x \rrbracket^{A^-} \{V\} &= \{\llbracket M/x \rrbracket^{A^-} V\} \\
\llbracket M/x \rrbracket^{A^-} \lambda N &= \lambda (\llbracket M/x \rrbracket^{A^-} N) \\
\llbracket M/x \rrbracket^{A^-} \langle \rangle^- &= \langle \rangle^- \\
\llbracket M/x \rrbracket^{A^-} \langle N_1, N_2 \rangle^- &= \langle (\llbracket M/x \rrbracket^{A^-} N_1), (\llbracket M/x \rrbracket^{A^-} N_2) \rangle^-
\end{aligned}$$

In the cases for η^+ (proof term $\langle z \rangle.N$) and \downarrow_L (proof term $x'.N$), the bound variables z and x' can always be α -converted to be different from both x and any variables free in M .

$$\boxed{\llbracket M/x \rrbracket^{A^-} S = S'} \quad (\text{part 3, } E = S)$$

$$\begin{aligned}
\llbracket M/x \rrbracket^{A^-} \text{NIL} &= \text{NIL} \\
\llbracket M/x \rrbracket^{A^-} \mathbf{pm} N &= \mathbf{pm} (\llbracket M/x \rrbracket^{A^-} N) \\
\llbracket M/x \rrbracket^{A^-} V; S &= (\llbracket M/x \rrbracket^{A^-} V); (\llbracket M/x \rrbracket^{A^-} S) \\
\llbracket M/x \rrbracket^{A^-} \pi_1; S &= \pi_2; (\llbracket M/x \rrbracket^{A^-} S) \\
\llbracket M/x \rrbracket^{A^-} \pi_2; S &= \pi_1; (\llbracket M/x \rrbracket^{A^-} S)
\end{aligned}$$

Leftist substitution. This is so named because it, rather unusually, breaks apart the first (and not the second) derivation. This is natural from the perspective of cut elimination: the second term N has an inversion it must do on the left, so just like we searched in rightist substitution for any (potential) use of the foc_L rule on x in the second term, we search in leftist substitution for uses of foc_R to derive A^+ in the first term.

$$\boxed{\llbracket M \rrbracket^{A^+} N = M'} \quad (\text{part 4, } E = M)$$

$$\begin{aligned}
\llbracket \mathbf{ret} V \rrbracket^{A^+} N &= (V \bullet N)^{A^+} \\
\llbracket x \circ S \rrbracket^{A^+} N &= x \circ (\llbracket S \rrbracket^{A^+} N) \\
\llbracket \langle z \rangle . M \rrbracket^{A^+} N &= \langle z \rangle . (\llbracket M \rrbracket^{A^+} N) \\
\llbracket x . M \rrbracket^{A^+} N &= x . (\llbracket M \rrbracket^{A^+} N) \\
\llbracket \mathbf{abort} \rrbracket^{A^+} N &= \mathbf{abort} \\
\llbracket [M_1, M_2] \rrbracket^{A^+} N &= [(\llbracket M_1 \rrbracket^{A^+} N), (\llbracket M_2 \rrbracket^{A^+} N)] \\
\llbracket \langle \rangle . M \rrbracket^{A^+} N &= \langle \rangle . (\llbracket M \rrbracket^{A^+} N) \\
\llbracket \times M \rrbracket^{A^+} N &= \times (\llbracket M \rrbracket^{A^+} N)
\end{aligned}$$

$$\boxed{\llbracket S \rrbracket^{A^+} N = S'} \quad (\text{part 4, } E = S)$$

$$\begin{aligned}
\llbracket \{M\} \rrbracket N &= \{\llbracket M \rrbracket N\} \\
\llbracket V; S \rrbracket N &= V; (\llbracket S \rrbracket N) \\
\llbracket \pi_1; S \rrbracket N &= \pi_1; (\llbracket S \rrbracket N) \\
\llbracket \pi_2; S \rrbracket N &= \pi_2; (\llbracket S \rrbracket N)
\end{aligned}$$

This completes the proof. The four parts of this theorem are named **cut+**, **cut-**, **rsubst**, and **lsubst** (respectively) in the accompanying Twelf development. \square

4. IDENTITY EXPANSION

A significant novelty of our presentation relative to existing work is our presentation of the identity expansion theorem; it is adapted from the identity expansion theorem given for *weak focusing* [Simmons and Pfenning 2011b], a less-restricted focusing calculus that does not require invertible rules to be applied eagerly. The familiar identity property for an unfocused sequent calculus states that, for all propositions A , there is a derivation $\Gamma, A \longrightarrow A$. Identity in an unfocused sequent calculus can generally be established by structural induction on the proposition A .

As with cut admissibility, there are two analogous identity properties for the focused sequent calculus. First, for all positive propositions A^+ there is a derivation of $\Gamma; A^+ \vdash A^+$. Second, for all negative propositions A^- there is a derivation $\Gamma, A^-; \cdot \vdash A^-$. As an exercise, you should convince yourself that this property

cannot be established directly by structural induction on A^+ or A^- . It doesn't work, in other words, to generalize the *init* rule from the unfocused sequent calculus (Figure 1) to get an identity principle for the focused sequent calculus. Instead, it is the suggestively named η^+ and η^- rules that generalize to admissible identity expansion principles:

$$\frac{\Gamma, \langle A^+ \rangle; \Omega \vdash U}{\Gamma; A^+, \Omega \vdash U} \text{ expand}^+ \quad \frac{\Gamma; \cdot \vdash \langle A^- \rangle}{\Gamma; \cdot \vdash A^-} \text{ expand}^-$$

When we introduced the η^+ and η^- rules, we said they reflected the idea that inversion should not suspend itself until reaching an atomic proposition. The existence of these admissible rules relaxes this requirement: we can optionally suspend inversion before the pattern matching process is exhausted. The non-atomic suspended propositions that appear when we suspend early appear to have a connection to the *complex values* in call-by-push-value [Levy 2004].

The premises of both of these rules are definitely not suspension-normal. Unlike cut admissibility, identity expansion is not at all restricted to suspension-normal sequents: non-atomic suspended propositions and focal substitution play an important role.

We associate positive identity expansion with the proof term $\eta^{A^+}(z.N)$ and negative identity expansion with the proof term $\eta^{A^-}(N)$, allowing us to annotate the admissible rules above:

$$\frac{\Gamma, z: \langle A^+ \rangle; \Omega \vdash N : U}{\Gamma; A^+, \Omega \vdash \eta^{A^+}(z.N) : U} \text{ expand}^+ \quad \frac{\Gamma; \cdot \vdash N : \langle A^- \rangle}{\Gamma; \cdot \vdash \eta^{A^-}(N) : A^-} \text{ expand}^-$$

Given identity expansion, the positive identity principle that $\Gamma; A^+ \vdash A^+$ holds for all A^+ is provable using positive identity expansion.

$$\frac{\frac{\overline{\Gamma, z: \langle A^+ \rangle \vdash z : [A^+]}}{\Gamma, z: \langle A^+ \rangle; \cdot \vdash \text{ret } z : A^+} \text{ foc}_R}{\Gamma; A^+ \vdash \eta^{A^+}(z.\text{ret } z) : A^+} \text{ expand}^+$$

The negative identity principle that $\Gamma, A^-; \cdot \vdash A^-$ holds for all A^- is similarly a corollary of negative identity expansion.

$$\frac{\frac{\overline{\Gamma, x: A^-; [A^-] \vdash \text{NIL} : \langle A^- \rangle}}{\Gamma, x: A^-; \cdot \vdash x \circ \text{NIL} : \langle A^- \rangle} \text{ foc}_L}{\Gamma, x: A^-; \cdot \vdash \eta^{A^-}(x \circ \text{NIL}) : A^-} \text{ expand}^-$$

THEOREM 3 IDENTITY EXPANSION.

- (1) For all A^+ , if $\Gamma, \langle A^+ \rangle; \Omega \vdash U$, then $\Gamma; A^+, \Omega \vdash U$.
- (2) For all A^- , if $\Gamma; \cdot \vdash \langle A^- \rangle$, then $\Gamma; \cdot \vdash A^-$.

PROOF. The proof is by induction and case analysis on the structure of the proposition A^+ or A^- .

We will present one case of part 1 and one case of part 2 line-by-line, and then present all of the cases using the language of proof terms.

Case (part 1). $A^+ = A^+ \wedge^+ B^+$

$$\begin{array}{ll}
\mathcal{D} :: \Gamma, \langle A^+ \wedge^+ B^+ \rangle; \Omega \vdash U & \text{given} \\
\mathcal{D}' :: \Gamma, \langle A^+ \rangle, \langle B^+ \rangle, \langle A^+ \wedge^+ B^+ \rangle; \Omega \vdash U & \text{by weakening on } \mathcal{D} \\
\mathcal{E}_1 :: \Gamma, \langle A^+ \rangle, \langle B^+ \rangle \vdash [A^+] & \text{by rule } id^+ \\
\mathcal{E}_2 :: \Gamma, \langle A^+ \rangle, \langle B^+ \rangle \vdash [B^+] & \text{by rule } id^+ \\
\mathcal{E} :: \Gamma, \langle A^+ \rangle, \langle B^+ \rangle \vdash [A^+ \wedge^+ B^+] & \text{by rule } \wedge_R^+ \text{ on } \mathcal{E}_1 \text{ and } \mathcal{E}_2 \\
\mathcal{F} :: \Gamma, \langle A^+ \rangle, \langle B^+ \rangle; \Omega \vdash U & \text{by focal substitution on } \mathcal{E} \text{ and } \mathcal{D}' \\
\mathcal{F}_1 :: \Gamma, \langle A^+ \rangle; B^+, \Omega \vdash U & \text{by i.h. (part 1) on } B^+ \text{ and } \mathcal{F} \\
\mathcal{F}_2 :: \Gamma; A^+, B^+, \Omega \vdash U & \text{by i.h. (part 1) on } A^+ \text{ and } \mathcal{F}_1 \\
\Gamma; A^+ \wedge^+ B^+, \Omega \vdash U & \text{by rule } \wedge_L^+ \text{ on } \mathcal{F}_2
\end{array}$$

Case (part 2). $A^- = A^+ \supset^- B^-$

$$\begin{array}{ll}
\mathcal{D} :: \Gamma; \cdot \vdash \langle A^+ \supset^- B^- \rangle & \text{given} \\
\mathcal{D}' :: \Gamma, \langle A^+ \rangle; \cdot \vdash \langle A^+ \supset^- B^- \rangle & \text{by weakening on } \mathcal{D} \\
\mathcal{E}_1 :: \Gamma, \langle A^+ \rangle \vdash [A^+] & \text{by rule } id^+ \\
\mathcal{E}_2 :: \Gamma, \langle A^+ \rangle; [B^-] \vdash \langle B^- \rangle & \text{by rule } id^- \\
\mathcal{E} :: \Gamma, \langle A^+ \rangle; [A^+ \supset^- B^-] \vdash \langle B^- \rangle & \text{by rule } \supset_L \text{ on } \mathcal{E}_1 \text{ and } \mathcal{E}_2 \\
\mathcal{F} :: \Gamma, \langle A^+ \rangle; \cdot \vdash \langle B^- \rangle & \text{by focal substitution on } \mathcal{D}' \text{ and } \mathcal{E} \\
\mathcal{F}_1 :: \Gamma, \langle A^+ \rangle; \cdot \vdash B^- & \text{by i.h. (part 2) on } B^- \text{ and } \mathcal{F} \\
\mathcal{F}_2 :: \Gamma; A^+ \vdash B^- & \text{by i.h. (part 1) on } A^- \text{ and } \mathcal{F}_1 \\
\Gamma; \cdot \vdash A^+ \supset^- B^- & \text{by rule } \supset_R \text{ on } \mathcal{F}_2
\end{array}$$

This suffices to show the line-by-line structure of the identity expansion theorem; other cases follow the same pattern. We will now give all the cases on the level of proof terms:

$$\boxed{\eta^{A^+}(v.N) = N'} \text{ (part 1)}$$

$$\begin{aligned}
\eta^{p^+}(z.N) &= \langle z \rangle.N \\
\eta^{\downarrow A^-}(z.N) &= x.([\text{thunk}(\eta^{A^-}(x \circ \text{NIL}))]/z]N) \\
\eta^{\perp}(z.N) &= \text{abort} \\
\eta^{A^+ \vee B^+}(z.N) &= [\eta^{A^+}(z_1.[\text{inl } z_1/z]N), \eta^{B^+}(z_2.[\text{inr } z_2/z]N)] \\
\eta^{\top}(z.N) &= \langle \rangle.([\langle \rangle^+/z]N) \\
\eta^{A^+ \wedge^+ B^+}(z.N) &= \times(\eta^{A^+}(z_1.(\eta^{B^+}(z_2.[\langle z_1, z_2 \rangle^+/z]N))))
\end{aligned}$$

$$\boxed{\eta^{A^-}(N) = N'} \text{ (part 2)}$$

$$\begin{aligned}
\eta^{p^-}(N) &= \langle N \rangle \\
\eta^{\uparrow A^+}(N) &= \{[N](\text{pm}(\eta^{A^+}(z.\text{ret } z)))\} \\
\eta^{A^+ \supset^- B^-}(N) &= \lambda(\eta^{A^+}(z.(\eta^{B^-}([N](z; \text{NIL})))))) \\
\eta^{\top}(N) &= \langle \rangle^- \\
\eta^{A^- \wedge^- B^-}(N) &= \langle \eta^{A^-}([N](\pi_1; \text{NIL})), \eta^{B^-}([N](\pi_2; \text{NIL})) \rangle^-
\end{aligned}$$

This completes the proof; the two parts of this theorem are named **expand+** and **expand-** (respectively) in the accompanying Twelf development. \square

5. FOCALIZATION

Theorem 4 in this section establishes the focalization property: it is possible to turn the derivation of any unfocused proof of an unpolarized sequent into a focused derivation for any polarized sequent that erases to the unpolarized one. This proof naturally factors into two parts. The first part is a series of *unfocused admissibility* lemmas, a family of admissible rules which serve to show that focused sequent calculus derivations can mimic unfocused derivations. The second part is a straightforward inductive proof that, if U is stable, $(\Gamma)^\circledast \longrightarrow (U)^\circledast$ implies $\Gamma; \cdot \vdash U$. Recall that $(-)^{\circledast}$, from Figure 6, is the erasure of polarization for contexts and succedents.

5.1 Unfocused admissibility

We think of unfocused admissibility as building an abstraction layer on top of focused, polarized logic. The proof of focalization then interacts with focused derivations entirely through the abstraction layer of unfocused admissibility.

It is possible to motivate unfocused admissibility independently of focalization. Consider the unfocused right rules for conjunction compared to the focused right rules for (positive) conjunction.

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge_R \quad \frac{\Gamma \vdash [A^+] \quad \Gamma \vdash [B^+]}{\Gamma \vdash [A^+ \wedge^+ B^+]} \wedge_R^+$$

The rules look similar, but their usage is quite different. To prove $A \wedge B$ we must prove A (possibly doing some work on the left first) and, in the other branch, we must prove B (possibly doing some work on the left first). To prove $A^+ \wedge^+ B^+$, we must decompose A^+ in one branch and B^+ in the other; there is no possibility of doing work on the left first. The admissible rule in polarized logic that actually matches the structure of the unfocused rule \wedge_R looks like this:

$$\frac{\Gamma; \cdot \vdash A^+ \quad \Gamma; \cdot \vdash B^+}{\Gamma; \cdot \vdash A^+ \wedge^+ B^+} \wedge_{uR}^+$$

The stable premises A^+ and B^+ ensure that, in both subderivations, it will be possible to do work on the left before decomposing A^+ or B^+ .

The unfocused admissibility lemmas could be established the slow, painful, and boring way – one induction over focused derivations per lemma. This more traditional approach is both technically and philosophically unsatisfying, however. It is technically unsatisfying because these theorems are long and annoying, and philosophically unsatisfying because cut admissibility and identity expansion are already supposed to capture global properties of the logic. We will instead establish unfocused admissibility directly from cut admissibility and identity expansion without the need for any additional inductive arguments; each unfocused admissibility proof is short, though dense. In Figure 9 we present the proof of \wedge_{uR}^+ as a derivation built using admissible rules.

The unfocused admissibility lemmas will be presented in terms of the admissible rules they justify, but their proofs will be presented entirely at the level of proof terms. In most cases, we will omit the propositions that annotate instances of cut admissibility. We must be careful about the interaction of cut admissibility and

$$\begin{array}{c}
\frac{\frac{\frac{\frac{\Gamma' \vdash [A^+]}{id^+} \quad \frac{\Gamma' \vdash [B^+]}{id^+}}{\Gamma' \vdash [A^+ \wedge^+ B^+]} \wedge_R^+}{\Gamma'; \cdot \vdash A^+ \wedge^+ B^+} foc_R}{\Gamma, \uparrow A^+, \langle B^+ \rangle; A^+ \vdash A^+ \wedge^+ B^+} expand^+ \\
\frac{\frac{\frac{\Gamma, \uparrow A^+, \langle B^+ \rangle; [\uparrow A^+] \vdash A^+ \wedge^+ B^+}{\Gamma, \uparrow A^+, \langle B^+ \rangle; \cdot \vdash A^+ \wedge^+ B^+} \uparrow_L}{\Gamma, \uparrow A^+, \langle B^+ \rangle; \cdot \vdash A^+ \wedge^+ B^+} foc_L}{\Gamma, \uparrow A^+, \langle B^+ \rangle; \cdot \vdash A^+ \wedge^+ B^+} expand^+ \\
\frac{\frac{\frac{\frac{\Gamma; \cdot \vdash A^+}{\Gamma; \cdot \vdash \uparrow A^+} \uparrow_R \quad \frac{\frac{\Gamma; \cdot \vdash B^+}{\Gamma, \uparrow A^+; \cdot \vdash B^+} D_2}{\Gamma, \uparrow A^+; \cdot \vdash B^+} weaken}{\Gamma, \uparrow A^+; \cdot \vdash A^+ \wedge^+ B^+} lsubst}{\Gamma; \cdot \vdash A^+ \wedge^+ B^+} rsubst}
\end{array}$$

Fig. 9. Unfocused admissibility rule \wedge_{uR}^+ as a derivation, where $\Gamma' = \Gamma, \uparrow A^+, \langle B^+ \rangle, \langle A^+ \rangle$.

identity expansion. The premises of $expand^+$ and $expand^-$ are not suspension-normal because they contain non-atomic suspended propositions, and cut admissibility is only defined on suspension-normal sequents. Because of this restriction on cuts, every unfocused admissibility lemma that makes use of a cut carries the implicit requirement that the given sequents be stable and suspension-normal.

In certain cases we do more work than necessary, such as in the left rule for $\uparrow\top^+$, which could alternatively be phrased as a use of weakening. This is done to match the structure of unfocused admissibility in substructural logics [Simmons 2012].

5.1.1 *Initial rules.* A positive atomic proposition can appear in the hypothetical context either as a shifted positive proposition $x:\uparrow p^+$ or as a suspended positive proposition $z:\langle p^+ \rangle$, and likewise for negative atomic propositions on the right. As a result, we need four initial rules to correspond to the single unfocused rule *init*. (We could cut these four rules down to two if we restricted erasure and focalization to suspension-free sequents instead of suspension-normal sequents.) Each of these unfocused admissibility lemmas are actually directly derivable.

$$\begin{array}{c}
\frac{\Gamma, x:p^-; \cdot \vdash initsusp_u^-(x) : \langle p^- \rangle}{initsusp_u^-(x) = x \circ \text{NIL}} \quad initsusp_u^- \\
\frac{\Gamma, x:p^-; \cdot \vdash init_u^-(x) : \downarrow p^-}{init_u^-(x) = \text{ret}(\text{thunk} \langle x \circ \text{NIL} \rangle)} \quad init_u^- \\
\frac{\Gamma, z:\langle p^+ \rangle; \cdot \vdash initsusp_u^+(z) : p^+}{initsusp_u^+(z) = \text{ret } z} \quad initsusp_u^+ \\
\frac{\Gamma, x:\uparrow p^+; \cdot \vdash initsusp_u^+(z) : p^+}{init_u^+(x) = x \circ \text{pm}(\langle z \rangle.\text{ret } z)} \quad init_u^+
\end{array}$$

5.1.2 *Disjunction.*

$$\frac{}{\Gamma, x:\uparrow\perp; \cdot \vdash \perp_{uL}(x) : U} \perp_{uL}$$

$$\perp_{uL}(x) = x \circ (\text{pm abort})$$

$$\frac{\Gamma; \cdot \vdash N_1 : A^+}{\Gamma; \cdot \vdash \vee_{uR1}(N_1) : A^+ \vee B^+} \vee_{uR1}$$

$$\vee_{uR1}(N_1) = \llbracket N_1 \rrbracket^{A^+} (\eta^{A^+} (z.\text{ret}(\text{inl } z)))$$

$$\frac{\Gamma; \cdot \vdash N_2 : B^+}{\Gamma; \cdot \vdash \vee_{uR2}(N_2) : A^+ \vee B^+} \vee_{uR2}$$

$$\vee_{uR2}(N_2) = \llbracket N_2 \rrbracket^{B^+} (\eta^{B^+} (z.\text{ret}(\text{inr } z)))$$

$$\frac{\Gamma, x_1:\uparrow A^+; \cdot \vdash N_1 : U \quad \Gamma, x_2:\uparrow B^+; \cdot \vdash N_2 : U}{\Gamma, x:\uparrow(A^+ \vee B^+); \cdot \vdash \vee_{uL}(x, x_1.N_1, x_2.N_2) : U} \vee_{uL}$$

$$\vee_{uL}(x, x_1.N_1, x_2.N_2) = x \circ \text{pm} (\llbracket N_{Id} \rrbracket [x_1.N_1, x_2.N_2])$$

where $N_{Id} = [\eta^{A^+} (z_1.\text{ret}(\text{inl}(\text{thunk}\{\text{ret } z_1\}))), \eta^{B^+} (z_2.\text{ret}(\text{inr}(\text{thunk}\{\text{ret } z_1\})))$ is a closed term of type $\downarrow(\uparrow A^+) \vee \downarrow(\uparrow B^+)$ introducing $A^+ \vee B^+$

5.1.3 *Positive conjunction.*

$$\frac{}{\Gamma; \cdot \vdash \top_{uR}^+ : \top^+} \top_{uR}^+$$

$$\top_{uR}^+ = \text{ret } \langle \rangle^+$$

$$\frac{\Gamma; \cdot \vdash N : U}{\Gamma, x:\uparrow\top^+; \cdot \vdash \top_{uL}^+(x, N) : U} \top_{uL}^+$$

$$\top_{uL}^+(x, N) = x \circ \text{pm} (\langle \rangle.N)$$

$$\frac{\Gamma; \cdot \vdash N_1 : A^+ \quad \Gamma; \cdot \vdash N_2 : B^+}{\Gamma; \cdot \vdash \wedge_{uR}^+(N_1, N_2) : A^+ \wedge^+ B^+} \wedge_{uR}^+$$

$$\wedge_{uR}^+(N_1, N_2) = \llbracket \{N_1\}/x_1 \rrbracket^{\uparrow A^+} (\llbracket N_2 \rrbracket^{B^+} N_{Id}(x))$$

where $N_{Id}(x) = \eta^{B^+} (z_2.x \circ \text{pm} (\eta^{A^+} (z_1.\text{ret} \langle z_1, z_2 \rangle^+)))$

is a term of type $A^+ \wedge^+ B^+$ introducing B^+ with x of type $\uparrow A^+$ free.

(This was the case given above as a derivation with admissible rules.)

$$\frac{\Gamma, x_1:\uparrow A^+, x_2:\uparrow B^+; \cdot \vdash N_1 : U}{\Gamma, x:\uparrow(A^+ \wedge^+ B^+); \cdot \vdash \wedge_{uL}^+(x, x_1.x_2.N_1) : U} \wedge_{uL}^+$$

$$\wedge_{uL}^+(x, x_1.x_2.N_1) = x \circ \text{pm} (\llbracket N_{Id} \rrbracket (\times x_1.x_2.N_1))$$

where $N_{Id} = \times (\eta^{A^+} (z_1.\eta^{B^+} (z_2.\text{ret} \langle \text{thunk}\{\text{ret } z_1\}, \text{thunk}\{\text{ret } z_2\} \rangle^+)))$

is a closed term of type $\downarrow\uparrow A^+ \wedge^+ \downarrow\uparrow B^+$ introducing $A^+ \wedge^+ B^+$.

5.1.4 *Implication.*

$$\frac{\Gamma, x_1: \uparrow A^+; \cdot \vdash N_1 : \downarrow B^-}{\Gamma; \cdot \vdash \supset_{uR} (x_1.N_1) : \downarrow (A^+ \supset B^-)} \supset_{uR}$$

$$\supset_{uR} (x_1.N_1) = \mathbf{ret} (\mathbf{thunk} (\llbracket \lambda x_1. \{N_1\} / x \rrbracket N_{Id}(x)))$$

where $N_{Id}(x) = \lambda(\eta^{A^+} (z.\eta^{B^-} x \circ (\mathbf{thunk} \{\mathbf{ret} z\}); (\mathbf{pm} x'.x' \circ \mathbf{NIL})))$
is a term introducing $A^+ \supset B^-$ with x of type $\downarrow \uparrow A^+ \supset \uparrow \downarrow B^-$ free.

$$\frac{\Gamma; \cdot \vdash N_1 : A^+ \quad \Gamma, x_2: B^-; \cdot \vdash N_2 : U}{\Gamma, x: A^+ \supset B^-; \cdot \vdash \supset_{uL} (N_1, x_2.N_2) : U} \supset_{uL}$$

$$\supset_{uL} (x, N_1, x_2.N_2) = \llbracket \llbracket N_1 \rrbracket^{A^+} N_{Id}(x) \rrbracket^{\downarrow B^-} x_2.N_2$$

where $N_{Id}(x) = \eta^{A^+} (z.\mathbf{ret} (\mathbf{thunk} (\eta^{B^-} (x \circ (z; \mathbf{NIL}))))))$
is a term of type $\downarrow B^-$ introducing A^+ with x of type $A^+ \supset B^-$ free.

5.1.4.1 *Negative conjunction.*

$$\frac{}{\Gamma; \cdot \vdash \top_{uR}^- : \downarrow \top^-} \top_{uR}^-$$

$$\top_{uR}^- = \mathbf{ret} (\mathbf{thunk} (\langle \rangle^-))$$

$$\frac{\Gamma; \cdot \vdash N_1 : \downarrow A^- \quad \Gamma; \cdot \vdash N_2 : \downarrow B^-}{\Gamma; \cdot \vdash \wedge_{uR}^-(N_1, N_2) : \downarrow (A^- \wedge^- B^-)} \wedge_{uR}^-$$

$$\wedge_{uR}^-(N_1, N_2) = \mathbf{ret} (\mathbf{thunk} (\llbracket \langle \{N_1\}, \{N_2\} \rangle^- / x \rrbracket N_{Id}(x)))$$

where $N_{Id}(x) = \langle \eta^{A^-} (x \circ \pi_1; \mathbf{pm} y.(y \circ \mathbf{NIL})), \eta^{B^-} (x \circ \pi_2; \mathbf{pm} y.(y \circ \mathbf{NIL})) \rangle^-$
is a term introducing $A^- \wedge^- B^-$ with x of type $\uparrow \downarrow A^- \wedge^- \uparrow \downarrow B^-$ free.

$$\frac{\Gamma, x_1: A^-; \cdot \vdash N_1 : U}{\Gamma, x: A^- \wedge^- B^-; \cdot \vdash \wedge_{uL1}^-(x, x_1.N_1) : U} \wedge_{uL1}^-$$

$$\wedge_{uL1}^-(x, x_1.N_1) = \llbracket \eta^{A^-} (x \circ \pi_1; \mathbf{NIL}) / x_1 \rrbracket^{A^-} N_1$$

$$\frac{\Gamma, x_2: B^-; \cdot \vdash N_2 : U}{\Gamma, x: A^- \wedge^- B^-; \cdot \vdash \wedge_{uL2}^-(x, x_2.N_2) : U} \wedge_{uL2}^-$$

$$\wedge_{uL2}^-(x, x_2.N_2) = \llbracket \eta^{B^-} (x \circ \pi_2; \mathbf{NIL}) / x_2 \rrbracket^{B^-} N_2$$

5.1.5 *Shift removal.* In order for the unfocused admissibility lemmas to form a complete abstraction boundary between the focused sequent calculus and the focalization theorem, we must account for the fact that many polarized propositions erase to the same proposition. For example, if $(A^+)^\bullet = P_1$ and $(B^-)^\bullet = P_2$, then

$$P_1 \supset P_2 = (\downarrow (A^+ \supset B^-))^\bullet = (\downarrow \downarrow (A^+ \supset B^-))^\bullet = (\downarrow \downarrow \downarrow (A^+ \supset B^-))^\bullet = \dots$$

and so on. To deal with deeply-shifted propositions in the completeness theorem, we will invoke a shift removal lemma. It is different from the other unfocused admissibility lemmas in that it mentions erasure and we prove it by induction over the structure of propositions.

LEMMA 1 SHIFT REMOVAL (POSITIVE). *If $(A^+)^\bullet = P$, there exists a B^+ , not of the form $\downarrow\uparrow C^+$, such that $(B^+)^\bullet = P$ and, for any Γ , $\Gamma; \cdot \vdash B^+$ implies $\Gamma; \cdot \vdash A^+$.*

LEMMA 2 SHIFT REMOVAL (NEGATIVE). *If $(A^-)^\bullet = P$, there exists a B^- , not of the form $\uparrow\downarrow C^-$, such that $(B^-)^\bullet = P$ and, for any Γ and U stable, $\Gamma, B^-; \cdot \vdash U$ implies $\Gamma, A^-; \cdot \vdash U$.*

PROOF. Both lemmas are by induction on the structure of the proposition A^+ or A^- . If the outermost structure of the proposition is two adjacent shifts, we invoke the induction hypothesis and apply either $\downarrow\uparrow_{uR}$ or $\uparrow\downarrow_{uL}$:

$$\frac{\Gamma; \cdot \vdash N_1 : A^+}{\Gamma; \cdot \vdash \downarrow\uparrow_{uR}(N_1) : \downarrow\uparrow A^+} \downarrow\uparrow_{uR}$$

$$\downarrow\uparrow_{uR}(N_1) = \text{ret}(\text{thunk}\{N_1\})$$

$$\frac{\Gamma, x_1 : A^-; \cdot \vdash N_1 : U}{\Gamma, x : \uparrow\downarrow A^-; \cdot \vdash \uparrow\downarrow_{uL}(x, x_1.N_1) : U} \uparrow\downarrow_{uL}$$

$$\uparrow\downarrow_{uL}(x, x_1.N_1) = x \circ \text{pm } x_1.N_1$$

In all cases where the outermost structure of the proposition is *not* made up of two adjacent shifts, we succeed immediately using the given derivation. These lemmas are called **rshifty** and **lshifty** in the accompanying Twelf development. \square

5.2 Proof of focalization

Since we have not defined proof terms corresponding to unfocused sequent calculus derivations, in the proof of the focalization we will return to the more traditional style of proof presentation.

THEOREM 4 FOCALIZATION. *If U is stable and Γ and U are suspension-normal, then $(\Gamma)^\circledast \longrightarrow (U)^\circledast$ implies $\Gamma; \cdot \vdash U$.*

The second condition, that Γ and U are suspension-normal, is not something we strictly need to state, as $(\Gamma)^\circledast$ and $(U)^\circledast$ are only defined on suspension-normal contexts and succedents.

PROOF. By induction on the structure of the given derivation \mathcal{D} . Each rule in Figure 1 decomposes one proposition, either $P = (A^+)^\bullet$ on the right or $P = (A^-)^\bullet$ on the left. We proceed by case analysis on the structure of A^+ and A^- ; by the shift removal lemmas, it suffices to consider the case where this formula is not double-shifted. We will show a few representative cases.

$$\text{Case. } A^+ = \top^+, \quad \mathcal{D} = \overline{(\Gamma)^\circledast \longrightarrow \top} \top_R$$

$$\Gamma; \cdot \vdash \top^+ \quad \text{by unfocused admissibility lemma } \top_{uR}^+$$

$$\text{Case. } A^+ = \downarrow\top^-, \quad \mathcal{D} = \overline{(\Gamma)^\circledast \longrightarrow \top} \top_R$$

$$\Gamma; \cdot \vdash \downarrow\top^- \quad \text{by unfocused admissibility lemma } \top_{uR}^-$$

$$\text{Case. } A^+ = B_1^+ \wedge^+ B_2^+, \quad \mathcal{D} = \frac{\frac{\mathcal{D}_1}{(\Gamma)^\circ \longrightarrow (B_1^+)^\bullet} \quad (\Gamma)^\circ \longrightarrow (B_2^+)^\bullet}{(\Gamma)^\circ \longrightarrow (B_1^+)^\bullet \wedge (B_2^+)^\bullet} \wedge_R$$

$$\begin{array}{ll} \mathcal{E}_1 :: \Gamma; \cdot \vdash B_1^+ & \text{by i.h. on } \mathcal{D}_1 \\ \mathcal{E}_2 :: \Gamma; \cdot \vdash B_2^+ & \text{by i.h. on } \mathcal{D}_2 \\ \Gamma; \cdot \vdash B_1^+ \wedge^+ B_2^+ & \text{by unfocused admissibility lemma } \wedge_{uR}^+ \text{ on } \mathcal{E}_1 \text{ and } \mathcal{E}_2 \end{array}$$

$$\text{Case. } A^+ = \downarrow(B_1^- \wedge^- B_2^-), \quad \mathcal{D} = \frac{\frac{\mathcal{D}_1}{(\Gamma)^\circ \longrightarrow (B_1^-)^\bullet} \quad (\Gamma)^\circ \longrightarrow (B_2^-)^\bullet}{(\Gamma)^\circ \longrightarrow (B_1^-)^\bullet \wedge (B_2^-)^\bullet} \wedge_R$$

$$\begin{array}{ll} \mathcal{E}_1 :: \Gamma; \cdot \vdash \downarrow B_1^- & \text{by i.h. on } \mathcal{D}_1 \\ \mathcal{E}_2 :: \Gamma; \cdot \vdash \downarrow B_2^- & \text{by i.h. on } \mathcal{D}_2 \\ \Gamma; \cdot \vdash \downarrow(B_1^- \wedge^- B_2^-) & \text{by unfocused admissibility lemma } \wedge_{uR}^- \text{ on } \mathcal{E}_1 \text{ and } \mathcal{E}_2 \end{array}$$

$$\text{Case. } A^- = \uparrow(B_1^+ \wedge^+ B_2^+), \quad \mathcal{D} = \frac{\frac{\mathcal{D}_1}{(\Gamma')^\circ, (B_1^+)^\bullet \wedge (B_2^+)^\bullet, (B_1^+)^\bullet \longrightarrow (U)^\circ}}{(\Gamma)^\circ, (B_1^+)^\bullet \wedge (B_2^+)^\bullet \longrightarrow (U)^\circ} \wedge_{L1}$$

$$\begin{array}{ll} \mathcal{E}_1 :: \Gamma', \uparrow(B_1^+ \wedge^+ B_2^+), \uparrow B_1^+; \cdot \vdash U & \text{by i.h. on } \mathcal{D}_1 \\ \mathcal{E}'_1 :: \Gamma', \uparrow(B_1^+ \wedge^+ B_2^+), \uparrow B_1^+, \uparrow B_2^+; \cdot \vdash U & \text{by weakening on } \mathcal{E}_1 \\ \Gamma', \uparrow(B_1^+ \wedge^+ B_2^+); \cdot \vdash U & \text{by unfocused admissibility lemma } \wedge_{uL}^+ \text{ on } \mathcal{E}'_1 \end{array}$$

$$\text{Case. } A^- = B_1^- \wedge^- B_2^-, \quad \mathcal{D} = \frac{\frac{\mathcal{D}_1}{(\Gamma')^\circ, (B_1^-)^\bullet \wedge (B_2^-)^\bullet, (B_1^-)^\bullet \longrightarrow (U)^\circ}}{(\Gamma)^\circ, (B_1^-)^\bullet \wedge (B_2^-)^\bullet \longrightarrow (U)^\circ} \wedge_{L1}$$

$$\begin{array}{ll} \mathcal{E}_1 :: \Gamma', B_1^- \wedge^- B_2^-, B_1^-; \cdot \vdash U & \text{by i.h. on } \mathcal{D}_1 \\ \Gamma', B_1^- \wedge^- B_2^-; \cdot \vdash U & \text{by unfocused admissibility lemma } \wedge_{uL1}^- \text{ on } \mathcal{E}_1 \end{array}$$

There are twelve other cases: four corresponding to the various ways the *init* sequent can be polarized, two corresponding to \wedge_{L2} that mirror the cases for \wedge_{L1} , and six corresponding to \perp_L , \vee_{R1} , \vee_{R2} , \vee_L , \supset_R , and \supset_L . All proceed in a similar fashion. This theorem is named **complete** in the accompanying Twelf development. \square

5.3 Corollaries of focalization

Consider this section a short victory lap. We have established cut admissibility and identity for the focused sequent calculus, as well as the focalization and defocalization properties, without reference to any properties of the unfocused sequent calculus other than weakening and exchange. If our goal was simply to prove cut admissibility and identity for the unfocused sequent calculus, then proving focused cut admissibility, identity expansion, de-focalization, and focalization would not be the easiest way to do so! But, given these four theorems, the standard metatheoretic results of the unfocused sequent calculus can be established as corollaries.

The only new thing we need is an arbitrary polarization strategy $(P)^\circ$ that translates unpolarized propositions to negatively polarized propositions. It is straightforward to then define $(\Gamma)^\circ$, the obvious lifting of this function to contexts.

COROLLARY 1. *If $\Gamma \longrightarrow P$ and $\Gamma, P \longrightarrow Q$, then $\Gamma \longrightarrow Q$.*

PROOF. Since $(-)^{\circ}$ is defined to be a partial inverse of erasure, the first given derivation is equally a derivation of $((\Gamma)^{\circ})^{\otimes} \rightarrow (\downarrow(P)^{\circ})^{\otimes}$, and the second given derivation is equally a derivation of $((\Gamma)^{\circ}, (P)^{\circ})^{\otimes} \rightarrow (\downarrow(Q)^{\circ})^{\otimes}$.

By focalization (Theorem 4), we have the focused derivations $(\Gamma)^{\circ}; \cdot \vdash \downarrow(P)^{\circ}$ and $(\Gamma)^{\circ}, (P)^{\circ}; \cdot \vdash \downarrow(Q)^{\circ}$, and by applying \downarrow_L to the second of these derivations we get $(\Gamma)^{\circ}; \downarrow(P)^{\circ} \vdash \downarrow(Q)^{\circ}$. Then, by cut admissibility (Theorem 2, part 4), we obtain a derivation of $(\Gamma)^{\circ}; \cdot \vdash \downarrow(Q)^{\circ}$, which by de-focalization (Theorem 1) gives us a derivation of $((\Gamma)^{\circ})^{\otimes} \rightarrow (\downarrow(Q)^{\circ})^{\otimes}$, which is the same thing as a derivation of $\Gamma \rightarrow Q$. \square

COROLLARY 2. *For all P , $\Gamma, P \rightarrow P$.*

PROOF. By the identity principle, which as discussed is a corollary of identity expansion (Theorem 3), we can obtain a derivation of $(\Gamma)^{\circ}, (P)^{\circ}; \cdot \vdash (P)^{\circ}$. By de-focalization (Theorem 1), this gives us a derivation of $((\Gamma)^{\circ}, (P)^{\circ})^{\otimes} \rightarrow ((P)^{\circ})^{\otimes}$, which is the same thing as a derivation of $\Gamma, P \rightarrow P$. \square

These corollaries (**unfocused-cut** and **unfocused-identity** in the accompanying Twelf development) are interesting primarily insofar as they establish the total dominance that the focused sequent calculus enjoys over the unfocused sequent calculus. We have performed precisely one induction over unpolarized propositions (implicitly, in the definition of $(-)^{\circ}$) and one induction over unfocused derivations (in the proof of focalization, Theorem 4). The cut admissibility and identity expansion lemmas for the focused sequent calculus are strong enough for the unfocused sequent calculus to inherit its metatheory from the force of the theorems in the focused setting.

6. CONCLUSION

We have presented two sequent calculi for different variants of propositional intuitionistic logic, an unfocused sequent calculus for unpolarized intuitionistic logic and a focused sequent calculus for polarized intuitionistic logic. We then proved a strong theorem about their equivalence at the level of derivability. The equivalence result follows from mechanized, structurally inductive proofs establishing internal soundness and completeness for the focused logic. That equivalence result implies the internal soundness and completeness of the unfocused logic.

We will close with a brief survey of existing techniques used to prove the focalization property, with an emphasis on intuitionistic logic.

6.1 Comparison to existing focalization proofs

The most prevalent technique by far has been to do things the long way. Andreoli's original presentation of a focused sequent calculus required a large and tedious series of invertibility lemmas; Andreoli described these lemmas as “long but not difficult” [Andreoli 1992]. Howe's dissertation presents a similar brute-force approach to the focalization property in the context of intuitionistic logics, including intuitionistic linear logic [Howe 1998]. In an unpublished note, Laurent described a refactored version of the focalization property for classical linear logic. Laurent staged the proof differently from Andreoli, introducing several intermediate refinements with some, but not all, of the restrictions of full focusing. Laurent's proof is conceptually

clearer than Andreoli’s, but it still requires tedious invertibility lemmas in order to establish the identity property [Laurent 2004].

The “grand tour” strategy of Liang and Miller stands somewhat alone as an attempt to piggyback on established focusing results, rather than proving new ones. Unfocused derivations are translated into classical linear logic derivations, which are then focused. It is then only necessary to show that focused derivations can be translated back out from the focused classical linear logic derivations [Liang and Miller 2009]. We believe most of instances of this strategy can be understood, in the context of our system, as specific polarization strategies, which (as partial inverses of erasure) are handled generically by our erasure-based proof of focalization.

The idea that focalization should arise as a consequence of the cut admissibility and identity properties for a focused logic originates from in Chaudhuri’s dissertation [Chaudhuri 2006]. Compared to this work, Chaudhuri’s reliance on the identity property is less direct, and his proof of identity was non-structural, relying on a global decomposition of contexts and propositions. Chaudhuri’s technique was generalized by Liang and Miller [2011] to any systems meeting a general set of criteria; these criteria encompass classical and linear logics. In comparison, the techniques in this paper have not yet been applied to classical logics, but have been shown to extend straightforwardly to substructural and modal logics [Simmons 2012].

A line of work by Reed proved focalization by adding extra structure to the logic being focused. Reed’s “token passing translation” obtains the necessary structure through the use of linearity and a distinguished linear atomic proposition [Reed 2008]. His work with Pfenning, which was aimed at giving a resource semantics for substructural logics, obtains the necessary structure through the use of first-order terms quotiented by an equivalence relation [Reed and Pfenning 2010]. These proofs avoid invertibility lemmas, but their technique is less direct than ours and may not be as amenable to formalization in existing logical frameworks.

A wildly different approach to focalization can be found in the context of Zeilberger’s *higher-order focusing* [Zeilberger 2008a]. This pattern-based presentation of logic entirely removes any mention of individual logical connectives from the core logic; negative and positive propositions are handled in a completely generic way, in line with synthetic presentations of focusing. This approach prevents tedious repetition by default; there aren’t enough rules left to tediously induct upon! Polarization strategy-based focalization for higher-order focusing has been formalized in the Agda proof assistant, and there do not appear to be any technical obstacles to mechanizing the erasure-based approach discussed by Zeilberger [2008b]. Higher-order focused proofs represent a significant departure from the style of presentation in this paper; in particular, higher-order proof terms are infinitary, which means the Agda mechanization cannot be ported straightforwardly in Twelf. It is unclear what impact Zeilberger’s strategy of de-functionalizing focused derivations (which makes them representable in Twelf and, more generally, by non-infinitary derivations) has on focalization [Zeilberger 2009a].

The broad outlines of this paper were first developed in conjunction with our study of ordered linear logic as a forward chaining logic programming language [Pfenning and Simmons 2009]. For the purposes of that paper, unfocused admissibility in a *weakly focused* sequent calculus – which did not force invertible rules to

be applied eagerly – was established the historic (long and tedious) way. A Twelf proof for weakly focused intuitionistic logic developed at the same time was the genesis of the structural identity expansion proof presented here [Simmons 2009]. Eventually, this Twelf proof was adapted back to ordered linear logic in a technical report that also introduced the idea of suspended propositions [Simmons and Pfenning 2011b]. Unfortunately, to prove full focalization it was still necessary to prove tedious invertibility lemmas [Simmons and Pfenning 2011a], meaning that the weak focusing technique gives no advantages beyond those provided by Laurent’s refactoring. We believe this article supersedes our work on weak focusing entirely.

Our novel presentation of identity expansion seems to be necessary to deal with positive propositions. In logics without any interesting positive structure, simpler techniques have been successfully applied to prove analogues of the focalization property. The first result in this line was Miller et al.’s work on *uniform proofs* which, like Andreoli’s seminal work, was motivated by logic programming [Miller et al. 1991]. We don’t intend to fully survey techniques applicable to settings with only negative connectives, but we will mention two such systems. The first system is Jagadeesan et al.’s λRCC , which introduced the idea of using positive or negative polarization of atoms to combine forward-chaining and backward-chaining logic programming; their focalization proof roughly resembles the one used by Miller et al. [Jagadeesan et al. 2005]. The second system is the framework in which Reed and Pfenning developed their constructive resource semantics. This system is notable for our purposes because its focalization proof almost exactly follows our development [Reed and Pfenning 2010]; it was not known at the time how to extend their proof to a language with non-trivial positive propositions.

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