Characterizing Continuous Time Random Walks on Ergodic Time Varying Graphs

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ABSTRACT

In this paper we study the behavior of a continuous time random walk (CTRW) on a time varying dynamic graph. We establish conditions under which the CTRW is a stationary and ergodic process. In general, the stationary distribution of the walker depends on the walker rate and is difficult to characterize. However, we characterize the stationary distribution in the following cases: i) the walker rate is significantly larger or smaller than the rate in which the graph changes (time-scale separation), ii) the walker rate is proportional to the degree of the node that it resides on (coupled dynamics), and iii) the degrees of vertices belonging to the same connected component are identical (structural constraints). We provide examples that illustrate our theoretical findings.

1. INTRODUCTION

During the last decade, there has been a wide interest in characterizing and modeling the structure of various networks, from neural networks, to the web, to Facebook friends. Real networks have a inherently dynamic structure in the sense that both vertices and edges come and go over some time-scale. However, most efforts consider the network as either a single static graph or as a pre-defined sequence of graph configurations.

Random walks are an important building blocks for characterizing networks. Their simple behavior on static networks has been explored to devise algorithms for various purposes, from ranking to searching (details in Section 5). However, very little is known about the long-term behavior of random walks on dynamic networks.

In this paper, we study continuous time random walks (CTRWs) on stationary and ergodic dynamic graphs. We make the following contributions towards this goal:

- We consider stationary and ergodic dynamic graphs where nodes are always present in the network but edges are allowed to come and go over time, including the cases where the network is formed by several connected components. We introduce the notion of T-connectivity and show that if the dynamic graph is stationary, ergodic and T-connected then the CTRW is also stationary and ergodic. In the full generality of our framework, the stationary distribution of the walker depends on the walker rate and is difficult to characterize. However,

- we characterize the stationary distribution of the random walk for several cases: *(i)* **Time-scale separation**: the walker rate is significantly larger or smaller than the rate in which the graph changes; *(ii)* **Coupled dynamics**: the walker rate is proportional to the degree of the node that it resides on; *(iii)* **Structural constraints**: the degrees of vertices within any connected components are identical (but can vary among different components).

- We evaluate numerically several examples to support our theoretical results and illustrate their applicability. We also present a simple DTN application that can be evaluated within our framwork.

The remainder of this paper is organized as follows. Section 2 presents the proposed modeling framework together with some definitions and properties. Section 3 presents the stationary distribution of CTRW when the walker rate (for being too fast or too slow in respect to the speed the graph changes) allows a time scale decomposition of the combined walker and graph processes; we also present conditions under which the CTRW stationary distribution is invariant under time scale changes. Section 4 presents the numerical examples and applications. In Section 5 we discuss the related work. Finally, Section 6 concludes the paper.

2. MODEL FORMULATION

In this section we define important concepts that will be used throughout this paper. We define a dynamics graph to be a simple marked point process. A continuous time random walk (CTRW) over a dynamic graph is also a marked point process. We define the concept of T-connectivity to express the ability of nodes to be connected in time. We also define conditions for stationarity of the graph process and the CTRW process. We start by defining the graph dynamics.

Definition 2.1 (Dynamic Graph). *The time evolution of the graphs under consideration is given by the (possibly simultaneous) addition and deletion of edges. Let V denote a fixed finite set of n nodes and let $\mathcal{A}$ denote a finite set of m adjacency matrices $\mathcal{A} = \{A_k\}_{k=1}^m$, where A_k is an $n \times n$ unweighted symmetric adjacency matrix. A dynamic graph is a simple random marked point process $\Psi = \{(X_i, S_i)\}_0^\infty$ where $X_i \in \mathcal{A}$ denotes the i-th graph configuration and S_i is the time that the network spends in that configuration.*

Because Ψ is simple, $P(S_i = 0) = 0$ for all i. We use G_k to denote the graph configuration that has adjacency matrix A_k, $k = 1, \ldots, m$. Throughout this work we use A_k and G_k interchangeably. To simplify our analysis we focus on unweighted adjacency matrices. However, matrix weights can be easily accounted for in the walker rates and thus our results are also applicable to weighted dynamic graphs. Unless stated otherwise we assume that $\{(X_i, S_i)\}_0^\infty$ is stationary and ergodic, and that $E[S_i] < \infty$ for all i. We define the process $A = \{A(t)\}$, $t \geq 0$ as $\{X_{i(t)}\}$ where

$$i(t) = \arg\min\{i \geq 0 : \sum_{j=0}^{i-1} S_j \leq t < \sum_{j=0}^{i} S_j\}, \forall t \geq 0$$

i.e., $A(t)$ denotes the adjacency matrix of the graph at time t. It follows from [20, Corollary 2.8] that $\{A(t)\}$ is a time asymptotic stationary process with distribution $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m)$. In what follows we restate an important classic result in probability theory:

Theorem 2.1 (Brikhoff-Khinchin). *If $\{A(t)\}$ is ergodic then for $k = 1, \ldots, m$,*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{A(x)=A_k\}} dx = \sigma_k\,, \quad P\text{-a.s.} \tag{1}$$

Theorem 2.1 states that if $\{A(t)\}$ is ergodic its time average converges almost surely. A proof of this classic result can be found in graduate level textbooks on probability, e.g. Shiryaev [19, pg. 409].

Another important property of ergodicity that we use throughout this work is the uniform convergence of the time average regardless of the initial conditions.

Proposition 2.1 (Uniform convergence). *If $\{A(t)\}_0^\infty$ is ergodic then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P(A(x) = A_k|V) dx = \sigma_k \tag{2}$$

for any set V such that $P(V) > 0$. Furthermore, the convergence is uniform with respect to V, namely, for every $\epsilon > 0$, there exists T_ϵ such that for every $t > T_\epsilon$ and for every Borel set V with $P(V) > 0$, then

$$\left| \frac{1}{t} \int_0^t (P(A(x) = A_k|V) - \sigma_k) dx \right| < \epsilon.$$

Proof. A short proof is presented. An extended proof can be found in our technical report [6]. Ergodicity implies (1). Integrating both sides of (1) w.r.t. $dP(\omega)$ for $\omega \in V$ and using Egorov's theorem [11, pg. 290, Theorem 12] gives (2) for uniformly in V because V has non-zero measure, i.e., $P(V) > 0$. □

It is possible for one or more graph configurations to consist of two or more disconnected components, in which case we have to be concerned about whether a walker can move from any node to any other node over time. In what follows we introduce the concept of connectivity over time.

Definition 2.2 (T-connectivity). *Two vertices u and v are said to be T-connected in $\{A(t)\}_{t\geq 0}$, if they are connected in the graph with adjacency matrix $A = \vee_k A_k$.*

Definition 2.3 (T-connected dynamic graph). *A dynamic graph is said to be T-connected if all pairs of nodes are T-connected.*

We now define a Continuous Time Random Walk (CTRW) over the dynamic graph.

Definition 2.4 (CTRW). *A continuous time random walk (CTRW) on a dynamic graph $\{A(t)\}_0^\infty$, is a process $\{(A(t), U(t))\}_0^\infty$, where $U(t) \in V$ is the position (vertex) of the walker at time t. The times between CTRW steps are independent and exponentially distributed. The rate at which the walker makes a step at vertex $U(t) = v$ when $A(t) = A_k$ is $\gamma_{k,v}$. At the time the walker leaves v, it chooses one of its currently connected neighbors in A_k (if any) uniformly at random. When v has no neighbors in A_k the walker stays at v until the next step event.*

Let Γ denote the set of walker rates associated with $\{A(t)\}$. We will find it useful to express $\gamma_{k,v}$ in the form $\gamma_{k,v} = \beta_{k,v}\gamma$, $k = 1, \ldots, m$ and $v \in V$. Walker rates of interest to us include $\beta_{k,v} = 1$ (denoted *CTRW with constant walker rate*) and $\beta_{k,v} = d_{k,v}$, where $d_{k,v}$ is the degree of v given adjacency matrix A_k (denoted *CTRW with degree dependent walker rate*).

The above framework is general enough to describe several more particular dynamic graph models, such as renewal processes and Markovian processes. In a Markovian process S_i is exponentially distributed and $P[X_i|X_{i-1}, X_{i-2}, \ldots] = P[X_i|X_{i-1}]$, $i = 0, 1, \ldots$.

Notation Summary

$\{A(t)\}$ or $\{(X_i, S_i)\}_0^\infty$	dynamic graph process
$\{(A(t), U(t))\}_0^\infty$	CTRW process
$\sigma = (\sigma_1, \ldots, \sigma_m)$	stationary distribution of $\{A(t)\}$
$\gamma_{k,v}$	*walker rate* of the CTRW at node v at configuration A_k

3. CHARACTERIZING RWS: STATIONARY BEHAVIOR

In this section we focus on the stationary behavior of a RW on a dynamic graph, in particular the steady state fraction of time the walker spends in each node of the network: $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$.

The steady state distribution $\boldsymbol{\pi}$ is trivial if we consider static connected unweighted graphs, i.e., $A(t) = A'$, $\forall t \geq 0$, where A' is a symmetric binary ($\{0, 1\}$) adjacency matrix of a connected graph. The stationary distribution is unique and given by

$$\pi_v = \frac{d_v/\gamma_v}{\sum_w d_w/\gamma_w}, \quad v \in V,$$

where d_v denotes the degree of node v and γ_v the walker rate at vertex v. Characterization on a dynamic graph is much more challenging. How can we characterize $\boldsymbol{\pi}$ on dynamic graphs? When does $\boldsymbol{\pi}$ converge for dynamic graphs? When can we give an expression for $\boldsymbol{\pi}$? We focus on three cases where we are able to characterize this behavior.

We begin our study of the stationary behavior of the process $\{U(t)\}$ with the following.

THEOREM 3.1. *If the dynamic graph process $\{(X_i, S_i)\}_{i\geq 0}$ is T-connected, stationary, and ergodic, then the process $\{(A(t), U(t))\}_{t\geq 0}$ is asymptotically stationary, and the stationary distribution is unique.*

Proof. We create a new marked point process, $\{(X'_i, S'_i)\}_0^\infty$ that is a superposition of the graph process $\{(X_j, S_j)\}_0^\infty$ and a Poisson process having rate $\gamma_{\max} = \max_{r\in\Gamma} r$. We associate the mark "0" with each point of the Poisson process, Hence $X'_i \in \{0\} \cup \{A_1, \ldots, A_m\}$. As both the graph and Poisson processes are event stationary, the new merged process is also event stationary [7, Section 1.3.5]. Let $t'_0 < t'_1 < \cdots < t'_i \leq \cdots$ be the times associated with this new process. Consider the process $\{(A'_i, U_i)\}_{i=0}^\infty$ where U_i denotes the walker position at time t'_i and A'_i denotes the adjacency matrix during the period $[t'_i, t'_{i+1})$. Note that $(A'_i, U_i) \in \mathcal{A} \times V$ takes values from a finite set. $\{(A'_i, U_i)\}$ is described by a stochastic recursion of the form $(A'_i, U_i) = \phi(U_{i-1}, X'_i, R_i)$ where

$$\begin{aligned} A'_i &= \phi_a(U'_{i-1}, X'_i, R_i) = \mathbf{1}\{X'_i = 0\}A'_{i-1} + \mathbf{1}\{X'_i \neq 0\}X'_i, \\ U_i &= \phi_b(U'_{i-1}, X'_i, R_i), \end{aligned}$$

for all $i = 0, \ldots$. Here X'_i, S'_i are as previously defined and $\{R_i\}$ is an iid sequence of uniformly distributed rvs in $[0, 1]$ independent of $\{(X'_i, S'_i)\}$. These auxiliary rvs are used to choose the neighbor to which the walker goes or to remain stationed at its current node. Note that $\{(X'_i, S'_i, R_i)\}$ is stationary. ϕ_b is defined so that when $X'_i \neq 0$, the walker does not move ($U_i = U_{i-1}$) but the graph changes to configuration X'_i. If $X'_i = 0$, the walker moves from U'_i with probability $\gamma_{X'_i, U_i}/\gamma_{\max}$, moving to one of its neighbors (in configuration A'_{i-1}) chosen uniformly at random (using R_i).

Theorem 1 in [13] states that if there exists a random subset, $B \subseteq \mathcal{A} \times V$ such that a sample path monotonicity condition ((5) in [13]) holds and the existence of a finite non-empty sample path absorbing set ((9) in [13]) exists, then the process $\{X'_i, S'_i, A'_i, U_i\}$ is event asymptotic stationary as is $\{U_i\}$. In our case, because our state space is finite, these conditions trivially hold by taking $B = \mathcal{A} \times V$. Since $\{X'_i, S'_i, A'_i, U_i\}$ is an event asymptotic stationary marked point process, $\{A'(t), U(t)\}$ is also asymptotially stationary. It follows from our construction that $\{A(t)\} = \{A'(t)\}$; hence $\{A(t), U(t)\}$ is also asymptotically stationary.

We now address the question of uniqueness through a coupling argument. We focus on a system where the graph process $\{A(t)\}$ is in the stationary regime for $t \geq 0$. Consider two random walks $\{\{U_1(t)\}_0^\infty$ and $\{U_2(t)\}\}_0^\infty$ that differ in their starting locations at time $t = 0$, $U_1(0) = u_1$ and $U_2(0) = u_2$. We are interested in establishing that the time, T at which they meet, is finite a.s.. After time T the processes couple, i.e., for $t > T$, $U_1(t) = U_2(t)$. This is possible because the times between steps are exponentially distributed random variables. Thus, when T is finite, the above coupling argument implies that $\{U_1(t)\}$ and $\{U_2(t)\}$, which we have shown to be time asymptotic stationary, have the same stationary distribution.

It is left to show is that T is finite. We sketch the argument here and relegate the details to the our technical report [6]. The basic idea is to identify intervals of time of length $T_0 < \infty$ starting at times $iT_0 \geq 0$, $i = 0, 1, \ldots$, and based on the ergodicity and time stationarity of the graph process to establish a lower bound, p_0, on the probability of two walkers coupling during interval $[iT_0, +(i+1)T_0]$. The probability that the walkers do not couple within the interval $[0, jT_0)$ is upper bounded by $(1 - p_0)^j$. Thus the walkers couple in finite time a.s..

If the graph process is not T-connected, then it is possible for the system to exhibit multiple stationary regimes that depend on the initial position of the walker. □

3.1 Stationary Behavior under Time-scale Separation

Consider a scenario where the walker is either much faster ($\gamma \to \infty$) or much slower ($\gamma \to 0$) relative to the rate that the graph changes configurations. In this case, we have a time-scale separation between the two processes and given this separation we can determine π for the RW.

3.1.1 The Fast Walker

Let us first assume that the walker rate is much larger than the rate at which the graph changes. For a sufficiently large γ, the steady state probabilities of the random walk $\boldsymbol{\pi}$ is a linear combination of the corresponding probabilities of the adjacency matrices $A_1, \ldots, A_m$. Theorem 3.2 formalizes this argument for the case that every adjacency matrix in $\mathcal{A}$ is connected. We will describe how to relax this assumption later.

In preparation, let $\gamma_{k,v} = \beta_{k,v}\gamma$ and let $\boldsymbol{\pi}^{(k)}(\gamma) = (\pi_1^{(k)}(\gamma), \ldots, \pi_n^{(k)}(\gamma))$ denote the steady state distribution of a random walk on the undirected graph with adjacency matrix A_k as a function of $\gamma > 0$. It is given by

$$\pi_v^{(k)}(\gamma) \equiv \pi^{(k)} = \frac{d_{k,v}/\beta_{k,v}}{\sum_{j\in V} d_{k,j}/\beta_{k,j}}, \quad v \in V; k = 1, \ldots, m. \tag{3}$$

independent of γ. Let

$$\boldsymbol{\pi}^{(k)}(\gamma, t, w) = (\pi_1^{(k)}(\gamma, t, w), \ldots, \pi_n^{(k)}(\gamma, t, w))$$

denote the distribution of the CTRW on A_k at time $t \geq 0$ starting from node w. Because the random walk with adjacency matrix A_k is described by a time-reversible Markov chain, $\boldsymbol{\pi}^{(k)}(\gamma, t, w)$ can be expressed as

$$\boldsymbol{\pi}^{(k)}(\gamma, t, w) = \boldsymbol{\pi}^{(k)} + \sum_{j=2}^{n} \mathbf{c}_{j,w}^{(k)} e^{\lambda_{kj}\gamma t}, \quad w \in V,\ t > 0. \tag{4}$$

where $0 = \lambda_{k1} > \lambda_{k2} \geq \cdots \geq \lambda_{km}$ are the eigenvalues associated with $Q_k(\gamma)/\gamma$ where $Q_k(\gamma)$ is the infinitesimal generator associated with the random walk with parameter γ on the graph with adjacency matrix A_k, and $\{\mathbf{c}_{j,w}^{(k)}\}$ are vectors related to the j-th eigenvector of the random walk and the initial condition that the walker begins at w.

THEOREM 3.2. *If the graph process is connected, stationary, and ergodic, and the configurations are always connected, then in the limit as $\gamma \to \infty$, the stationary distribution $\boldsymbol{\pi}$ of the random walk is given by*

$$\boldsymbol{\pi} = \sum_{k=1}^{m} \sigma_k \boldsymbol{\pi}^{(k)}. \tag{5}$$

Proof. We show that the walker steady state distribution $\boldsymbol{\pi}(\gamma) \to \boldsymbol{\pi}$ as $\gamma \to \infty$ where $\boldsymbol{\pi}$ is given in (5).

We focus on the i-th graph configuration, X_i. Let $F_i^{(k)}(x) = P(S_i \le x | X_i = A_k)$ and define $\boldsymbol{\eta}_i^{(k)}(\gamma)$ to be the stationary distribution of the CTRW while the graph is in state $X_i = A_k$. Let $P_{i,w}(\gamma)$ denote the initial walker distribution when the process first enters graph configuration X_i. $\boldsymbol{\eta}_i^{(k)}(\gamma)$ is defined as

$$\begin{aligned} &\boldsymbol{\eta}_i^{(k)}(\gamma) \\ &= \sum_{w\in V} P_{i,w}(\gamma) \int_0^\infty \frac{1}{t}\int_0^t \boldsymbol{\pi}^{(k)}(\gamma,x,w)dx\, dF_i^{(k)} \\ &= \boldsymbol{\pi}^{(k)} + \sum_{w\in V} P_{i,w}(\gamma)\int_0^\infty \frac{1}{t}\int_0^t \sum_{j=2}^n \mathbf{c}_w^{(k)} e^{\lambda_{kj}\gamma x} dx\, dF_i^{(k)} \end{aligned}$$

where the second equality follows from (4). We focus on the second term, henceforth denoted as C_γ, which we show goes to zero as $\gamma \to \infty$. We focus first on the singularity of the $1/t$ term due to the first integral starting from zero,

$$\begin{aligned} |C_\gamma| &< F_i^{(k)}(\gamma^{-1/4})\mathbf{e} \\ &+ \sum_{w\in V} P_{i,w}(\gamma)\int_{\gamma^{-1/4}}^\infty \frac{1}{t}\int_0^t \sum_{j=2}^n |\mathbf{c}_w^{(k)}| e^{\lambda_{kj}\gamma x} dx\, dF_i^{(k)} \end{aligned}$$

where $|\mathbf{c}|$ is the vector whose components are the absolute values of the components of $\mathbf{c}$ and $\mathbf{e}$ is a vector of all ones. Evaluating the second integral and recognizing that $1/t \le 1/\ell^{1/4}$ for $t \ge 1/\gamma^{1/4}$ yields

$$\begin{aligned} |C_\ell| &< F_i^{(k)}(\gamma^{-1/4})\mathbf{e} \\ &+ \frac{1}{\gamma^{1/4}}\sum_{w\in V} P_{i,w}(\gamma)\sum_{j=2}^n \int_{\gamma^{-1/4}}^\infty \frac{|\mathbf{c}_w^{(k)}|}{\lambda_{kj}\gamma}(e^{\lambda_{kj}\gamma t}-1)dF_i^{(k)} \\ &= F_i^{(k)}(\gamma^{-1/4})\mathbf{e} + \frac{1}{\gamma^{1/4}}\sum_{w\in V} P_{i,w}(\gamma)\sum_{j=2}^n \frac{|\mathbf{c}_w^{(k)}|}{(\lambda_{kj}\gamma)^2} e^{\lambda_{kj}\gamma^{3/4}} \\ &+ \frac{1}{\gamma^{1/4}}\sum_{w\in V} P_{i,w}(\gamma)\sum_{j=2}^n \frac{|\mathbf{c}_w^{(k)}|}{-\lambda_{kj}\gamma}(1-F_i^{(k)}(\gamma^{-1/4})) \end{aligned}$$

All three terms above go to zero as $\gamma \to \infty$. Consequently $C_\ell \to 0$ and

$$\lim_{\gamma\to\infty} \boldsymbol{\eta}_i^{(k)}(\gamma) = \boldsymbol{\pi}^{(k)}, \quad \forall k$$

This holds for all i; therefore it holds when the graph is in steady state and removal of the conditioning on the graph configuration yields (5). □

We now focus on the case where one or more of the graph configurations consists of disconnected components. We relabel the nodes in each graph configuration in order to easily identify the disconnected components. For each of the original m adjacency matrices, A_k, we rearrange the n nodes into subsets of connected components. In other words, consider graph configuration G_k associated with adjacency matrix A_k. Partition the set of nodes in G_k into o_k sets, each containing only connected nodes. The o_k sets correspond to o_k adjacency matrices, $\{A_{k,1},\dots,A_{k,o_k}\}$ and graph configurations $\{G_{k,1},\dots,G_{k,o_k}\}$. Let $V_{k,l}$ denote the set of nodes in configuration $G_{k,l}$.

Let $\boldsymbol{\psi}(\gamma) = (\psi_{1,1}(\gamma),\dots,\psi_{m,o_m}(\gamma))$ denote the vector of probabilities that the walker is in the different components of all of the configurations when the rate parameter is γ. Because the CTRW process is ergodic, this vector exists and is given by

$$\psi_{k,l}(\gamma) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sum_{v\in V_{k,l}} \mathbf{1}_{\{A(s)=A_k, U(s)=v\}} ds, \quad \forall k,l$$

Define

$$\boldsymbol{\psi} = \lim_{\gamma\to\infty} \boldsymbol{\psi}(\gamma)\,. \tag{6}$$

We will describe conditions under which $\boldsymbol{\psi}$ can be computed shortly. In what follows if $\boldsymbol{\psi}$ exists then we show how to obtain the stationary distribution $\boldsymbol{\pi}$ of the random walk in the limit as $\gamma \to \infty$.

Let $\boldsymbol{\pi}^{(k,l)}$ be the steady state distribution of a random walk on the undirected graph with adjacency matrix $A_{k,l}$. Similar to equation (3)[1],

$$\boldsymbol{\pi}_v^{(k,l)} = \frac{d_{k,v}/\beta_{k,v}}{\sum_{j\in V_{k,l}} d_{k,j}/\beta_{k,j}}, \quad v \in V_{k,l},$$

and $k = 1,\dots,m, l = 1,\dots,o_k$.

We let $\boldsymbol{\pi}^{(k)}$ be the concatenation of vectors $\boldsymbol{\pi}^{(k,l)}$, that is, $\boldsymbol{\pi}^{(k)} = (\boldsymbol{\pi}^{(k,1)}\|\dots\|\boldsymbol{\pi}^{(k,o_k)})$ and

$$\hat{\boldsymbol{\pi}}^{(k)} = (\psi_{k,1}\boldsymbol{\pi}^{(k,1)}\|\dots\|\psi_{k,o_k}\boldsymbol{\pi}^{(k,o_k)}).$$

Note that vectors $\hat{\boldsymbol{\pi}}^{(k)}$ for all $1 \le k \le m$ have the same cardinality.

We have the following result.

THEOREM 3.3. *If the dynamic graph is T-connected, stationary, and ergodic, and $\boldsymbol{\psi}$ exists, then in the limit as $\gamma \to \infty$, the stationary distribution $\boldsymbol{\pi}$ of the random walk when graph configurations may be disconnected is given by*

$$\boldsymbol{\pi} = \sum_{k=1}^m \hat{\boldsymbol{\pi}}^{(k)}. \tag{7}$$

Proof. The proof is similar to that for the case where all graphs are connected. □

In general $\boldsymbol{\psi}$ is difficult to compute. The difficulty here lies in that the walker state at time t_0 can now depend on $\{A(t)\}_0^{t_0}$, something that was not possible when all configurations were connected. However, it is easily characterized when the underlying transitions between configurations are described by a Markov chain and the times that the graph remains in a configuration correspond to mutually independent sequences of iid random variables; one sequence for each configuration. Let $P = [p_{ij}]$ denote the $m \times m$ transition probability matrix for the graph configurations and, with an abuse of notation, let $\{S_{k,i}\}$ $k = 1,\dots m$ denote the mutually independent iid sequences of configuration holding times for the graph configurations. To be stationary and ergodic, P is aperiodic. Let $0 < E[S_k] < \infty$ denote the mean configuration holding time for configuration A_k, $k = 1,\dots,m$.

We focus now on transitions that the walker makes between connected components in two different graph configurations, say the j_1-th connected component in configuration G_{k_1} and the j_2-th connected component in configuration G_{k_2}. We define the matrix $\hat{P} = [\hat{p}_{k_1,j_1;k_2,j_2}]$ as follows

$$\hat{p}_{k_1,j_1;k_2,j_2} = p_{k_1,k_2}\frac{\sum_{v\in V_{k_1,j_1}\cap V_{k_2,j_2}} d_{k_1,v}/\beta_{k_1,v}}{\sum_{w\in V_{k_1,j_1}} d_{k_1,w}/\beta_{k_1,w}}. \tag{8}$$

[1]If the denominator is zero (i.e., there are isolated nodes) we simplify our notation assuming the ratio $0/0 = 1$.

The first term accounts for transitions between graph configurations and the second term accounts for the walker dynamics. Here $\hat{P}$ can be thought of as the transition probability matrix for a discrete time Markov chain that characterizes the subgraphs visited by a random walk at graph transitions in the limit as $\gamma \to \infty$. This chain is irreducible provided the graph is T-connected. Let $\boldsymbol{\psi}^* = (\psi^*_{1,1}, \dots, \psi^*_{m,l_m})$ denote the stationary distribution of this MC. The earlier introduced probability distribution $\boldsymbol{\psi}$ can be expressed in terms of $\boldsymbol{\psi}^*$ as follows

$$\psi_{k,l} = \frac{\psi^*_{k,l} E[S_k]}{\sum_{i=1}^m \sum_{j=1}^{o_i} \psi^*_{i,j} E[S_i]}, \quad A_k \in \mathcal{A}; l = 1, \dots, o_k \quad (9)$$

Note that the above characterization depends on the independence and identical distribution assumptions of the configuration holding times. This, along with (7) fully characterizes the stationary distribution of the walker in the fast walker regime for the Markovian environment.

3.1.2 The Fast Graph Dynamics

In this section we consider the walker steady state, $\boldsymbol{\pi}$, at the other timescale decomposition, namely where the graph dynamics speed up relative to the walker. Consider a walker with the set of walker rates Γ walking a dynamic graph Ψ. We represent the graph speed up by the process $\Psi_a = \{(X_i, S_i^{(a)}\}$ that is related to Ψ by $S_i^{(a)} = aS_i$, $i = 1, \dots$, for all $a > 0$, and characterize the CTRW on Ψ_a as $a \to \infty$. We do this in two steps. We fist consider an observer of Ψ_a, who makes observations according to a Poisson process and determine conditions underwhich the obsever is guaranteed to independent instances of the graph with probability given by the stationary distribution of the graph. We then couple the walker with the observer to characterize the stationary distribution of the walker.

We introduce a renewal process $\{W_j\}_1^\infty$ where W_j denotes the time between the $(j-1)$-th and j-th observations with CDF $G(x)$ (with PDF g(x)) satisfying the following assumption.

Assumption 3.1. *$g_a(x) := dP(W_i < x)/dx = ag(ax)$, $a > 0$, where $g(x)$ is continuous, non-increasing, nonnegative, with $g(0) < \infty$, $\int_0^\infty g(x)dx = 1$, $\int_0^\infty xg(x)dx := D < \infty$.*

Note that these conditions are satisfied if W_i is exponentially distributed with parameter $\gamma < \infty$. They are also satisfied if W_i has a Pareto distribution with Pareto index strictly larger than one. Let $g'(x) = dg/dx$. We will observe Ψ_a at the renewal points. Let $A^{(j)}$ denote the graph configuration at the j-th renewal point.

Lemma 3.1. *If the graph process is ergodic and the observation process satisfies Assumption 3.1, then for any $i \geq 0$, $k = 1, \dots, K$,*

$$\lim_{a\to\infty} P(A^{(i+1)} = A_k \,|\, A^{(j)} = A_j, j = 1, \dots, i) = \sigma_k. \quad (10)$$

Proof.

Recall that $A^{(j)} := A(W_1 + \cdots + W_j)$, $j \geq 1$. Conditioning on $W_j = y_j$ for $j = 1, \dots, i$ and $W_{i+1} = x$ and using the independence assumption between the process $\{A(t), t \geq 0\}$ and the iid $(W_j)_j$ with pdf $g_a(\cdot)$, gives

$$\begin{aligned} &P(A^{(i+1)} = A_k \,|\, A^{(j)} = A_{l_j}, j = 1, \dots, i) - \sigma_k = \\ &\quad \int_{y_1,\dots,y_i \in I} \int_{x=0}^\infty (P(A(x_i + x) = A_k \,|\, A(x_j) = A_{l_j}, j = 1, \dots, i) \\ &\quad -\sigma_k) g_a(x)\,dx\, g_a(y_1) \cdots g_a(y_i) dy_1 \cdots dy_i \end{aligned} \quad (11)$$

with $x_j := \sum_{l=1}^j y_j$ for $j = 1, \dots, i$ and $I := [0, \infty)^i$.

From now on $\mathbf{x}^i = (x_1, \dots, x_i)$.

Define

$$\begin{aligned} V(\mathbf{x}^i) &= \{\omega : A(x_j, \omega) = A_{l_j}, j = 1, \dots, i)\,, \\ f(u, \mathbf{x}^i) &= P(A(u) = A_k \,|\, V(\mathbf{x}^i)) - \sigma_k\,, \\ F(x, \mathbf{x}^i) &= \int_0^x f(x_i + u, \mathbf{x}^i) du. \end{aligned}$$

Note that $|f(u, \mathbf{x}^i)| \leq 1$ for any $u, \mathbf{x}^i$ so that $|F(x, \mathbf{x}^i)| \leq x$ for any $x, \mathbf{x}^i$.

Integrating by parts and using the definition of $g_a(x)$ in Assumption 3.1 yields

$$\begin{aligned} &\int_0^\infty f(x + x_i, \mathbf{x}^i) g_a(x) dx \quad (12) \\ &\quad = \Big[g_a(x) F(x, \mathbf{x}^i) \Big]_{x=0}^\infty - \int_0^\infty F(x, \mathbf{x}^i) g'_a(x) dx \\ &\quad = \lim_{x\to\infty} g_a(x) F(x, \mathbf{x}^i) - a^2 \int_0^\infty F(x, \mathbf{x}^i) g'(ax) dx \\ &\quad = -a^2 \int_0^\infty F(x, \mathbf{x}^i) g'(ax) dx. \quad (13) \end{aligned}$$

Due to Assumption 3.1 $\lim_{x\to\infty} xg(x) = 0$ or, equivalently, that $\lim_{x\to\infty} xag(ax) = 0$, which together with $|F(x, \mathbf{x}^i)| \leq x$ shows that $\lim_{x\to\infty} g_a(x) F(x, \mathbf{x}^i) = 0$. Another useful property that follows from Assumption 3.1 is

$$\int_0^\infty x g'(ax) dx = \left(xg(ax)\big|_0^\infty - \int_0^\infty g(ax) dx \right) /a = -a^{-2}.$$

In what follows we denote $C = -a^{-2}$.

Combining (11) and (13) gives (remember that $g'(x) \leq 0$)

$$\begin{aligned} &P(A^{(i+1)} = A_k \,|\, A^{(j)} = A_{l_j}, j = 1, \dots, i) - \sigma_k \\ &\quad = -a^2 \int_{y_1,\dots,y_i \in I} \int_{x=0}^\infty F(x, \mathbf{x}^i) g'(ax) dx \prod_{j=1}^i g_a(y_j) dy_j. \end{aligned} \quad (14)$$

We are left with proving that the r.h.s. of (14) goes to zero as $a \to 0$.

Since

$$\begin{aligned} F(x, \mathbf{x}^i) &= \int_0^x f(x_i + u, \mathbf{x}^i) du = \int_{x_i}^{x + x_i} f(u, \mathbf{x}^i) du \\ &= \int_0^{x + x_i} f(u, \mathbf{x}^i) du - \int_0^{x_i} f(u, \mathbf{x}^i) du \end{aligned}$$

we may rewrite (14) as

$$P(A^{(i+1)} = A_k \,|\, A^{(j)} = A_j, j = 1, \dots, i) - \sigma_k = \phi_1 + \phi_2 \quad (15)$$

with

$$\phi_1 := - a^2 \int_{y_1,\ldots,y_i\in I} \int_{x=0}^{\infty} \left(\int_{u=0}^{x+x_i} f(u,\mathbf{x}^i)du\right) g'(ax)dx \times \prod_{j=1}^{i} g_a(y_j)dy_j$$

$$\begin{aligned}\phi_2 :=& a^2 \int_{y_1,\ldots,y_i\in I} \left(\int_{u=0}^{x_i} f(u,\mathbf{x}^i)du\right)\left(\int_{x=0}^{\infty} g'(ax)dx\right) \times \prod_{j=1}^{i} g_a(y_j)dy_j \\ =& - ag(0)\int_{y_1,\ldots,y_i\in I}\left(\int_{u=0}^{x_i} f(u,\mathbf{x}^i)du\right)\prod_{j=1}^{i} g_a(y_j)dy_j \end{aligned} \tag{16}$$

since $\int_0^\infty g'(ax)dx = -g(0)/a$ (see Assumption 3.1).

Fix $\epsilon > 0$ and consider first ϕ_2. By (2) we know that there exists $T_{2,\epsilon}$, denoted as T_2 from now on, such that for all $x_i > T_2$, $|(1/x_i)\int_0^{x_i} f(u,\mathbf{x}^i)du| < \epsilon/(4ig(0)D)$. Using Proposition 2.1 we note that T_2 does not depend on the set $V(\mathbf{x}^i)$ since the convergence in (2) is uniform w.r.t. the set V. Hence,

$$\begin{aligned}|\phi_2| =& ag(0)\Bigg(\int_{\substack{y_1,\ldots,y_i\in I\\ y_1+\cdots+y_i\le T_2}} \left(\int_{u=0}^{x_i} f(u,\mathbf{x}^i)du\right)\prod_{j=1}^{i} g_a(y_j)dy_j \\ &+ \int_{\substack{y_1,\ldots,y_i\in I\\ y_1+\cdots+y_i> T_2}} \left(\frac{1}{x_i}\int_{u=0}^{x_i} f(u,\mathbf{x}^i)du\right) x_i \prod_{j=1}^{i} g_a(y_j)dy_j\Bigg) \\ \le& ag(0)T_2 + a\frac{\epsilon}{4iD}\int_{y_1,\ldots,y_i\in I}\left(\sum_{j=1}^{i} y_j\right)\prod_{j=1}^{i} g_a(y_j)dy_j \\ =& ag(0)T_2 + a\frac{\epsilon}{4iD}\sum_{j=1}^{i}\int_{y_j=0}^{\infty} y_j g_a(y_j)dy_j \\ =& ag(0)T_2 + \frac{\epsilon}{4}, \end{aligned} \tag{17}$$

by using $-1 \le f(\cdot,\cdot) \le 1$, the definition of x_i ($= \sum_{j=1}^{i} y_j$) and the assumption that $\int_0^\infty x g_a(x)dx = D/a$.

We conclude from the above that for all $0 < a < \epsilon/(4ag(0)T_2)$

$$|\phi_2| \le \frac{\epsilon}{2}. \tag{18}$$

Consider now ϕ_1. By (2) we know that there exists $T_{1,\epsilon}$, denoted as T_1 from now on, such that for all $x + x_i > T_1$, $|(1/(x+x_i))\int_0^{x+x_i} f(u,\mathbf{x}^i)du| < \epsilon/(4(C + ig(0)D))$, uniformly in $V(\mathbf{x}^i)$.
Let $J = \{(y_1,\ldots,y_i,x) \in [0,\infty)^{i+1}\}$. We have

$$\begin{aligned}&|\phi_1| \\ &\le - a^2 \int_{\substack{(y_1,\ldots,y_i,x)\in J\\ x+x_i\le T_1}} \int_{u=0}^{x+x_i} f(u,\mathbf{x}^i)du g'(ax)dx \prod_{j=1}^{i} g_a(y_j)dy_j \\ &\quad - a^2 \int_{\substack{(y_1,\ldots,y_i,x)\in J\\ x+x_i> T_1}} \left(\frac{1}{x+x_i}\int_{u=0}^{x+x_i} f(u,\mathbf{x}^i)du\right) \\ &\quad \times (x+x_i)g'(ax)\,dx \prod_{j=1}^{i} g_a(y_j)dy_j \\ &\le - a^2 T_\epsilon^1 \int_{\substack{(y_1,\ldots,y_i,x)\in J\\ x+x_i\le T_1}} g'(ax)dx \prod_{j=1}^{i} g_a(y_j)dy_j \qquad (19) \\ &\quad - \frac{a^2\epsilon}{4(C+ig(0)D)}\int_{\substack{(y_1,\ldots,y_i,x)\in J\\ x+x_i> T_1}} (x+x_i)g'(ax)dx\prod_{j=1}^{i} g_a(y_j)dy_j \\ &\le - a^2 T_\epsilon^1 \int_{(y_1,\ldots,y_i,x)\in J} g'(ax)dx \prod_{j=1}^{i} g_a(y_j)dy_j \\ &\quad - \frac{a^2\epsilon}{4(C+ig(0)D)}\int_{(y_1,\ldots,y_i,x)\in J} (x+x_i)g'(ax)dx\prod_{j=1}^{i} g_a(y_j)dy_j \\ &= ag(0)T_1 + \frac{\epsilon}{4} \end{aligned}$$

by using the identities $\int_0^\infty g'(ax)dx = -g(0)/a$, and where (19) is a consequence of $\int_0^{x+x_i} f(u,\mathbf{x}^i)du \le T_1$.

We conclude from the above that for all $0 < a < \epsilon/(4g(0)T_1)$

$$|\phi_1| \le \frac{\epsilon}{2}. \tag{20}$$

Combining (18)-(20) shows that for all $0 < a < \epsilon/(4g(0)\max\{T_1,T_2\})$ (note in passing that $T_1 \ge T_2$ from their very definition)

$$|P(A^{(i+1)} = A_k \mid A^{(j)} = A_{l_j}, j = 1,\ldots,i) - \sigma_k| < \epsilon$$

which completes the proof. □

Application of the chain rule yields the following result.

Proposition 3.1. *The sequence $\{A^{(j)}\}_1^\infty$ is iid with distribution $P(A^{(j)} = A_k) = \sigma_k$, $k = 1,\ldots,m$.*

It is now straightforward to describe the behavior of a constant rate walker. Take the observation process to be Poisson with rate γ. We couple the times that the walker takes a step to the observation times in the observation process. Let $U^{(j)}$ denote the position of the walker immediately after the j-th step, $j = 1,\ldots$ with initial position at time $t = 0$ $U^{(0)}$. The following equation describes the behavior of $\pi_u^{(j)} = P(U^{(j)} = u)$

$$\begin{aligned}\pi_u^{(j)} &= \sum_k\sum_v P(A^{(j)} = A_k, U(j-1) = v)A_{k,vu}/d_{k,v}, \\ &= \sum_k P(A^{(j)} = A_k)\sum_v P(U(j-1) = v)A_{k,vu}/d_{k,v}\end{aligned}$$

where $d_{k,v}$ is the degree of node $v \in V$ in graph configuration A_k. The second equality follows from Proposition 3.1 and the fact that $U^{(j-1)}$ depends on $A^{(j-1)}$ and that $A^{(j)}$ is independent of $U^{(j-1)}$. It follows that the stationary distribution of $\pi^{(j)}$, π is described by

$$\pi = \pi P$$

where

$$P_{vu} = \sum_k \sigma_k A_{k,vu}/d_{k,v}$$

Because of the PASTA property, $\boldsymbol{\pi} = \pi$.

The case where the walker rate depends on the node and graph configuration in which the walker resides yields a similar characterization.

PROPOSITION 3.2. *Let $\{U(t)\}$ be a CTRW with walker rates $\vec{\beta} = (\beta_{k,v})_{k=1,\dots,m,\, v\in V}$. Let $A^{(j)}$ (resp. $U^{(j)}$) denote the state of $\{A(t/a)\}$ (resp. $\{U(t)\}$) at the time of the j-th walker step, where $A^{(0)} = A(0)$ (resp. $U^{(0)} = U(0)$). Then, in the limit as $a \to \infty$, $\boldsymbol{\pi}$ satisfies*

$$\boldsymbol{\pi} = \boldsymbol{\pi} P$$

where

$$P_{vu} = \begin{cases} \sum_k \sigma_k \frac{\beta_{k,v}}{\max_{k,w}\beta_{k,w}} \frac{A_{k,vu}}{d_{k,v}}, & v \neq u \\ \sum_k \sigma_k \big(1 - \frac{\beta_{k,v}}{\max_{k,w}\beta_{k,w}}\big), & v = u \end{cases} \quad (21)$$

Proof. We introduce a Poisson process with parameter $\beta_{max} = \max(\beta_{k,v})$ that observes $\{A(t/a)\}$. We now couple the walker steps to this process. Note that, depending on the graph configuration and position, the walker may not take a step and may remain at its position at that time. Because of Proposition 3.1, the position of the walker at the observation times of this Poisson process can be written as (21) and because of PASTA property, (21) describes the stationary distribution of the walker on a fast graph. □

Proposition 3.2 shows the steady state distribution of a walker with state dependent rates to be just a function of $\boldsymbol{\sigma}$ (the stationary distribution of $\{A(t)\}$), the set of configurations $\mathcal{A}$, and the walker rates $\vec{\beta}$.

3.2 Time-scale Invariant Stationary Distribution

Consider a CTRW $\{U(t)\}_0^\infty$ with non-zero walker rates $\vec{\gamma}(\gamma) = (\gamma\beta_{k,v})_{v\in V,k=1,\dots,m}$, on a stationary and ergodic T-connected graph $\{A(t)\}$. In this section we turn our attention to a sufficient condition where the CTRW stationary distribution is invariant to the walker time scale γ. We believe that our results can be extend to a larger class of dynamic graphs that include non stationary graphs.

The key insight into our sufficient condition is the following: If there exists a $\boldsymbol{\pi}$ that is the CTRW stationary distribution given any (static) configuration $A_1, \dots, A_m$, then once the CTRW reaches distribution $\boldsymbol{\pi}$ it remains with distribution $\boldsymbol{\pi}$ independent of the graph dynamics. The question is whether the CTRW always converges to distribution $\boldsymbol{\pi}$. We see that this is true if $\{A(t)\}$ is T-connected.

We first present the notation used in this section. Let

$$Q(A_k, \vec{\gamma}(\gamma)) = \begin{cases} A_k(i,j)\gamma\beta_{k,i}/d_{k,i} & \text{if } i \neq j \\ -\sum_{j\in V} A_k(i,j)\gamma\beta_{k,i}/d_{k,i} & \text{if } i = j\,, \end{cases}$$

where $A_k(i,j)$ is the element (i,j) of A_k, be the infinitesimal generator of $\{U(t)\}$ given configuration A_k, $k = 1, \dots, m$. Let $\vec{\gamma}$ denote $\vec{\gamma}(1)$.

ASSUMPTION 3.2 (FIXED POINT $\boldsymbol{\pi}^\star$). *Let $\mathcal{A}$ be a set of graph configurations and $\vec{\gamma}$ be a set of walker rates such that exists a $\boldsymbol{\pi}^\star \in [0,1]^n$, $\sum_{v\in V} \boldsymbol{\pi}^\star(v) = 1$, that is a fixed point solution to*

$$\mathbf{0} = \boldsymbol{\pi}^\star Q(A, \vec{\gamma})\,, \quad \forall A \in \mathcal{A}\,. \quad (22)$$

In what follows we show that if $\mathcal{A}$ and $\vec{\gamma}$ satisfy Assumption 3.2, then $\lim_{t\to\infty} P[U(t) = v] = \boldsymbol{\pi}^\star(v)$, $\forall v \in V$. Moreover, we show that $\boldsymbol{\pi}^\star$ is independent of γ. We are now ready for the main result of this section.

THEOREM 3.4. *Let $\{A(t)\}$ with graph configuration set $\mathcal{A}$ be stationary, ergodic, and T-connected. Let $\{U(t)\}$ be a CTRW on $\{A(t)\}$ with walker rates $\vec{\gamma}(\gamma) = \{\gamma\beta_{k,v}\}, v \in V, k = 1, \dots, m$, $\gamma > 0$. If $\mathcal{A}$ and $\vec{\gamma}(1)$ satisfy Assumption 3.2, then*

$$\lim_{t\to\infty} P[U(t) = v] = \boldsymbol{\pi}^\star(v)\,, \quad \forall v \in V,$$

where $\boldsymbol{\pi}^\star$ solves (22). Moreover, $\boldsymbol{\pi}^\star$ does not depend on γ.

Proof. For now assume $\gamma = 1$. Let $\Pi(t) = (P[U(t) = v])_{v\in V}$. The Kolmogorov forward equation gives

$$\frac{d\Pi(t)}{dt} = \Pi(t) Q(A(t), \vec{\gamma}(1)). \quad (23)$$

From Assumption 3.2 there exists $\boldsymbol{\pi}^\star$ is that is a solution to (22). Hence, $\Pi(t) = \boldsymbol{\pi}^\star$ is also a solution to (23) where $d\Pi(t)/dt = 0$. It follows from Theorem 3.1 that there is no other solution to (23) and therefore $\lim_{t\to\infty} P[U(t) = v] = \boldsymbol{\pi}^\star(v)$, $\forall v \in V$. To show that $\boldsymbol{\pi}^\star$ does not depend on the assumption $\gamma = 1$, note that for any $\alpha > 0$

$$\mathbf{0}\alpha = \boldsymbol{\pi}^\star Q(A_k, \vec{\gamma}(1))\alpha = \boldsymbol{\pi}^\star Q(A_k, \vec{\gamma}(\alpha)).$$

□

Examples of adjacency matrix sets $\mathcal{A} = \{A_k : k = 1, \dots, m\}$ that satisfy Assumption 3.2 for a constant rate walker, $\vec{\gamma}(\gamma) = (\gamma, \dots, \gamma)$, where obtaining $\boldsymbol{\pi}^\star$ is straightforward include:

- **Regular graphs**: A_i, $i = 1, \dots, m$, consists of $C_i \geq 1$ connected components where the j-th connected component ($j = 1, \dots, C_i$) is a $d_j^{(i)}$-regular graph ($d_j^{(i)} >= 0$).
- **Nodes $v \in V$ alternate between isolated and connected with constant degree, i.e., $d_k(v) \in \{0, d(v)\}$, $d(v) > 0$, $k = 1, \dots, m$.** Figure 1 illustrates a dynamic graph that satisfies these requirements.

Conditions imposed on the walker rates $\vec{\gamma}(\gamma)$ can also guarantee that Assumption 3.2 is valid for any set of graph configurations $\mathcal{A}$. As for instance in the following proposition stated without a proof:

PROPOSITION 3.3 (**Degree proportional walker**). *Let $\mathcal{A} = \{A_k : k = 1, \dots, m\}$ be a set of graph configurations. If the walker rates are $\vec{\gamma}(\gamma) = (\gamma d_{k,v})_{v\in V,k=1,\dots,m}$, where $d_{k,v}$ is the degree of node v at configuration k. Then Theorem 3.4 is satisfied. Moreover, $\boldsymbol{\pi}^\star = (\frac{1}{n}, \dots, \frac{1}{n})$.*

Proposition 3.3 has an interesting application. We can uniformly sample nodes (in time) without knowing the underlying topology or graph dynamics $\{A(t)\}$, as long as $\{A(t)\}$ is stationary, ergodic, and T-connected. So far we have focused on conditions that allow us to obtain the stationary distribution of the walker. In what follows we present some case studies solved numerically.

4. CASE STUDIES

In previous sections we characterize the stationary behavior of a RW on a broad class of graph dynamic processes $\{(X_i, S_i)\}_0^\infty$. In this section, we mostly focus on random walks over Markovian dynamic graphs, i.e. the cases where $\{X_i\}_0^\infty$ can be modeled by a Markov chain and $\{S_i\}_0^\infty$ are iid exponentially distributed random variables. We comment on how the results of previous sections can be immediately

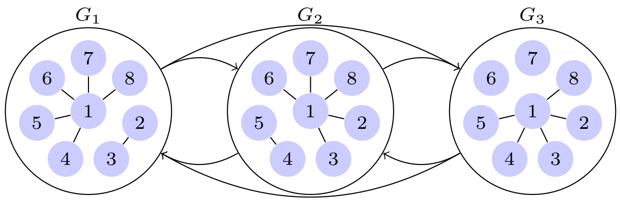


Figure 1: Illustration of a dynamic graph whose node degrees are either kept constant in all graph configurations or there are isolated nodes. We make no assumption about the holding times of each graph configuration except that the graph is T-connected.

obtained for CTRWs on Markovian dynamic graphs and also provide several examples and numerical evaluation for such cases. Later in the section, we show an application of a slow CTRW that applies to general graph dynamics and general walker inter step time distribution.

4.1 Markovian dynamic graph models

The constant rate CTRW over the dynamic graph process can be represented as a continuous-time Markov process $\{Q(t)\}_0^\infty$ where $Q(t) \in V \times \mathcal{A}$. It is convenient to work with two distinct partitions for the infinitesimal generator matrix of $\{Q(t)\}$, $\mathbf{Q}$: $\mathcal{P}^{(g)}$ and $\mathcal{P}^{(w)}$, where $\mathcal{P}^{(g)}$ denotes the partition in respect to the graph dynamics and $\mathcal{P}^{(w)}$ is the partition in respect to the walker dynamics, defined as follows. First define the subsets of states $\mathcal{S}_k^{(g)}$, $k = 1, \ldots, m$ (m is the number of different graphs) where $\mathcal{S}_k^{(g)}$ contains all the states in the MC corresponding to the graph G_k. We denote this partition (that matches the arrangement of Figure 2(a)) as $\mathcal{P}^{(g)}$. Note that transitions among blocks depend only on the transition rates associated with the graph dynamics. We consider a second partition $\mathcal{P}^{(w)}$ with subsets of states $\mathcal{S}_k^{(w)}$, $k = 1, \ldots, n$ (n is the total number of vertices in any of the graphs), where subset $\mathcal{S}_k^{(w)}$ contains all states in which the walker is at vertex k. Within a block we have the graph dynamics and transitions among blocks represent the walker dynamics. Note that transitions between blocks depend solely on γ, but not on the rates that govern the graph dynamics.

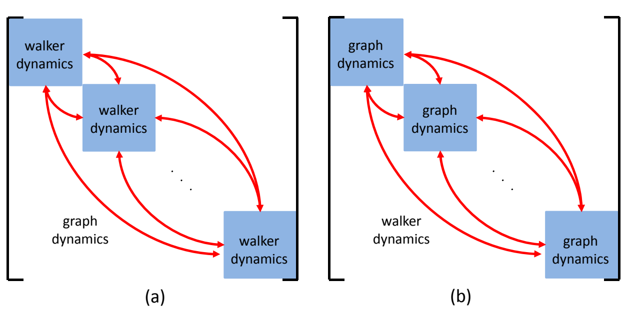


Figure 2: Two different matrix arrangements for CTRW on Markov evolving graphs.

The results for the limiting cases in which $\gamma \to 0$ and $\gamma \to \infty$ (time scale separation), obtained in Theorem 3.2 and Proposition 3.2 can also be immediately obtained from the decomposition theory for Markov chains, by considering partitions $\mathcal{P}^{(g)}$ and $\mathcal{P}^{(w)}$ for each case, respectively. In these cases the resulting model is *nearly completely decomposable* (NCD). It is of interest to take a closer look at this alternative approach.

In an NCD system, the transition rates between states in different partitions are much smaller as compared to the rate of events that are responsible for jumping from one state to another in the same partition. Therefore, each time the system enters subset $\mathcal{S}_i$ it stays there long enough so that the effect of the transient behavior in subset $\mathcal{S}_i$ vanishes and the expected number of visits in a state of $\mathcal{S}_i$ in each visit to the subset becomes independent of which state $\mathcal{S}_i$ was entered.

Using Courtois' results [5] after partitioning the model according to $\mathcal{P}^{(g)}$ when $\gamma \to 0$ and $\mathcal{P}^{(w)}$ when $\gamma \to \infty$, we can *fold back* the outgoing rates of each of the states in any block partition in Figure 2(a) (or 2(b), respectively) into the corresponding state, transforming the generator matrix of the original problem into a completely decomposable matrix. The solution of each block after the transformation, is an approximation of the steady state probabilities conditioned on the walker being in that block. The steady state distribution $\boldsymbol{\pi}$ for the states in the CTRW Markov model immediately follows after aggregating each block into a single state, solving the aggregated model, and unconditioning the conditioned state probabilities using the solution of the aggregated model.

It is interesting to observe that, from Simon-Ando results [21], the solution obtained can always meet a given level of accuracy for sufficiently large (respectively, small) γ. In addition, efficient procedures exist for obtaining approximate solutions or exact solution when $\mathbf{Q}$ has special structures.

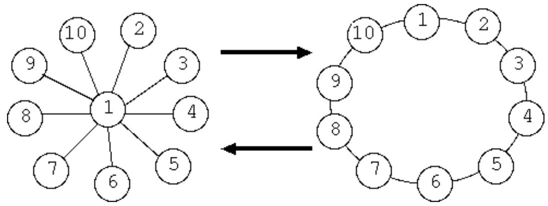
4.1.1 Star-circle example

We start our examples by considering a very simple example, consisting of just two graph snapshots: a star and a circle, as illustrated in Figure 3a. The graph dynamics transitions from one graph to the other with rates $\lambda_{12} = \lambda_{21} = 1$. Thus, the average time in each graph is 1/2. Note however, that edges (1,2) and (1,10) are always present, since they exist in both configurations.

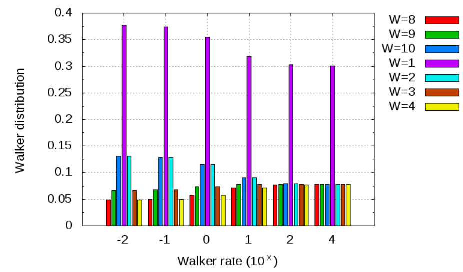
We investigate the steady state solution of the CTRW on this dynamic graph. Figure 3b shows the numerical solution of the steady state distribution of the random walk as a function of the walker rate (for clarity, only a subset of the states are shown). Interestingly, the stationary distribution of the walker depends on the walker rate and converges to some distribution as the walker moves faster or slower. Indeed, the numerical results obtained are in agreement with the theoretical distributions for the fast and slow walker given in Sections 3.1.1 and 3.1.2, respectively. Finally, we note that in a graph with n nodes, node one alternates between degrees $n-1$ and two. As n increases the dependence on the walker speed is magnified.

4.2 Edge Markovian model and examples

In this section we consider some particularities of random walks on a special class of dynamic graphs called edge Markovian graphs. Start with a fixed adjacency matrix A



(a) The star-circle graph dynamics.



(b) Steady state distribution of walker as a function of walker rate (nodes 5, 6 ,7 not shown for clarity).

Figure 3: The star-circle graph dynamics and behavior of CTRW as a function of walker rate.

and attach an independent On-Off process at each edge with exponentially distributed holding times. In edge Markovian graphs, edges alternate between being present and absent from the graph according to independent On-Off processes. Let E be the set of edges in the graph described by A. Let $\Lambda_0(e)$ and $\Lambda_1(e)$ denote the rate at which edge $e \in E$ changes from the On to the Off state and from the Off to the On state, respectively, which can vary from edge to edge.

A few observations on the model follows. Let A_k be a particular configuration of the dynamic graph model. In particular, the edge Markovian model induces a total of $m = 2^{|E|}$ configurations, which represent all possible labeled subgraphs over an edge set with $|E|$ edges. Moreover, consider any transition between two configurations induced by the model. The graphs corresponding to these two configurations differ by exactly one edge, since the on-off processes associated with the edges are continuous in time. Moreover, the rate associated with this graph transition is given by the corresponding rate of the edge (either its On rate or Off rate).

Let q_{uv} denote the steady state fraction of time that edge $(u, v) \in E$ is On, which is simply given by

$$q_{uv} = \Lambda_1(u, v)/(\Lambda_0(u, v) + \Lambda_1(u, v)).$$

Let σ_k denote the steady state fraction of time that the edge Markovian model spends in configuration A_k, $k = 1, \ldots, 2^{|E|}$. In particular, we have

$$\sigma_k = \prod_{v \in V} \prod_{u \in V} \Big(A_{k,uv} \sqrt{q_{uv}} + (\mathbf{1}\{(u, v) \in E\} - A_{k,uv}) \sqrt{1 - q_{uv}} \Big), \tag{24}$$

where the squared root comes from the fact that G is undirected (thus A_k is symmetric, $k = 1, \ldots, |E|$). We note that edge Markovian graphs are of interest due to their simple description and structure. However, as we will soon see in our numerical results, the steady state distribution of a constant rate random walk on this model depends on the walker rate. It is an open problem whether the walker steady state distribution can be obtained in closed form.

4.2.1 Edge Markovian: $A = K_3$

Consider the edge Markovian graph model over a complete graph with 3 nodes. The number of different configurations is $2^3 = 8$, out of which 4 have at least one isolated vertex (graph not connected). Moreover, let $\Lambda_1(1, 2) = \Lambda_0(1, 2) = 10^4$, $\Lambda_1(1, 3) = \Lambda_0(1, 3) = 1$, $\Lambda_1(2, 3) = \Lambda_0(2, 3) = 1$. Thus, $p_e = 0.5$ for every edge (i.e., all edges have the same time average), and thus, all configurations have the same time average probability of 1/8, as given by equation (24).

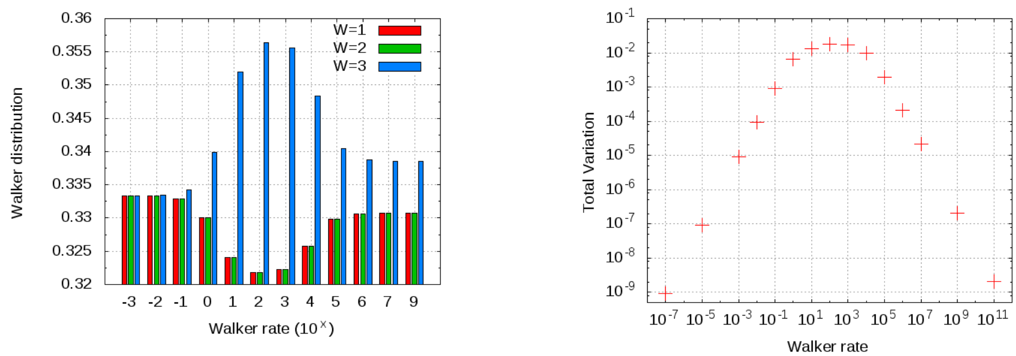
Figure 4a shows the exact steady state distribution of the random walk as a function of the walker rate. Interestingly, while all edges have the same time average and all configurations A_k, $k = 1, \ldots, 2^{|E|}$, have the same σ_k, the walker steady state distribution still depends on the walker rate. Moreover, the behavior of the fast and slow walker is different. While the slow walker will converge to a uniform distribution over the nodes, the fast walker will always favor node 3, the node not incident to the fastest changing edge $(1, 2)$. Despite the relatively small differences in the walker distribution ($P[W = 3]$ varies by 7%), the point is to illustrate that such differences can arise even in small and simple models.

Figure 4b also shows the total variation between our theoretical results for the fast walker and for the fast graph and the actual distribution obtained exactly. For walker rates greater than one, the theoretical results for the fast walker were used (Section 3.1.1), while for rates smaller than one the results for the fast graph were used (Section 3.1.2). Note that as the walker slows down or speeds up, the total variation decreases fast (graph in log-log scale). As expected, our numerical results for the fast and slow walker agree with our asymptotic results. When the timescales of the walker and graph dynamics are similar, the difference between the asymptotic cases and the steady state distribution is large, as expected.

Moreover, if we consider the "degree-time distribution" of a given node, namely, the fraction of time that node $v \in V$ has degree $0, 1, \ldots$, we note that all three nodes have identical degree-time distributions but the fraction of time the walker spends on each node varies. This indicates that the degree-time distribution is insufficient to characterize the walker steady state. This is also like true in transient metrics as well.

4.2.2 Edge Markovian on a 6-node kite

As a last example, consider the edge Markovian process where A is the "kite graph" illustrated in Figure 5a. In particular, let all thin edges be distributed according to the On-Off process defined by $\Lambda_1(e) = \Lambda_0(e) = 1$, and let all thick edges be distributed to the On-Off process defined by



(a) Steady state distribution of walker as a function of walker rate.

(b) Total variation between theoretical results for fast and slow walker when compared to exact walker distribution.

Figure 4: Characteristics of random walks on a K_3 edge Markovian graph model.

$\Lambda_1(e) = 100, = \Lambda_0(e) = 10$. Note that in this example, besides having the same time degree distribution (and time average), all nodes are incident on identical and independent On-Off processes. In particular, every node $v \in V$ is incident to two edges that follow independent instances of the slower and symmetric On-Off process and one edge that follows the faster and asymmetric process. Thus, in some sense, every node is identical with respect to the stochastic processes that govern the dynamics of its incident edges. Surprisingly, even in this case the behavior of the walker depends on its rate, as shown in Figure 5b. This indicates the difficulty of characterizing the exact behavior of random walks in general graph dynamics, even when limited to edge Markovian models. Clearly, the structure of the graph plays an important role, as illustrated in this example. Finally, note that fast and slow walkers have different behavior.

4.3 A simple vehicular DTN

In this section we consider an application of our modeling framework to a simple vehicular disruption-tolerant networks (DTN) model. Although unrealistic, the model captures some essential characteristics of DTNs. Consider a set of buses equipped with wireless routers moving around according to their routes. Two buses establish communication when they are within the coverage radius of the wireless routers. Buses belonging to the same line (route) move from one bus stop to another following a predefined sequence of stops in a circular fashion. The following notation is used to describe the model: $S = \{s_1, \ldots, s_o\}$ is the set of bus stops across all bus lines, where o is the number of different stops; $L_i \in S^{n_i}$ is a vector with the sequence of stops for bus line i, and n_i is the number of stops at bus line i; l denotes the number of different bus lines and b_i is the number of different buses operating in line $i = 1, \ldots, l$; Assume that both ξ^i_k, the amount of time bus i stay at a stop time k, and ζ^i_{kl}, the time it takes a bus i to move from stop k to l are exponentially distributed random variables, but can have different parameter values for any i, k, l. In addition, buses move independently of each other, including those in the same line.

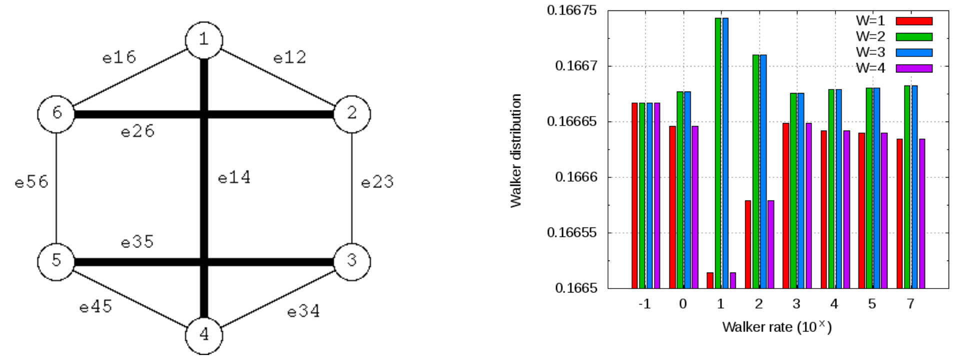
The bus routes and the coverage radius of the wireless router allow two or more buses to exchange information when buses are at the same stop. Thus, if two or more different bus lines share at least one bus stop in their route, then buses from these lines will be able to communicate at the shared bus stops. Moreover, two or more buses from the same line can communicate in any stop of their line, as they can always meet at these stops. Finally, we assume that the communication radius is smaller than the distance between bus stops, such that buses can only communicate when located at a shared bus stop.

Figure 6a shows an example with $o = 11$ bus stops, three bus lines ($l = 3$), defined by $L_1 = (s_1, s_2, s_3, s_4, s_5)$, $L_2 = (s_3, s_4, s_6, s_7, s_8)$ and $L_3 = (7, 9, 10, 11)$, and four buses: $b_1 = 1$, $b_2 = 1$, $b_3 = 2$ (line three has two buses). Note that lines 1 and 2 share two bus stops (s_3 and s_4) and that lines 2 and 3 share one bus stop (s_7).

Consider a continuous-time random walker (CTRW) moving around the buses with rate γ. The goal is to determine the fraction of time that the walker spends in each bus or in each bus line. This problem can be formulated and solved using the modeling framework proposed in this paper. The first step is to construct a dynamic connectivity graph model from the movement of the buses in their respective lines. In particular, each bus is a node in the graph, since this corresponds to a possible location for the random walk. Moreover, each possible configuration of buses on their stops will define a connectivity graph, where nodes (buses) in the same stop are all within communication radio of one another. Note that each connectivity graph is composed of connected components that are all cliques (fully connected subgraph), since all buses in the same stop can communicate.

Consider the example in Figure 6a and the possible connectivity graphs that can be created, which are illustrated in Figure 6b. The connectivity graph has four nodes, corresponding to the four buses. Each bus can be in a different stop, thus yielding a connectivity graph with no edges. Also, bus in line 2 can be at stop s_7 at the same time as the two buses from line 3, yielding a connectivity graph where these three buses are all connected. Finally, note that not all graphs with four nodes are possible, since different lines may not have stops in common such as lines 1 and 3, for instance.

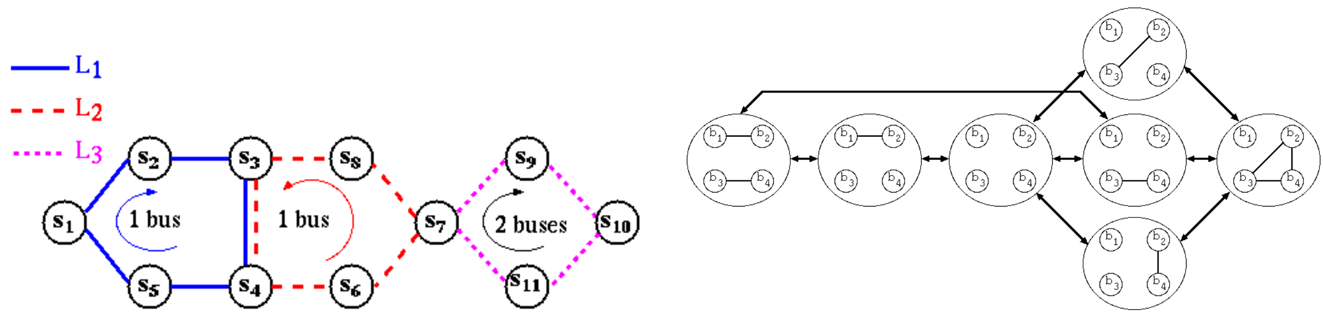
The transitions between graph configurations are shown in Figure 6b. In our model, buses cannot simultaneously leave a stop. As a consequence, the number of allowed transitions is reduced. Once the dynamic graph model is constructed,



(a) The edge Markovian model over the 6-node kite graph where edges belong to one of two On-Off processes.

(b) Steady state distribution of walker as a function of walker rate (nodes 5 and 6 not shown, but are identical to nodes 2 and 3).

Figure 5: Characteristics of random walks on a 6-node kite edge Markovian graph model.



(a) Example of a vehicular DTN with eleven bus stops, three bus lines and four buses (line 3 has 2 buses).

(b) Connectivity graphs and transitions between them according to the bus system illustrated in Figure 6a.

Figure 6: Example of a simple bus system (a) and the induced dynamic graph model (b).

we shall define the state holding times for each graph configuration. This is known if the holding times at bus stations and the amount of time it takes a bus to move from a station to another are exponentially distributed. However, this is non-trivial in the general case, since buses can move along their routes without changing the connectivity graph. Moreover, totally different bus configurations over the set of stops can lead to the same connectivity graph. Since our modeling framework makes no assumption on the state (static graph) holding times of the dynamic graph model we can extend the exponential assumption considering general distributions for the holding times *at each graph configuration*, assuming only that the expected holding time is finite for all static graphs and that dynamic graph process is stationary, ergodic and T-connected.

The model constructed from this scenario matches the case studied in Section 3.2 in which the stationary distribution of the random walk is time-scale invariant and uniform over the set of nodes in the graph, independent of graph dynamics (see Theorem 3.4). This occurs since every connected component of every possible connectivity graph is a clique, thus, having identical degree within each component. Therefore, the fraction of time the walker spends in any given bus is simply $1/\sum_j b_j$, while the fraction of time spent in bus line i is simply $b_i/\sum_j b_j$, independent of the walker rate and graph dynamics! Such result could then be used to devised principled random walk-based mechanisms for searching for information or sampling properties in such systems. In addition, our results indicate that the random walk spends 1/4 of the time in each bus and 1/2 of the time in bus line 3.

5. RELATED WORK

Random walks have been widely used to understand and characterize graphs due to their well understood steady state behavior. By leveraging the steady state distribution of random walks, principled mechanisms for characterizing and estimating vertex-related properties have been devised [8, 12, 14, 16, 17].

It follows that random walks can potentially be used to understand and characterize dynamic graphs. In fact, efforts in this direction concerning time-independent dynamic graphs (i.e., each snapshot is independent of the previous) have appeared in the literature [3, 9, 10, 15], mainly in the context of determining upper and lower bounds for the cover time of random walks. More recently, proposals to define time-dependent dynamic graph models as well as characterize random walks in them have also appeared in the literature [1,2,4]. However, these efforts have focused on discrete-time dynamic graph models with a goal of computing the cover time of random walks either in special graph structures [2], in specific dynamic graph models [4], or through numerical evaluations [1]. Our work differs from these in the sense that we consider continuous-time dynamic graph models and continuous-time random walks with the goal of analytically characterizing the steady state behavior of the walker. Moreover, our prior work on this topic considered only Markovian dynamics and characterized the steady state behavior of the walker only under time-scale separation [18].

Finally, random walks have also been used as a sampling mechanism to estimate characteristics of vertices (e.g., fraction of vertices of a particular kind) in large static graphs [8, 12, 17]. More recently, efforts to measure characteristics of vertices in dynamic graphs have also appeared in the literature [16, 22]. However, these are mostly preliminary and exploratory papers, indicating potential pitfalls and biases introduced by fast changing dynamic graphs. In contrast, our works is a first step at providing a theoretical foundation that can then be applied to estimate characteristics of vertices in dynamic graphs.

6. CONCLUSION

Understanding the long-term behavior of CTRW over dynamic graphs is an important step in measuring and characterizing dynamic graphs. Differently from the static case, CTRWs on dynamic graphs have an non-trivial behavior. However, as we have shown, several situations are amenable to theoretical analysis, allowing us to precisely determine its steady state distribution. Such situations may arise naturally in some networks or be induced in others. For example, the constant degree constraint is a good approximation for P2P networks where peers maintain a nearly fixed number of connections. Hence, Theorem 3.4 helps explain why sampling these dynamic networks using random walks leads to meaningful results [22].

We believe that the topic of CTRWs over dynamic graphs offer a host of interesting open questions. For instance, for a given class of dynamic graphs, can our asymptotic results for the fast walker and fast graph be used to bound the stationary distribution of a CTRW with arbitrary rate? Or would the asymptotic results offer little or even no information about the steady state distribution in these scenarios?

7. ACKNOWLEDGMENTS

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8. REFERENCES

[1] Utku Günay Acer, Petros Drineas, and Alhussein A. Abouzeid. Random walks in time-graphs. In *Proceedings of the Second International Workshop on Mobile Opportunistic Networking*, MobiOpp'10, pages 93–100, 2010.

[2] C. Avin, M. Koucky, and Z. Lotker. How to explore a fast-changing world. (on the cover time of dynamic graphs). In *Proc. ACM ICALP*, pages 121–132, 2008.

[3] Andrea E. F. Clementi, Francesco Pasquale, Angelo Monti, and Riccardo Silvestri. Communication in dynamic radio networks. In *Proc. ACM PODC'07*, pages 205–214. ACM, 2007.

[4] Andrea E.F. Clementi, Claudio Macci, Angelo Monti, Francesco Pasquale, and Riccardo Silvestri. Flooding time in edge-markovian dynamic graphs. In *Proc. ACM PODC'08*, pages 213–222. ACM, 2008.

[5] Pierre-Jacques Courtois. *Decomposability, Queueing and Computer Systems Applications*. Academic Press, 1976.

[6] Daniel Figueiredo, Philippe Nain, Bruno Ribeiro, Edmundo de Souza e Silva, and Don Towsley. Characterizing random walks on general time varying graphs. Technical report, UMass Amherst, 2012.

[7] P. Franken, D. Konig, U. Arndt, and V. Schmidt. *Queues and Point Processes*. J. Wileyl, 1982.

[8] M. Gjoka, M. Kurant, C. Butts, and A. Markopoulou. A walk in Facebook: Uniform sampling of users in online social networks. In *Proc. IEEE INFOCOM*, 2010.

[9] S.M. Hedetniemi, S.T. Hedetniemi, and A.L.Liestman. A survey of gossiping and broadcasting in communication networks. *Networks*, 18(4):319–349, 1988.

[10] D. Kempe and J. Kleinberg. Protocols and impossibility results for gossip-based communication mechanisms. In *Proc. of Symposium on Foundations of Computer Science (FOCS)*, pages 471–480, 2002.

[11] A. N. Kolmogorov and S. V. Fomin. *Introductory Real Analysis*. Dover, 1975.

[12] Laurent Massoulié, Erwan Le Merrer, Anne-Marie Kermarrec, and Ayalvadi Ganesh. Peer counting and sampling in overlay networks: random walk methods. In *Proceedings of the twenty-fifth annual ACM symposium on Principles of distributed computing*, PODC '06, pages 123–132, New York, NY, USA, 2006. ACM.

[13] P. Moyal. A generalized backwards scheme for solving non monotonic stochastic recursions. *Arxiv preprint arXiv:1009.1235*, Sept. 2010.

[14] M.E.J. Newman. A measure of betweenness centrality based on random walks. *Social Networks*, 27(1):39–54, 2005.

[15] B. Pittel. On spreading a rumor. *SIAM Journal on Applied Mathematics*, 47(1):213–223, 1987.

[16] A. H. Rasti, M. Torkjazi, R. Rejaie, N. Duffield, W. Willinger, and D. Stutzbach. Respondent-driven sampling for characterizing unstructured overlays. In *Proc. of IEEE INFOCOM'09 Mini-Conference*, 2009.

[17] B. Ribeiro and D. Towsley. Estimating and sampling graphs with multidimensional random walks. In *Proc. of the ACM SIGCOMM Internet Measurement Conference*, Nov 2010.

[18] Bruno Ribeiro, Daniel Figueiredo, Edmundo de Souza e Silva, and Don Towsley. Characterizing continuous-time random walks on dynamic networks. *SIGMETRICS Perform. Eval. Rev.*, 39(1):343–344, June 2011.

[19] Albert N. Shiryaev. *Probability*. Springer, 2nd edition, 1995.

[20] Karl Sigman. *Stationary Marked Point Processes: An Intuitive Approach*. Chapman & Hall, 1995.

[21] Herbert A. Simon and Albert Ando. Aggregation of variables in dynamic systems. *Econometrica*, 29:111–138, 1961.

[22] Daniel Stutzbach, Reza Rejaie, Nick Duffield, Subhabrata Sen, and Walter Willinger. On unbiased sampling for unstructured peer-to-peer networks. *IEEE/ACM Trans. Netw.*, 17:377–390, April 2009.

APPENDIX

A. PROOF: T IS FINITE

It follows from the fact that the graph process is time stationary and ergodic that there exists $T_k > 0$ s.t.

$$\frac{1}{t}\int_x^{x+t} \mathbf{1}(A(s) = A_k)ds > \sigma_k/2\,, \quad x \geq 0,\ t \geq T_k$$

independent of x. Choose $k_0 = \text{argmin}_k\{\sigma_k\}$ and $T' = \max\{T_k\}$. Thus

$$\frac{1}{T'}\int_x^{x+T'} \mathbf{1}(A(s) = A_k)ds > \sigma_{k_0}/2\,, \quad x \geq 0.$$

Consequently the graph process spends at least $\sigma_{k_0}T'/2$ units of time in configuration k during interval $[x, x+T')$ regardless of the state of the graph process at $t = x$. Now consider the situation where $U_1(x) = u$, $U_2(x) = w$, and $A(x) = A_k$. Let $A_k = A_{l_1}, A_{l_2}, \ldots, A_{l_j}$ be a sequence of graphs such that there is a temporal path between u in A_{l_1} and w in A_{l_j}. This requires that there be paths within each graph configuration, $p_i = (v_{i,0}, v_{i,1}, \ldots, v_{i,n_i})$ that satisfy $v_{1,0} = u$, $v_{i+1,0} = v_{i,n_i}$, $i = 1, \ldots, j-1$, and $v_{j,n_j} = w$. Let $h_{u,w}$ denote the number of physical hops on this path and $j = H_{u,w}$ the number of configurations in the sequence. We focus now on the event that walker 1 progresses from node u to node w in the interval $[x, x+jT')$ jT' by progressing across path p_i during $[x+(i-1)T', x+iT')$ while walker 2 remains at node w. The probability of this event, $p_{u,w}$ is bounded from below by

$$p_{u,w} \geq e^{-\gamma_{\max}H_{\max}T'}e^{-\gamma_{\max}H_{\max}T'(1-\sigma_{k_0}/2)} \times \big(\frac{1}{d_{\max}}\big)^{h_{\max}}\prod_{i=1}^{j}P\big(\sum_{\ell=0}^{n_i-1}Z_{i,v_\ell} < T'\sigma_{k_0}/2 \leq \sum_{\ell=0}^{n_i}Z_{i,v_\ell}\big)$$

Here $H_{\max} = \max_{u,w}H_{u,w}$, $h_{\max} = \max_{u,w}h_{u,w}$, and $Z_{i,v}$ denotes the time between a walker arriving to node v in configuration A_{l_i} and taking its next step. This time is exponentially distributed with rate $\gamma_{l_i,v}$ and there exists some $q > 0$ such that

$$P\big(\sum_{\ell=0}^{n_i-1}Z_{i,v_\ell} < T'\sigma_{k_0}/2 \leq \sum_{\ell=0}^{n_i}Z_{i,v_\ell}\big) > q$$

for all u, w. Hence

$$p_{u,w} \geq p_0 \equiv e^{-2\gamma_{\max}H_{\max}T'}\big(\frac{1}{d_{\max}}\big)^{h_{\max}}q$$

Last, $T_0 \equiv H_{\max}T'$.

B. UNIFORM CONVERGENCE

Proof. Fix $\epsilon > 0$. From Egorov's theorem [11, pg. 290, Theorem 12] there exists a measurable set B_ϵ with $P(B_\epsilon) < \epsilon$ and $T_{B_\epsilon,\epsilon} > 0$, such that for all $t > T_{B_\epsilon,\epsilon}$ (we write $A(x,\omega)$ for $A(x)$ to emphasize that $A(x)$ is a rv, P-measurable)

$$\left|\frac{1}{t}\int_0^t \mathbf{1}_{A(x,\omega)=A_k}dx - \sigma_k\right| < \epsilon$$

for all $\omega \in \mathcal{F}$ and for all $\mathcal{F} \in \Omega - B_\epsilon$. Integrating w.r.t. $dP(\omega)$ for $w \in V$ gives

$$(\sigma_k - \epsilon)P(V) < \int_V dP(\omega)\frac{1}{t}\int_0^t(\mathbf{1}_{\{A(x,\omega))=A_k\}}dx < (\sigma_k + \epsilon)P(V) \tag{25}$$

for all $t > T$. By Fubini's theorem

$$\begin{aligned}\int_V\int_0^t \mathbf{1}_{\{A(x,\omega))=A_k\}}dxdP(\omega) &= \int_0^t\int_\Omega \mathbf{1}_{\omega\in V}\mathbf{1}_{\{A(x,\omega)=A_k\}}dP(\omega)dx \\ &= \int_0^t P(\{A(x,\omega) = A_k\} \cap V)dx \\ &= P(V)\int_0^t P(A(x) = A_k|V)dx. \quad (26)\end{aligned}$$

Combining (25) and (26) give

$$(\sigma_k - \epsilon)P(V) < \frac{P(V)}{t}\int_0^t P(A(x,\omega) = A_k|V)dx < (\sigma_k + \epsilon)P(V)$$

for all $t > T$. If $P(V) > 0$ we may divide the above inequalities by $P(V)$ to get

$$\sigma_k - \epsilon < \frac{1}{t}\int_0^t P(A(x,\omega) = A_k|V)dx < \sigma_k + \epsilon$$

for all $t > T$. This shows that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t P(A(x) = A_k|V)dx = \sigma_k$$

uniformly w.r.t. to any set V such that $P(V) > 0$. The uniform convergence follows from the fact that $T_{B_\epsilon,\epsilon}$ does not depend on V. $\square$