3 Mobile Robot Kinematics

3.1 Introduction

Kinematics is the most basic study of how mechanical systems behave. In mobile robotics, we need to understand the mechanical behavior of the robot both in order to design appropriate mobile robots for tasks and to understand how to create control software for an instance of mobile robot hardware.

Of course, mobile robots are not the first complex mechanical systems to require such analysis. Robot manipulators have been the subject of intensive study for more than thirty years. In some ways, manipulator robots have been much more complex than early mobile robots: a standard welding robot may have five or more joints, whereas early mobile robots were simple differential drive machines. In recent years, the robotics community has achieved a fairly complete understanding of the kinematics and even the dynamics (i.e. relating to force and mass) of robot manipulators [30,31].

The mobile robotics community poses many of the same kinematic questions as the robot manipulator community. A manipulator robot's workspace is crucial because it defines the range of possible positions that can be achieved by its end-effector relative to its fixture to the environment. A mobile robot's workspace is equally important because it defines the range of possible poses that the mobile robot can achieve in its environment. The robot arm's controllability defines the manner in which active engagement of motors can be used to move from pose to pose in the workspace. Similarly, a mobile robot's controllability defines possible paths and trajectories in its workspace. Robot dynamics places additional constraints on workspace and trajectory due to mass and force considerations. The mobile robot is also limited by dynamics; for instance, a high center of gravity limits the practical turning radius of a fast, car-like robot because of the danger of rolling.

But the chief difference between a mobile robot and a manipulator arm also introduces a significant challenge for *position estimation*. A manipulator has one end fixed to the environment. Measuring the position of an arm's end effector is simply a matter of understanding the kinematics of the robot and measuring the position of all intermediate joints. The manipulator's position is thus always computable by looking at current sensor data. But a mobile robot is a self-contained automaton that can wholly move with respect to its environment. There is no direct way to measure a mobile robot's position instantaneously. Instead, one must integrate the motion of the robot over time. Add to this the inaccuracies of *motion estimation* due to slippage and it is clear that measuring a mobile robot's position precisely is an extremely challenging task.

The process of understanding the motions of a robot begins with the process of describing the contribution each wheel provides for motion. Each wheel has a role in enabling the whole robot to move. By the same token, each wheel also imposes constraints on the robot's motion, for example refusing to skid laterally. In the following section, we introduce notation that allows expression of robot motion in a global reference frame as well as the robot's local reference frame. Then, using this notation, we demonstrate the construction of simple forward kinematic models of motion, describing how the robot as a whole moves as a function of its geometry and individual wheel behavior. Next, we formally describe the kinematic constraints of individual wheels, and then combine these kinematic constraints to express the whole robot's kinematic constraints. With these tools, one can evaluate the paths and trajectories that define the robot's maneuverability.

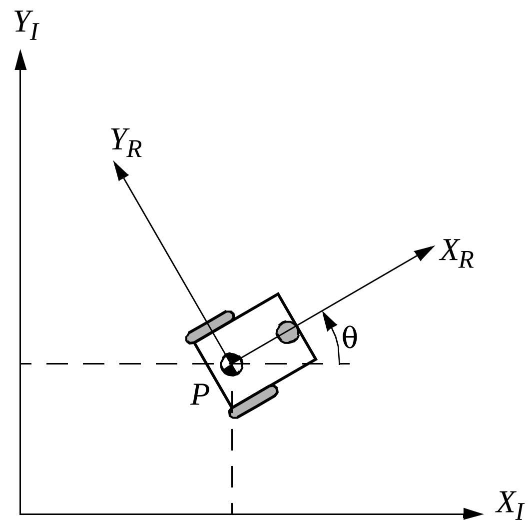

Fig 3.1 The global reference frame and the robot local reference frame

3.2 Kinematic Models and Constraints

Deriving a model for the whole robot's motion is a bottom-up process. Each individual wheel contributes to the robot's motion and, at the same time, imposes constraints on robot motion. Wheels are tied together based on robot chassis geometry, and therefore their constraints combine to form constraints on the overall motion of the robot chassis. But the forces and constraints of each wheel must be expressed with respect to a clear and consistent reference frame. This is particularly important in mobile robotics because of its self-contained and mobile nature; a clear mapping between global and local frames of reference is required. We begin by defining these reference frames formally, then using the resulting formalism to annotate the kinematics of individual wheels and whole robots. Throughout this process we draw extensively on the notation and terminology presented in [35].

3.2.1 Representing robot position

Throughout this analysis we model the robot as a rigid body on wheels, operating on a horizontal plane. The total dimensionality of this robot chassis on the plane is three, two for position in the plane and one for orientation along the vertical axis, which is orthogonal to the plane. Of course, there are additional degrees of freedom and flexibility due to the wheel axles, wheel steering joints and wheel castor joints. However by robot *chassis* we refer only to the rigid body of the robot, ignoring the joints and degrees of freedom internal to the robot and its wheels.

In order to specify the position of the robot on the plane we establish a relationship between the global reference frame of the plane and the local reference frame of the robot as in figure 3.1. The axes X_I and Y_I define an arbitrary inertial basis on the plane as the global reference frame from some origin *O*: $\{X_I, Y_I\}$. To specify the position of the robot, choose a point *P* on the robot chassis as its position reference point. The basis $\{X_R, Y_R\}$ defines two axes relative to *P* on the robot chassis and is thus the robot's local reference frame. The position of

P in the global reference frame is specified by coordinates x and y, and the angular difference between the global and local reference frames is given by θ. We can describe the pose of the robot as a vector with these three elements. Note the use of the subscript I to clarify the basis of this pose as the global reference frame:

$$\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \tag{3.1}$$

To describe robot motion in terms of component motions, it will be necessary to map motion along the axes of the global reference frame to motion along the axes of the robot's local reference frame. Of course, the mapping is a function of the current pose of the robot. This mapping is accomplished using the *orthogonal rotation matrix*:

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

This matrix can be used to map motion in the global reference frame $\{X_I, Y_I\}$ to motion in terms of the local reference frame $\{X_R, Y_R\}$. This operation is denoted by $(\theta)\dot{\xi}_I$ because the computation of this operation depends on the value of θ:

$$\dot{\xi}_R = R(\frac{\pi}{2})\dot{\xi}_I \tag{3.3}$$

For example, consider the robot in Figure 3.2. For this robot, because $\theta = \frac{\pi}{2}$ we can easily compute the instantaneous rotation matrix R:

$$R(\frac{\pi}{2}) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.4}$$

Given some velocity $(\dot{x}, \dot{y}, \dot{\theta})$ in the global reference frame we can compute the components of motion along this robot's local axes X_R and Y_R. In this case, due to the specific angle of the robot, motion along X_R is equal to $\dot{y}$ and motion along Y_R is $-\dot{x}$:

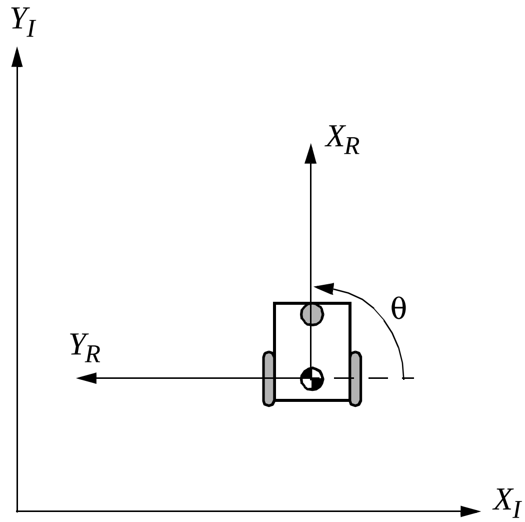

Fig 3.2 The mobile robot aligned with a global axis

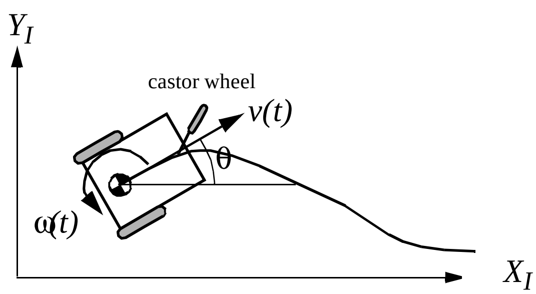

Fig 3.3 A differential drive robot in its global reference frame

$$\dot{\xi}_R = R\left(\frac{\pi}{2}\right)\dot{\xi}_I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{bmatrix} \tag{3.5}$$

3.2.2 Forward kinematic models

In the simplest cases, the mapping described by Equation (3.3) is sufficient to generate a formula that captures the forward kinematics of the mobile robot: how does the robot move, given its geometry and the speeds of its wheels? More formally, consider the example shown in Figure 3.3.

This differential drive robot has two wheels, each with diameter r. Given a point P centered between the two drive wheels, each wheel is a distance l from P. Given r, l, θ and the spinning speed of each wheel, $\dot{\varphi}_1$ and $\dot{\varphi}_2$, a forward kinematic model would predict the robot's overall speed in the global reference frame:

$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(l,\ r,\ \theta,\ \dot{\varphi}_1,\ \dot{\varphi}_2) \tag{3.6}$$

From Equation (3.3) we know that we can compute the robot's motion in the global reference frame from motion in its local reference frame: $\dot{\xi}_I = R(\theta)^{-1}\dot{\xi}_R$. Therefore, the strategy will be to first compute the contribution of each of the two wheels in the local reference frame, $\dot{\xi}_R$. For this example of a differential-drive chassis, this problem is particularly straightforward.

Suppose that the robot's local reference frame is aligned such that the robot moves forward along $+X_R$, as shown in Figure (3.1). First consider the contribution of each wheel's spinning speed to the translation speed at P in the direction of $+X_R$. If one wheel spins while the other wheel contributes nothing and is stationary, since P is halfway between the two wheels, it will move instantaneously with half the speed: $\dot{x}_{r1} = \frac{1}{2}r\dot{\varphi}_1$ and $\dot{x}_{r2} = \frac{1}{2}r\dot{\varphi}_2$. In a differential drive robot, these two contributions can simply be added to calculate the $\dot{x}_R$ component of $\dot{\xi}_R$. Consider, for example, a differential robot in which each wheel spins with equal speed but in opposite directions. The result is a stationary, spinning robot. As expected, $\dot{x}_R$ will be 0 in this case. The value of $\dot{y}_R$ is even simpler to calculate. Neither wheel can contribute to sideways motion in the robot's reference frame, and so $\dot{y}_R$ is always 0. Finally, we must compute the rotational component $\dot{\theta}_R$ of $\dot{\xi}_R$. Once again, the contributions of each wheel can be computed independently and just added. Consider the right wheel (we will call this wheel 1). Forward spin of this wheel results in *counter-clockwise* rotation at point P. Recall that if wheel 1 spins alone, the robot pivots around wheel 2. The rotation velocity ω_1 at P can be computed because the wheel is instantaneously moving along the arc of a circle of radius $2l$: $\omega_1 = \frac{r\dot{\varphi}_1}{2l}$

The same calculation applies to the left wheel, with the exception that forward spin results in *clockwise* rotation at point P:

$$\omega_2 = \frac{-r\dot{\varphi}_2}{2l} \tag{3.7}$$

Combining these individual formulae yields a kinematic model for the differential-drive example robot:

$$\dot{\xi}_I = R(\theta)^{-1}\begin{bmatrix} \frac{r\dot{\varphi}_1}{2} + \frac{r\dot{\varphi}_2}{2} \\ 0 \\ \frac{r\dot{\varphi}_1}{2l} + \frac{-r\dot{\varphi}_2}{2l} \end{bmatrix} \tag{3.8}$$

We can now use this kinematic model in an example. However we must first compute $R(\theta)^{-1}$. In general calculating the inverse of a matrix may be challenging. In this case, however, it is easy because it is simply a transform from $\dot{\xi}_R$ to $\dot{\xi}_I$ rather than vice-versa:

$$R(\theta)^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.9}$$

Suppose that the robot is positioned such that $\theta = \frac{\pi}{2}$, and $r=1$ and $l=1$. If the robot engages its wheels unevenly, with speeds $\dot{\varphi}_1 = 4$ and $\dot{\varphi}_2 = 2$, we can compute its velocity in the global reference frame:

$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \tag{3.10}$$

So, this robot will move instantaneously along the y axis of the global reference frame with speed 3 while rotating with speed 1. This approach to kinematic modeling can provide information about the motion of a robot given its component wheel speeds in straightforward cases. However, we wish to determine the space of possible motions for each robot chassis design. To do this, we must go further, describing formally the constraints on robot motion imposed by each wheel. The next section begins this process by describing constraints for various wheel types, then the rest of this chapter provides tools for analyzing the characteristics and workspace of a robot given these constraints.

3.2.3 Wheel kinematic constraints

The first step to a kinematic model of the robot is to express constraints on the motions of individual wheels. Just as shown in the previous section, the motions of individual wheels can later be combined to compute the motion of the robot as a whole. As discussed in Chapter 2, there are four basic wheel types with widely varying kinematic properties. Therefore, we begin by presenting sets of constraints specific to each wheel type.

However several important assumptions will simplify this presentation. We assume that the

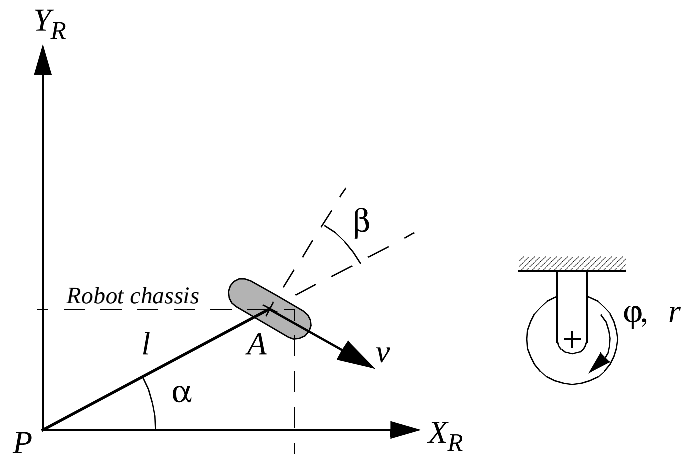

Fig 3.4 Standard fixed wheel and its parameters

plane of wheel always remains vertical and that there is in all cases one single point of contact between the wheel and the ground plane. Furthermore we assume that there is no sliding at this single point of contact. That is, the wheel undergoes motion only under conditions of pure rolling and rotation about the vertical axis through the contact point.

Under these assumptions, we present two constraints for every wheel type. The first constraint enforces the concept of rolling contact- that the wheel must roll when motion takes place in the appropriate direction. The second constraint enforces the concept of no lateral slippage- that the wheel must not slide orthogonal to the wheel plane.

3.2.3.1 Fixed standard wheel

The fixed standard wheel has no vertical axis of rotation for steering. Its angle to the chassis is thus fixed, and it is limited to motion back and forth along the wheel plane and rotation around its contact point with the ground plane. Figure 3.4 depicts a fixed standard wheel *A* and indicates its position pose relative to the robot's local reference frame $\{X_R, Y_R\}$. The position of *A* is expressed in polar coordinates by distance l and angle α. The angle of the wheel plane relative to the chassis is denoted by β, which is fixed since the fixed standard wheel is not steerable. The wheel, which has radius r, can spin over time, and so its rotational position around its horizontal axle is a function of time t: $\varphi(t)$.

The rolling constraint for this wheel enforces that all motion along the direction of the wheel plane must be accompanied by the appropriate amount of wheel spin so that there is pure rolling at the contact point:

$$\left[\sin(\alpha+\beta) \;\; -\cos(\alpha+\beta) \;\; (-l)\cos\beta\right] R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0 \tag{3.11}$$

The first term of the sum denotes the total motion along the wheel plane. The three elements of the vector on the left represents mappings from each of $\dot{x}$, $\dot{y}$, $\dot{\theta}$ to their contributions for motion along the wheel plane. Note that the $R(\theta)\dot{\xi}_I$ term is used to transform the motion parameters $\dot{\xi}_I$ that are in the global reference frame $\{X_I, Y_I\}$ into motion parameters in the local reference frame $\{X_R, Y_R\}$ as shown in example equation (3.5). This is necessary because all other parameters in the equation, α, β, l, are in terms of the robot's local reference frame. This motion along the wheel plane must be equal, according to this constraint, to the motion accomplished by spinning the wheel, $r\dot{\varphi}$.

The sliding constraint for this wheel enforces that the component of the wheel's motion orthogonal to the wheel plane must be zero:

$$\begin{bmatrix}\cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta\end{bmatrix} R(\theta)\dot{\xi}_I = 0 \tag{3.12}$$

For example, suppose that wheel A is in a position such that $\{(\alpha = 0), (\beta = 0)\}$. This would place the contact point of the wheel on X_I with the plane of the wheel oriented parallel to Y_I. If $\theta = 0$ then the sliding constraint (3.12) reduces to:

$$\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{\theta}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{\theta}\end{bmatrix} = 0 \tag{3.13}$$

This constrains the component of motion along X_I to be zero and since X_I and X_R are parallel in this example, the wheel is constrained from sliding sideways, as expected.

3.2.3.2 Steered standard wheel

The steered standard wheel differs from the fixed standard wheel only in that there is an additional degree of freedom: the wheel may rotate around a vertical axis passing through the center of the wheel and the ground contact point. The equations of position for the steered standard wheel (figure 3.5) are identical to that of the fixed standard wheel shown in figure 3.4 with one exception. The orientation of the wheel to the robot chassis is no longer a single fixed value, β, but instead varies as a function of time: $\beta(t)$. The rolling and sliding constraints are:

$$\begin{bmatrix}\sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta\end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0 \tag{3.14}$$

$$\begin{bmatrix}\cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta\end{bmatrix} R(\theta)\dot{\xi}_I = 0 \tag{3.15}$$

These constraints are identical to those of the fixed standard wheel because, unlike $\dot{\varphi}$, $\dot{\beta}$

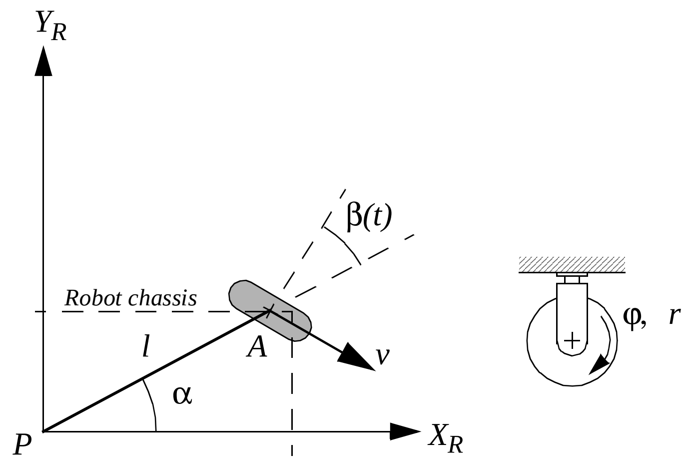

Fig 3.5 Steered standard wheel and its parameters

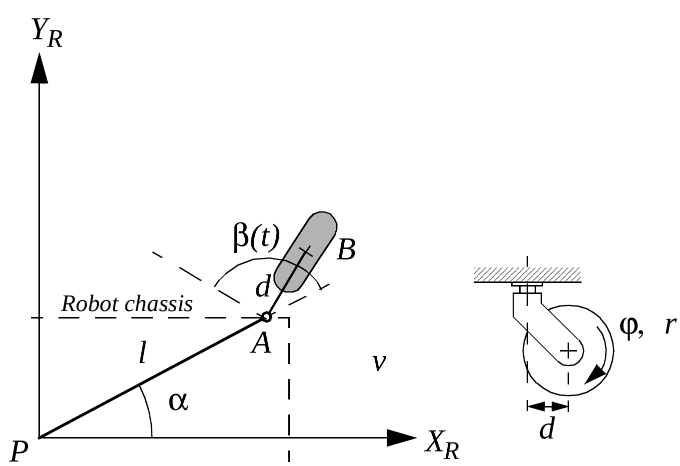

Fig 3.6 Castor wheel and its parameters

does not have a direct impact on the instantaneous motion constraints of a robot. It is only by integrating over time that changes in steering angle can affect the mobility of a vehicle. This may seem subtle, but is a very important distinction between change in steering position, $\dot{\beta}$, and change in wheel spin, $\dot{\varphi}$.

3.2.3.3 Castor wheel

Castor wheels are able to steer around a vertical axis. However, unlike the steered standard wheel, the vertical axis of rotation in a castor wheel does not pass through the ground contact point. Figure 3.6 depicts a castor wheel, demonstrating that formal specification of the castor wheel's position requires an additional parameter.

The wheel contact point is now at position *B*, which is connected by a rigid rod *AB* of fixed length *d* to point *A*. *A* fixes the location of the vertical axis about which *B* steers, and this point *A* has a position specified in the robot's reference frame as in Fig. 3.6. We assume that

Fig 3.7 Office chair with five castor wheels

the plane of the wheel is aligned with *AB* at all times. Similar to the steered standard wheel, the castor wheel has two parameters that vary as a function of time. $\varphi(t)$ represents the wheel spin over time as before. $\beta(t)$ denotes the steering angle and orientation of *AB* over time.

For the castor wheel, the rolling constraint is identical to before because the offset axis plays no role during motion that is aligned with the wheel plane:

$$\begin{bmatrix}\sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta\end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0 \tag{3.16}$$

The castor geometry does, however, have significant impact on the sliding constraint. The critical issue is that the lateral force on the wheel occurs at point *A* because this is the attachment point of the wheel to the chassis. Because of the offset ground contact point relative to *A*, the constraint that there be zero lateral movement would be wrong. Instead, the constraint is much like a rolling constraint, in that appropriate rotation of the vertical axis must take place:

$$\begin{bmatrix}\cos(\alpha+\beta) & \sin(\alpha+\beta) & (d+l)\sin\beta\end{bmatrix} R(\theta)\dot{\xi}_I + d\dot{\beta} = 0 \tag{3.17}$$

In equation (3.17), any motion orthogonal to the wheel plane must be balanced by an equivalent and opposite amount of castor steering motion. This result is critical to the success of castor wheels because by setting the value of $\dot{\beta}$ any arbitrary lateral motion can be acceptable. In a steered standard wheel, the steering action does not by itself cause a movement of the robot chassis. But in a castor wheel the steering action itself moves the robot chassis because of the offset between the ground contact point and the vertical axis of rotation.

More concisely, it can be surmised from Equations (3.16) and (3.17) that, given *any* robot chassis motion $\dot{\xi}_I$, there exists some value for spin speed $\dot{\varphi}$ and steering speed $\dot{\beta}$ such that the constraints are met. Therefore, a robot with only castor wheels can move with any velocity in the space of possible robot motions. We term such systems *omnidirectional*.

A real-world example of such a system is the five-castor wheel office chair shown in Figure 3.7. Assuming that all joints are able to move freely, you may select any motion vector on the plane for the chair and push it by hand. Its castor wheels will spin and steer as needed to

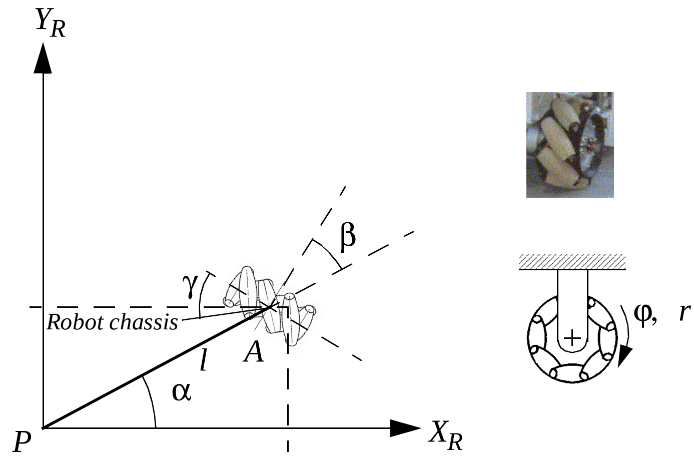

Fig 3.8 *Swedish wheel and its parameters*

achieve that motion without contact point sliding. By the same token, if each of the chair's castor wheels housed two motors, one for spinning and one for steering, then a control system would be able to move the chair along any trajectory in the plane. Thus, although the kinematics of castor wheels is somewhat complex, such wheels do not impose any real constraints on the kinematics of a robot chassis.

3.2.3.4 Swedish wheel

Swedish wheels have no vertical axis of rotation, yet are able to move *omnidirectionally* like the castor wheel. This is possible by adding a degree of freedom to the standard fixed wheel. Swedish wheels consist of a standard fixed wheel with rollers attached to the wheel perimeter with axes that are antiparallel to the main axis of the fixed wheel component. The exact angle γ between the roller axes and the main axis can vary, as shown in figure 3.8.

For example, given a swedish 45 degree wheel, the motion vectors of the principal axis and the roller axes can be drawn as in Fig. 3.8. Since each axis can spin clockwise or counterclockwise, one can combine any vector along one axis with any vector along the other axis. These two axes are not necessarily independent (except in the case of the swedish 90); however, it is visually clear that any desired direction of motion is achievable by choosing the appropriate two vectors.

The pose of a swedish wheel is expressed exactly as in a fixed standard wheel, with the addition of a term, γ, representing the angle between the main wheel plane and the axis of rotation of the small circumferential rollers. This is depicted in Fig. 3.8 within the robot's reference frame.

Formulating the constraint for a swedish wheel requires some subtlety. The instantaneous constraint is due to the specific orientation of the small rollers. The axis around which these rollers spin is a zero component of velocity at the contact point. That is, moving in that direction without spinning the main axis is not possible without sliding. The motion constraint that is derived looks identical to the rolling constraint for the fixed standard wheel in Equation (3.11) except that the formula is modified by adding γ such that the effective direction along which the rolling constraint holds is along this zero component rather than along the wheel plane:

$$\begin{bmatrix}\sin(\alpha+\beta+\gamma) & -\cos(\alpha+\beta+\gamma) & (-l)\cos(\beta+\gamma)\end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi}\cos\gamma = 0 \tag{3.18}$$

Orthogonal to this direction the motion is not constrained because of the free rotation $\dot{\varphi}_{sw}$ of the small rollers.

$$\begin{bmatrix}\cos(\alpha+\beta+\gamma) & \sin(\alpha+\beta+\gamma) & l\sin(\beta+\gamma)\end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi}\sin\gamma - r_{sw}\dot{\varphi}_{sw} = 0 \tag{3.19}$$

The behavior of this constraint and thereby the swedish wheel changes dramatically as the value γ varies. Consider $\gamma = 0$. This represents the swedish 90 degree wheel. In this case, the zero component of velocity is in line with the wheel plane and so Equation (3.18) reduces exactly to Equation (3.11), the fixed standard wheel rolling constraint. But because of the rollers, there is no sliding constraint orthogonal to the wheel plane (eq. (3.19). By varying the value of $\dot{\varphi}$, any desired motion vector can be made to satisfy Equation (3.18) and therefore the wheel is omnidirectional. In fact, this special case of the swedish design results in fully decoupled motion, in that the rollers and the main wheel provide orthogonal directions of motion.

At the other extreme, consider $\gamma = \pi/2$. In this case, the rollers have axes of rotation that are parallel to the main wheel axis of rotation. Interestingly, if this value is substituted for γ in Equation (3.18) the result is the fixed standard wheel sliding constraint, Equation (3.12). In other words, the rollers provide no benefit in terms of lateral freedom of motion since they are simply aligned with the main wheel. However, in this case the main wheel never needs to spin and therefore the rolling constraint disappears. This is a degenerate form of the swedish wheel and therefore we assume in the remainder of this chapter that $\gamma \neq \pi/2$.

3.2.3.5 Spheric wheel

The final wheel type, a ball or spheric wheel, places no direct constraints on motion (figure 3.9). Such a mechanism has no principal axis of rotation, and therefore no appropriate rolling or sliding constraints exist. As with castor wheels and swedish wheels, the spheric wheel is clearly omnidirectional and places no constraints on the robot chassis kinematics. Therefore Equation (3.20) simply describes the roll rate of the ball in the direction of motion v_A of point A of the robot.

$$\begin{bmatrix}\sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta\end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0 \tag{3.20}$$

By definition the wheel rotation orthogonal to this direction is zero.

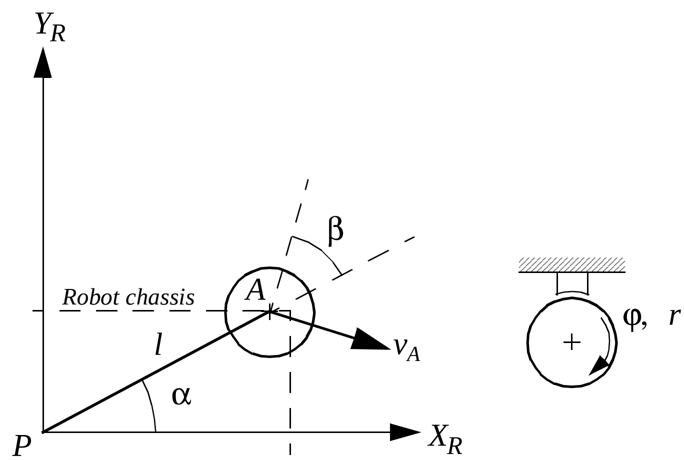

Fig 3.9 Spheric wheel and its parameters

$$\left[\cos(\alpha+\beta)\ \sin(\alpha+\beta)\ l\sin\beta\right]R(\theta)\dot{\xi}_I = 0 \tag{3.21}$$

As can be seen, the equations for the spheric wheel are exactly the same as for the fixed standard wheel. However, the interpretation of equation (3.21) is different. The omnidirectional spheric wheel can have any arbitrary direction of movement, where the motion direction given by β is a free variable deduced from equation (3.21). Consider the case that the robot is in pure translation in the direction of Y_R. Then equation (3.21) reduces to $\sin(\alpha+\beta) = 0$, thus $\beta = -\alpha$ which makes sense for this special case.

3.2.4 Robot kinematic constraints

Given a mobile robot with M wheels we can now compute the kinematic constraints of the robot chassis. The key idea is that each wheel imposes zero or more constraints on robot motion, and so the process is simply one of appropriately combining all of the kinematic constraints arising from all of the wheels based on the placement of those wheels on the robot chassis.

We have categorized all wheels into five categories: fixed and steerable standard wheels, castor wheels, swedish wheels and spheric wheels. But note from the wheel kinematic constraints in equations (3.16) - (3.18) that the castor wheel, swedish wheel and spheric wheel impose *no* kinematic constraints on the robot chassis, since $\dot{\xi}_I$ can range freely in all of these cases due to the internal wheel degrees of freedom.

Therefore only fixed standard wheels and steerable standard wheels have impact on robot chassis kinematics and therefore require consideration when computing the robot's kinematic constraints. Suppose that the robot has a total of N standard wheels, comprising N_f fixed standard wheels and N_s steerable standard wheels. We use $\beta_s(t)$ to denote the variable steering angles of the N_s steerable standard wheels. In contrast, β_f refers to the orientation of the

N_f fixed standard wheels as depicted in Fig. 3.4. In the case of wheel spin, both the fixed and steerable wheels have rotational positions around the horizontal axle that vary as a function of time. We denote the fixed and steerable cases separately as $\varphi_f(t)$ and $\varphi_s(t)$, and use $\varphi(t)$ as an aggregate matrix that combines both values:

$$\varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix} \tag{3.22}$$

The rolling constraints of all wheels can now be collected in a single expression:

$$J_1(\beta_s)R(\theta)\dot{\xi}_I - J_2\dot{\varphi} = 0 \tag{3.23}$$

This expression bears a strong resemblance to the rolling constraint of a single wheel, but substitutes matrices in lieu of single values, thus taking into account all wheels. J_2 is a constant diagonal NxN matrix whose entries are radii r of all standard wheels. $J_1(\beta_s)$ denotes a matrix with projections for all wheels to their motions along their individual wheel planes:

$$J_1(\beta_s) = \begin{bmatrix} J_{1f} \\ J_{1s}(\beta_s) \end{bmatrix} \tag{3.24}$$

Note that $J_1(\beta_s)$ is only a function of β_s and not β_f. This is because the orientations of steerable standard wheels vary as a function of time, whereas the orientations of fixed standard wheels are constant. J_{1f} is therefore a constant matrix of projections for all fixed standard wheels. It has size ($N_f \times 3$), with each row consisting of the three terms in the three-matrix from Equation (3.11) for each fixed standard wheel. $J_{1s}(\beta_s)$ is a matrix of size ($N_s \times 3$), with each row consisting of the three terms in the three-matrix from Equation (3.14) for each steerable standard wheel.

In summary, Equation (3.23) represents the constraint that all standard wheels must spin around their horizontal axis an appropriate amount based on their motions along the wheel plane so that rolling occurs at the ground contact point.

We use the same technique to collect the sliding constraints of all standard wheels into a single expression with the same structure as Equations (3.12) and (3.15):

$$C_1(\beta_s)R(\theta)\dot{\xi}_I = 0 \tag{3.25}$$

$$C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix} \tag{3.26}$$

C_{1f} and C_{1s} are $(N_f \times 3)$ and $(N_s \times 3)$ matrices whose rows are the three terms in the three-matrix of Equation (3.12) and Equation (3.15) for all fixed and steerable standard wheels. Thus Equation (3.25) is a constraint over all standard wheels that their components of motion orthogonal to their wheel planes must be zero. This sliding constraint over all standard wheels has the most significant impact on defining the overall maneuverability of the robot chassis, as explained in the next section.

3.2.5 Examples: Robot kinematic models and constraints

In Section (3.2.2), we presented a forward kinematic solution for $\dot{\xi}_I$ in the case of a simple differential drive robot by combining each wheels' contribution to robot motion. We can now use the tools presented above to construct the same kinematic expression by direct application of the rolling constraints every wheel type. We proceed with this technique applied again to the differential drive robot, enabling verification of the method as compared to the results of Section (3.2.2). Then, we proceed to the case of three-wheeled omnidirectional robot.

3.2.5.1 A differential drive robot example

First, refer to Equation (3.23). This equation relates robot motion to the rolling constraints, $J_1(\beta_s)$ and the wheel spin speed of the robot's wheels, $\dot{\varphi}$. Isolating $\dot{\xi}_I$ yields the following expression:

$$\dot{\xi}_I = R(\theta)^{-1} J_1(\beta_s)^{-1} J_2 \dot{\varphi} \tag{3.27}$$

Once again, consider the differential drive robot in Figure 3.3. We will construct $J_1(\beta_s)$ directly from the rolling constraints of each wheel. The castor is unpowered and is free to move in any direction, so we ignore this third point of contact altogether. The two remaining drive wheels are not steerable, and therefore $J_1(\beta_s)$ simplifies to J_{1f}. To employ the Fixed standard wheel's rolling constraint formula, Equation (3.11), we must first identify each wheel's values for α and β. Suppose that the robot's local reference frame is aligned such that the robot moves forward along $+X_R$, as shown in Figure (3.1). In this case, for the right wheel $\alpha = -\frac{\pi}{2}$, $\beta = \pi$ and for the left wheel $\alpha = \frac{\pi}{2}$, $\beta = 0$. Note the value of β for the right wheel is necessary to ensure that positive spin causes motion in the $+X_R$ direction (figure 3.4). Now we can compute the J_{1f} matrix using the matrix terms from Equation (3.11):

$$J_{1f} = \begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \end{bmatrix} \tag{3.28}$$

Substitution into Equation (3.27) yields the kinematic equation specific to our differential

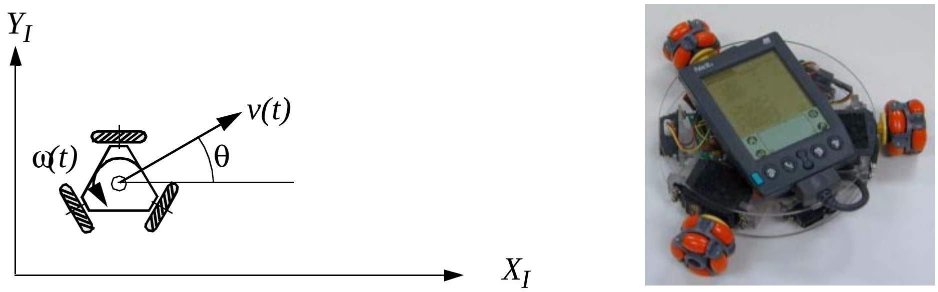

Fig 3.10 A three wheel omni-drive robot developed by Carnegie Mellon University (www.cs.cmu.edu/~pprk).

drive robot:

$$\dot{\xi}_I = R(\theta)^{-1}\begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \end{bmatrix}^{-1} J_2\dot{\varphi} \tag{3.29}$$

In order to use this formula to calculate $\dot{\xi}_I$ for a specific differential drive robot's wheel spin speeds requires matrix inversion of both $R(\theta)$ and J_{1f}. $R(\theta)^{-1}$ is easy to compute, as demonstrated by Equation (3.9). J_{1f} does not technically have an inverse because it is non-square in this case. However, it does have a *pseudoinverse* such that $_{1f}\cdot\ J_{1f}^{-1}\cdot\ J_1 = J_1$. General methods for calculating the inverse or pseudoinverse of matrices are well known but will not be discussed further here. For details see Golub and Kahan [92]. But note the similarity of Equation (3.29) to Equation (3.8). Recall that J_2 is a diagonal matrix containing r for each wheel. Rewriting Equation (3.8) yields:

$$\dot{\xi}_I = R(\theta)^{-1}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \\ \frac{1}{2l} & -\frac{1}{2l} \end{bmatrix} J_2\dot{\varphi} \tag{3.30}$$

As is evident by comparing Equation (3.29) and Equation (3.30), we already have J_{1f}^{-1}. This demonstrates that, for the simple differential drive case, the combination of wheel rolling constraints describe the kinematic behavior, based on our manual calculation in Section 3.2.2.

3.2.5.2 An omnidirectional robot example

Consider the omni-wheel robot show in Figure 3.10. This robot has three swedish 90

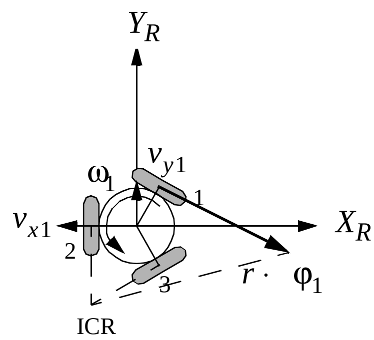

Fig 3.11 The local reference frame plus detailed parameters for wheel 1

wheels, arranged radially symmetrically, with the rollers perpendicular to each main wheel.

First we must impose a specific local reference frame upon the robot. We do so by choosing point P at the center of the robot, then aligning the robot with the local reference frame such that X_R is co-linear with the axis of wheel 2. Figure 3.11 shows the robot and its local reference frame arranged in this manner.

We assume that the distance between each wheel and P is l, and that all three wheels have the same radius, r. Once again, the value of $\dot{\xi}_I$ can be computed as a combination of the rolling constraints of the robot's three omnidirectional wheels, as in Equation (3.27). As with the differential drive robot, since this robot has no steerable wheels, $J_1(\beta_s)$ simplifies to J_{1f}:

$$\dot{\xi}_I = R(\theta)^{-1} J_{1f}^{-1} J_2 \dot{\varphi} \tag{3.31}$$

We calculate J_{1f} using the matrix elements of the rolling constraints for the swedish wheel, given by Equation (3.18). But to use these values, we must establish the values α, β, γ for each wheel. Referring to Figure (3.8), we can see that $\gamma = 0$ for the Swedish 90 wheel. Note that this immediately simplifies Equation (3.18) to Equation (3.11), the rolling constraints of a fixed standard wheel. Given our particular placement of the local reference frame, the value of α for each wheel is easily computed: $\left(\alpha_1 = \frac{\pi}{3}\right)$, $(\alpha_2 = \pi)$, $\left(\alpha_3 = -\frac{\pi}{3}\right)$. Furthermore, $\beta = 0$ for all wheels because the wheels are tangent to the robot's circular body. Constructing and simplifying J_{1f} using Equation (3.11) yields:

$$J_{1f} = \begin{bmatrix} \sin\frac{\pi}{3} & -\cos\frac{\pi}{3} & -l \\ 0 & -\cos\pi & -l \\ \sin-\frac{\pi}{3} & -\cos-\frac{\pi}{3} & -l \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & -l \\ 0 & 1 & -l \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & -l \end{bmatrix} \tag{3.32}$$

Once again, computing the value of $\dot{\xi}_I$ requires calculating the inverse, J_{1f}^{-1}, as needed in Equation (3.31). One approach would be to apply rote methods for calculating the inverse of a 3x3 square matrix. A second approach would be to compute the contribution of each swedish wheel to chassis motion as shown in Section (3.2.2). We leave this process as an exercise for the enthusiast. Once the inverse is obtained, $\dot{\xi}_I$ can be isolated:

$$\dot{\xi}_I = R(\theta)^{-1} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3l} & -\frac{1}{3l} & -\frac{1}{3l} \end{bmatrix} J_2\dot{\varphi} \tag{3.33}$$

Consider a specific omnidrive chassis with l=1 and r=1 for all wheels. The robot's local reference frame and global reference frame are aligned, so that $\theta = 0$. If wheels 1, 2 and 3 spin at speeds $(\varphi_1 = 4)$, $(\varphi_2 = 1)$, $(\varphi_3 = 2)$, what is the resulting motion of the whole robot? Using the equation above, the answer can be calculated readily:

$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{3}} \\ -\frac{4}{3} \\ -\frac{7}{3} \end{bmatrix} \tag{3.34}$$

So, this robot will move instantaneously along the x axis with positive speed and along the y axis with negative speed while rotating clockwise. We can see from the above examples that robot motion can be predicted by combining the rolling constraints of individual wheels.

The sliding constraints comprising $C_1(\beta_s)$ can be used to go even further, enabling us to evaluate the maneuverability and workspace of the robot rather than just its predicted motion. Next, we examine methods for using the sliding constraints, sometimes in conjunction with rolling constraints, to generate powerful analyses of the maneuverability of a robot chassis.

3.3 Mobile Robot Maneuverability

The kinematic mobility of a robot chassis is its ability to directly move in the environment. The basic constraint limiting mobility is the rule that every wheel must satisfy its sliding constraint. Therefore, we can formally derive robot mobility by starting from Equation (3.25).

In addition to instantaneous kinematic motion, a mobile robot is able to further manipulate its position, over time, by steering steerable wheels. As we will see in Section (3.3.3), the overall maneuverability of a robot is thus a combination of the mobility available based on the kinematic sliding constraints of the standard wheels, plus the additional freedom contributed by steering and spinning the steerable standard wheels.

3.3.1 Degree of mobility

Equation (3.25) imposes the constraint that every wheel must avoid any lateral slip. Of course, this hold separately for each and every wheel, and so it is possible to specify this constraint separately for fixed and for steerable standard wheels:

$$C_{1f}R(\theta)\dot{\xi}_I = 0 \tag{3.35}$$

$$C_{1s}(\beta_s)R(\theta)\dot{\xi}_I = 0 \tag{3.36}$$

For both of these constraints to be satisfied, the motion vector $R(\theta)\dot{\xi}_I$ must belong to the *null space* of the projection matrix $C_1(\beta_s)$, which is simply a combination of C_{1f} and C_{1s}. Mathematically, the null space of $C_1(\beta_s)$ is the space N such that for any vector n in N $C_1(\beta_s)n = 0$. If the kinematic constraints are to be honored, then the motion of the robot must always be within this space N. The kinematic constraints (3.35) and (3.36) can also be demonstrated geometrically using the concept of a robot's Instantaneous Center of Rotation (ICR).

Consider a single standard wheel. It is forced by the sliding constraint to have zero lateral motion. This can be shown geometrically by drawing a *zero motion line* through its horizontal axis, perpendicular to the wheel plane (Figure 3.12). At any given instant, wheel motion along the zero motion line must be zero. In other words, the wheel must be moving instantaneously along some circle of radius R such that the center of that circle is located on the *zero motion line*. This center point, called the *Instantaneous Center of Rotation*, may lie anywhere along the zero motion line. When R is at infinity, the wheel moves in a straight line.

A robot such as the Ackerman vehicle in Figure 3.12a can have several wheels, but must always have a single *ICR*. Because all of its zero motion lines meet at a single point, there is a single solution for robot motion, placing the *ICR* at this meet point.

This *ICR* geometric construction demonstrates how robot mobility is a function of the number of constraints on the robot's motion, not the number of wheels. In Figure 3.12b, the bi-

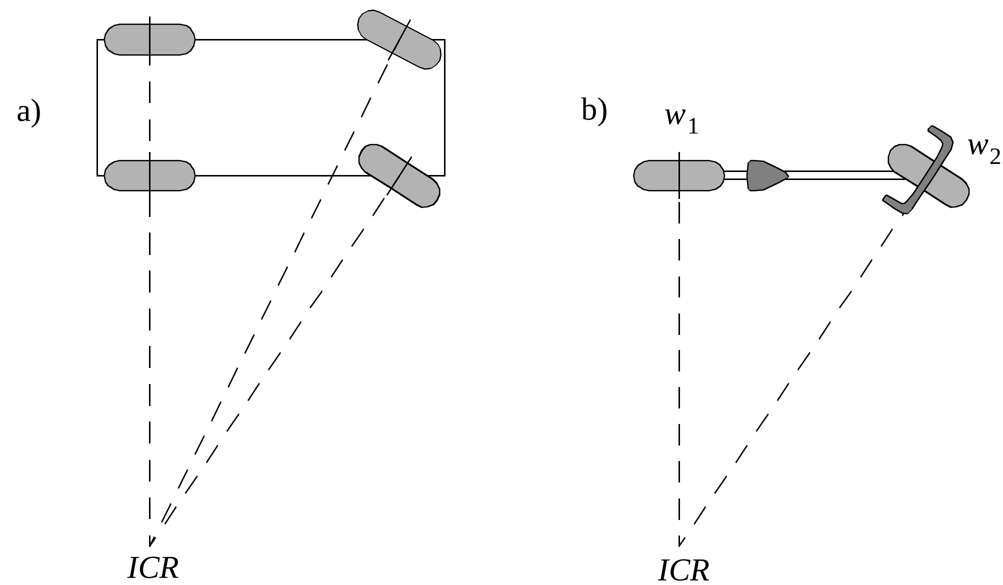

Fig 3.12 *a) 4-wheel with car-like Ackermann steering*
b) bicycle

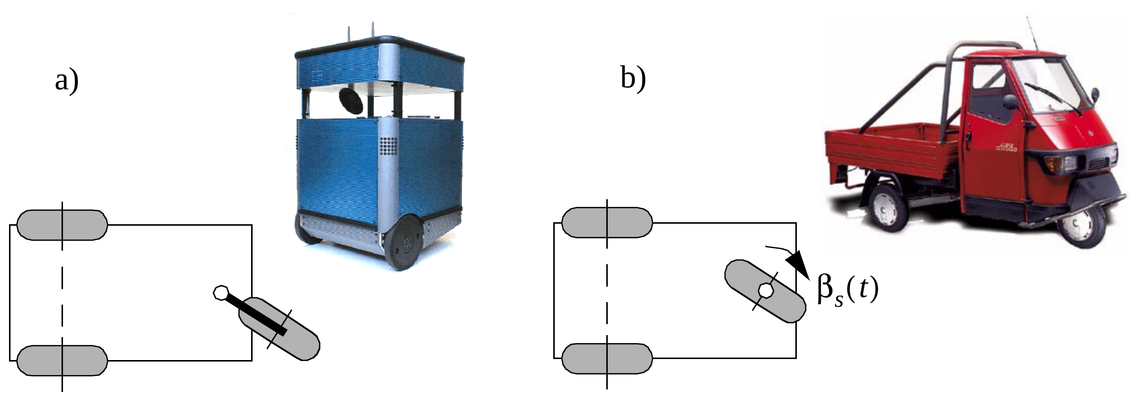

Fig 3.13 *a) Differential Drive Robot with two individually motorized wheels and a castor wheel, e.g. the Pygmalion robot at EPFL*
b)Tricycle with two fixed standard wheels and one steered standard wheel, e.g. Piaggio mini transporter

cycle shown has two wheels w_1 and w_2. Each wheel contributes a constraint, or a zero motion line. Taken together the two constraints result in a single point as the only remaining solution for the *ICR*. This is because the two constraints are independent, and thus each further constrains overall robot motion.

But in the case of the differential drive robot in Figure 3.13a, the two wheels are aligned along the same horizontal axis. Therefore, the *ICR* is constrained to lie along a line, not at a specific point. In fact, the second wheel imposes no additional kinematic constraints on robot motion since its zero motion line is identical to the first wheel's. Thus although the bicycle and differential drive chassis have the same number of non-omnidirectional wheels, the former has two independent kinematic constraints while the latter has only one.

The Ackerman vehicle of Figure 3.12a demonstrates another way in which a wheel may be unable to contribute an independent constraint to the robot kinematics. This vehicle has two steerable standard wheels. Given the instantaneous position of just one of these steerable wheels and the position of the fixed rear wheels, there is only a single solution for the *ICR*. The position of the second steerable wheel is absolutely constrained by the *ICR*. Therefore, it offers no independent constraints to robot motion.

Robot chassis kinematics is therefore a function of the set of *independent* constraints arising from all standard wheels. The mathematical interpretation of independence is related to the *rank* of a matrix. Recall that the rank of a matrix is the smallest number of independent rows or columns. Equation (3.25) represents all sliding constraints imposed by the wheels of the mobile robot. Therefore $rank\left[C_1(\beta_s)\right]$ is the number of independent constraints.

The greater the number of independent constraints, and therefore the greater the rank of $C_1(\beta_s)$, the more constrained is the mobility of the robot. For example, consider a robot with a single fixed standard wheel. Remember that we consider only standard wheels. This robot may be a unicycle or it may have several swedish wheels; however it has exactly one fixed standard wheel. The wheel is at position specified by parameters α, β, l relative to the robot's local reference frame. $C_1(\beta_s)$ is comprised of C_{1f} and C_{1s}. However, since there are no steerable standard wheels C_{1s} is empty and therefore $C_1(\beta_s)$ contains only C_{1f}. Because there is one fixed standard wheel, this matrix has rank one and therefore this robot has a single independent constrain on mobility:

$$C_1(\beta_s) = C_{1f} = \left[\cos(\alpha+\beta)\ \sin(\alpha+\beta)\ l\sin\beta\right] \tag{3.37}$$

Now let us add an additional fixed standard wheel to create a differential drive robot by constraining the second wheel to be aligned with the same horizontal axis as the original wheel. Without loss of generality, we can place point *P* at the midpoint between the centers of the two wheels. Given α_1, β_1, l_1 for wheel w_1 and α_2, β_2, l_2 for wheel w_2, it holds geometrically that $\{(l_1 = l_2), (\beta_1 = \beta_2 = 0), (\alpha_1 + \pi = \alpha_2)\}$. Therefore, in this case, the matrix $C_1(\beta_s)$ has two constraints but a rank of one:

$$C_1(\beta_s) = C_{1f} = \begin{bmatrix} \cos(\alpha_1) & \sin(\alpha_1) & 0 \\ \cos(\alpha_1+\pi) & \sin(\alpha_1+\pi) & 0 \end{bmatrix} \tag{3.38}$$

Alternatively, consider the case when w_2 is placed in the wheel plane of w_1 but with the same orientation, as in a bicycle with the steering locked in the forward position. We again place point *P* between the two wheel centers, and orient the wheels such that they lie on axis x_1. This geometry implies that $\{(l_1 = l_2), (\beta_1 = \beta_2 = \pi/2), (\alpha_1 = 0), (\alpha_2 = \pi)\}$ and, therefore, the matrix $C_1(\beta_s)$ retains two independent constraints and has a rank of two:

$$C_1(\beta_s) = C_{1f} = \begin{bmatrix} \cos(\pi/2) & \sin(\pi/2) & l_1\sin(\pi/2) \\ \cos(3\pi/2) & \sin(3\pi/2) & l_1\sin(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & l_1 \\ 0 & -1 & l_1 \end{bmatrix} \tag{3.39}$$

In general, if $rank\left[C_{1f}\right] > 1$ then the vehicle can, at best, only travel along a circle or along a straight line. This configuration means that the robot has two or more independent constraints due to fixed standard wheels that do not share the same horizontal axis of rotation. Because such configurations have only a degenerate form of mobility in the plane, we do not consider them in the remainder of this chapter. Note, however, that some degenerate configurations such as the four wheeled slip-skid steering system are useful in certain environments, such as on loose soil and sand, even though they fail to satisfy sliding constraints. Not surprisingly, the price that must be paid for such violations of the sliding constraints is that dead-reckoning based on odometry becomes less accurate and power efficiency is reduced dramatically.

In general, a robot will have zero or more fixed standard wheels and zero or more steerable standard wheels. We can therefore identify the possible range of rank values for any robot: $0 \le rank\left[C_1(\beta_s)\right] \le 3$. Consider the case $rank\left[C_1(\beta_s)\right] = 0$. This is only possible if there are zero independent kinematic constraints in $C_1(\beta_s)$. In this case there are neither fixed nor steerable standard wheels attached to the robot frame: $N_f = N_s = 0$.

Consider the other extreme, $rank\left[C_1(\beta_s)\right] = 3$. This is the maximum possible rank since the kinematic constraints are specified along three degrees of freedom (i.e. the constraint matrix is three columns wide). Therefore, there cannot be more than three independent constraints. In fact, when $rank\left[C_1(\beta_s)\right] = 3$ then the robot is completely constrained in all directions and is, therefore, degenerate since motion in the plane is totally impossible.

Now we are ready to formally define a robot's *degree of mobility* δ_m:

$$\delta_m = dimN\left[C_1(\beta_s)\right] = 3 - rank\left[C_1(\beta_s)\right] \tag{3.40}$$

The dimensionality of the Null Space ($dimN$) of matrix $C_1(\beta_s)$ is a measure of the number of degrees of freedom of the robot chassis that can be immediately manipulated through changes in wheel velocity. It is logical therefore that δ_m must range between 0 and 3.

Consider an ordinary differential drive chassis. On such a robot there are two fixed standard wheels sharing a common horizontal axis. As discussed above, the second wheel adds no independent kinematic constraints to the system. Therefore, $rank\left[C_1(\beta_s)\right] = 1$ and $\delta_m = 2$. This fits with intuition: a differential drive robot can control both the rate of its change in orientation and its forward/reverse speed, *simply by manipulating wheel velocities*. In other words, its *ICR* is constrained to lie on the infinite line extending from its wheels' horizontal axles.

In contrast, consider a bicycle chassis. This configuration consists of one fixed standard wheel and one steerable standard wheel. In this case, each wheel contributes an independent sliding constraint to $C_1(\beta_s)$. Therefore, $\delta_m = 1$. Note that the bicycle has the same total number of non-omnidirectional wheels as the differential drive chassis, and indeed one of its wheels is steerable. Yet it has one less degree of mobility. Upon reflection this is appropriate. A bicycle only has control over its forward/reverse speed by direct manipulation of wheel velocities. Only by steering can the bicycle change its *ICR*.

As expected, based on Equation (3.40) any robot consisting only of omnidirectional wheels such as swedish or spheric wheels will have the maximum mobility, $\delta_m = 3$. Such a robot can directly manipulate all three degrees of freedom.

3.3.2 Degree of steerability

The degree of mobility defined above quantifies the degrees of controllable freedom based on changes to wheel velocity. Steering can also have an eventual impact on a robot chassis pose ξ, although the impact is indirect because after changing the angle of a steerable standard wheel, the robot must move for the change in steering angle to have impact on pose.

As with mobility, we care about the number of independently controllable steering parameters when defining the *degree of steerability* δ_s:

$$\delta_s = rank\left[C_{1s}(\beta_s)\right] \tag{3.41}$$

Recall that in the case of mobility, an increase in the rank of $C_1(\beta_s)$ implied more kinematic constraints and thus a less mobile system. In the case of steerability, an increase in the rank of $_{1s}(\beta_s)$ implies more degrees of steering freedom and thus greater eventual maneuverability. Since $C_1(\beta_s)$ includes $C_{1s}(\beta_s)$, this means that a steered standard wheel can both decrease mobility and increase steerability: its particular orientation at any instant imposes a kinematic constraint, but its ability to change that orientation can lead to additional trajectories.

The range of δ_s can be specified: $0 \le \delta_s \le 2$. The case $\delta_s = 0$ implies that the robot has no steerable standard wheels, $N_s = 0$. The case $\delta_s = 1$ is most common when a robot configuration includes one or more steerable standard wheels.

For example, consider an ordinary automobile. In this case $N_f = 2$ and $N_s = 2$. But the fixed wheels share a common axle and so $rank\left[C_{1f}\right] = 1$. The fixed wheels and any one of the steerable wheels constrain the *ICR* to be a point along the line extending from the rear axle. Therefore, the second steerable wheel cannot impose any independent kinematic con-

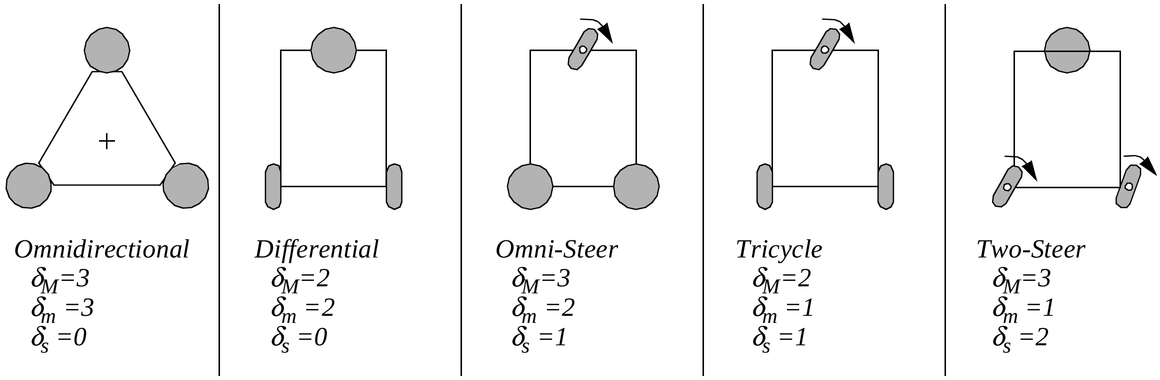

Fig 3.14 *The five basic types of three-wheel configurations. The spheric wheels can be replaced by castor or swedish wheels without influencing the maneuverability. More configurations with various numbers of wheels are found chapter 2.*

straint and so $rank\left[C_{1s}(\beta_s)\right] = 1$. In this case $\delta_m = 1$ and $\delta_s = 1$.

The case $\delta_s = 2$ is only possible in robots with no fixed standard wheels: $N_f = 0$. Under these circumstances, it is possible to create a chassis with two separate steerable standard wheels, like a pseudo-bicycle (or the two-steer) in which both wheels are steerable. Then, orienting one wheel constrains the *ICR* to a line while the second wheel can constrain the *ICR* to any point along that line. Interestingly, this means that the $\delta_s = 2$ implies that the robot can place its *ICR* anywhere on the ground plane.

3.3.3 Robot maneuverability

The overall degrees of freedom that a robot can manipulate, called the *degree of maneuverability* δ_M, can be readily defined in terms of mobility and steerability:

$$\delta_M = \delta_m + \delta_s \tag{3.42}$$

Therefore maneuverability includes both the degrees of freedom that the robot manipulates directly through wheel velocity and the degrees of freedom that it indirectly manipulates by changing the steering configuration and moving. Based on the investigations of the previous sections, one can draw the basic types of wheel configurations. They are depicted in figure 3.14

Note that two robots with the same δ_M are not necessarily equivalent. For example differential drive and tricycle geometries (fig. 3.13) have equal maneuverability $\delta_M = 2$. In differential drive all maneuverability is the result of direct mobility because $\delta_m = 2$ and $\delta_s = 0$. In the case of a tricycle the maneuverability results from steering also: $\delta_m = 1$ and $\delta_s = 1$. Neither of these configurations allows the *ICR* to range anywhere on the plane. In both cases, the *ICR* must lie on a predefined line with respect to the robot reference

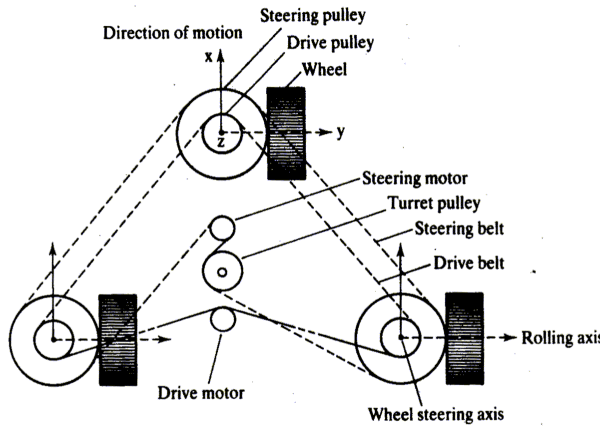

Fig 3.15 *The synchro drive configuration*

frame. In the case of differential drive, this line extends from the common axle of the two fixed standard wheels, with the differential wheel velocities setting the *ICR* point on this line. In a tricycle, this line extends from the shared common axle of the fixed wheels, with the steerable wheel setting the *ICR* point along this line.

More generally, for any robot with $\delta_M = 2$ the *ICR* is always constrained to lie on a line and for any robot with $\delta_M = 3$ the *ICR* can be set to any point on the plane.

One final example will demonstrate the use of the tools we have developed above. One common robot configuration for indoor mobile robotics research is the *synchro drive* configuration (Figure 3.15). Such a robot has two motors and three wheels that are locked together. One motor provides power for spinning all three wheels while the second motor provides power for steering all three wheels. In a three-wheeled synchro drive robot $N_f = 0$ and $N_s = 3$. Therefore $rank\left[C_{1s}(\beta_s)\right]$ can be used to determine both δ_m and δ_s. The three wheels do not share a common axle, therefore two of the three contribute independent sliding constraints. The third must be dependent on these two constraints for motion to be possible. Therefore $rank\left[C_{1s}(\beta_s)\right] = 2$ and $\delta_m = 1$. This is intuitively correct. A synchro drive robot with the steering frozen manipulates only one degree of freedom, consisting of traveling back and forth on a straight line.

However an interesting complication occurs when considering δ_s. Based on Equation (3.41) the robot should have $\delta_s = 2$. Indeed, for a three-wheel-steering robot with the geometric configuration of a synchro drive robot this would be correct. However, we have additional information: in a synchro drive configuration a single motor steers all three wheels using a belt drive. Therefore, although ideally, if the wheels were independently steerable, then the system would achieve $\delta_s = 2$, in the case of synchro-drive the drive system further constrains the kinematics such that in reality $\delta_s = 1$. Finally, we can compute maneuverability based on these values: $\delta_M = 2$ for a synchro drive robot.

This result implies that a synchro drive robot can only manipulate, in total, two degrees of freedom. In fact, if the reader reflects on the wheel configuration of a synchro drive robot it will become apparent that there is no way for the chassis orientation to change. Only the *x*-*y* position of the chassis can be manipulated and so, indeed, a synchro-drive robot has only two degrees of freedom, in agreement with our mathematical conclusion.

3.4 Mobile Robot Workspace

For a robot, maneuverability is equivalent to its control degrees of freedom. But the robot is situated in some environment, and the next question is to situate our analysis in the environment. We care about the ways in which the robot can use its control degrees of freedom to position itself in the environment. For instance, consider the Ackerman vehicle, or automobile. The total number of control degrees of freedom for such a vehicle $\delta_M = 2$, one for steering and the second for actuation of the drive wheels. But what is the total degrees of freedom of the vehicle in its environment? In fact it is three: the car can position itself on the plane at any *x*,*y* point and with any angle θ.

Thus identifying a robot's space of possible configurations is important because surprisingly it can exceed δ_M. In addition to *workspace*, we care about how the robot is able to move between various configurations: what are the types of paths that it can follow and, furthermore, what are its possible trajectories through this configuration space? In the remainder of this discussion, we move away from inner kinematic details such as wheels and focus instead on the robot chassis pose and the chassis degrees of freedom. With this in mind, let us place the robot in the context of its workspace now.

3.4.1 Degrees of Freedom

In defining the workspace of a robot, it is useful to first examine its *admissible velocity space*. Given the kinematic constraints of the robot, its velocity space describes the independent components of robot motion that the robot can control. For example, the velocity space of a unicycle can be represented with two axes, one representing the instantaneous forward speed of the unicycle and the second representing the instantaneous change in orientation, $\dot{\theta}$, of the unicycle.

The number of dimensions in the velocity space of a robot is the number of independently achievable velocities. This is also called the *differentiable degrees of freedom*. A robot's differential degrees of freedom (*DDOF*) is *always* equal to its degree of mobility δ_m. For example, a bicycle has the following degree of maneuverability:
$\delta_M = \delta_m + \delta_s = 1 + 1 = 2$. The *DDOF* of a bicycle is indeed 1.

In contrast to a bicycle, consider an *omnibot*, a robot with three swedish wheels. We know that in this case there are zero standard wheels and therefore $\delta_M = \delta_m + \delta_s = 3 + 0 = 3$. So, the omnibot has three differential degrees of freedom. This is appropriate, given that because such a robot has no kinematic motion constraints, it is able to independently set all three pose variables: $\dot{x},\ \dot{y},\ \dot{\theta}$.

Given the difference in differential degrees of freedom between a bicycle and an omnibot, consider the overall degrees of freedom in the workspace of each configuration. The omnibot can achieve any pose (x, y, θ) in its environment and can do so by directly achieving the goal positions of all three axes simultaneously because DDOF=3. Clearly, it has a workspace with DOF=3.

Can a bicycle achieve any pose (x, y, θ) in its environment? It can do so, but achieving some goal points may require more time and energy than an equivalent omnibot. For example, if a bicycle configuration must move laterally one meter, the simplest successful maneuver would involve either a spiral or a back and forth motion similar to *parallel parking* of automobiles. Nevertheless, a bicycle can achieve any (x, y, θ) and therefore the workspace of a bicycle has DOF=3 as well.

Clearly there is an inequality relation at work: $DDOF \le \delta_M \le DOF$. Although the dimensionality of a robot's workspace is an important attribute, it is clear from the example above that the particular paths available to a robot matter as well. Just as workspace *DOF* governs the robot's ability to achieve various poses, so the robot's *DDOF* governs its ability to achieve various paths.

3.4.2 Holonomic Robots

In the robotics community, when describing the path space of a mobile robot, often the concept of holonomy is used. The term *holonomic* has broad applicability to several mathematical areas, including differential equations, functions and constraint expressions. In mobile robotics, the term refers specifically to the kinematic constraints of the robot chassis. A *holonomic robot* is a robot that has zero non-holonomic kinematic constraints. Conversely, a *non-holonomic robot* is a robot with one or more non-holonomic kinematic constraints.

A *holonomic kinematic constraint* can be expressed as an explicit function of position variables only. For example, in the case of a mobile robot with a single fixed standard wheel, a holonomic kinematic constraint would be expressible using $\alpha_1, \beta_1, l_1, r_1, \varphi_1, x, y, \theta$ only. Such a constraint may not use derivatives of these values, such as $\dot{\varphi}$ or $\dot{\xi}$. A *non-holonomic kinematic constraint* requires a differential relationship, such as the derivative of a position variable. Furthermore, it cannot be integrated to provide a constraint in terms of the position variables only. Because of this latter point of view, non-holonomic systems are often called *non-integrable* systems.

Consider the fixed standard wheel sliding constraint:

$$\left[\cos(\alpha+\beta) \;\; \sin(\alpha+\beta) \;\; l\sin\beta\right] R(\theta)\dot{\xi}_I = 0 \tag{3.43}$$

This constraint must use robot motion $\dot{\xi}$ rather than pose ξ because the point is to constrain robot motion perpendicular to the wheel plane to be zero. The constraint is non-integrable, depending explicitly on robot motion. Therefore, the sliding constraint is a non-holonomic constraint. Consider a bicycle configuration, with one fixed standard wheel and one steerable standard wheel. Because the fixed wheel sliding constraint will be in force for such a robot, we can conclude that the bicycle is a non-holonomic robot.

But suppose that one locks the bicycle steering system, so that it becomes two fixed standard wheels with separate but parallel axes. We know that $\delta_M = 1$ for such a configuration. Is it non-holonomic? Although it may not appear so because of the sliding and rolling constraints, the locked bicycle is actually holonomic. Consider the workspace of this locked bicycle. It consists of a single infinite line along which the bicycle can move (assuming the steering was frozen straight-ahead). For formulaic simplicity, assume that this infinite line is aligned with X_I in the global reference frame and that $\{\beta_{1,2} = \pi/2, \alpha_1 = 0, \alpha_2 = \pi\}$. In this case the sliding constraints of both wheels can be replaced with an equally complete set of constraints on the robot pose: $\{y = 0, \theta = 0\}$. This eliminates two non-holonomic constraints, corresponding to the sliding constraints of the two wheels.

The only remaining non-holonomic kinematic constraints are the rolling constraints for each wheel:

$$\left[-\sin(\alpha+\beta) \;\; \cos(\alpha+\beta) \;\; l\cos\beta\right] R(\theta)\dot{\xi}_I + r\dot{\varphi} = 0 \tag{3.44}$$

This constraint is required for each wheel to relate the speed of wheel spin to the speed of motion projected along the wheel plane. But in the case of our locked bicycle, given the initial rotational position of a wheel at the origin, φ_o, we can replace this constraint with one that directly relates position on the line, x, with wheel rotation angle, φ: $\varphi = (x/r) + \varphi_o$.

The locked bicycle is an example of the first type of holonomic robot- where constraints do exist but are all holonomic kinematic constraints. This is the case for all holonomic robots with $\delta_M < 3$. The second type of holonomic robot exists when there are no kinematic constraints, i.e. $N_f = 0$ and $N_s = 0$. Since there are no kinematic constraints, there are also no non-holonomic kinematic constraints and so such a robot is always holonomic. This is the case for all holonomic robots with $\delta_M = 3$.

An alternative way to describe a holonomic robot is based on the relationship between the differential degrees of freedom of a robot and the degrees of freedom of its workspace: *a robot is holonomic if and only if DDOF = DOF*. Intuitively, this is because it is only through non-holonomic constraints (imposed by steerable or fixed standard wheels) that a robot can achieve a workspace with degrees of freedom exceeding its differential degrees of freedom, *DOF*>*DDOF*. Examples include differential drive and bicycle/tricycle configurations.

In mobile robotics, useful chassis generally must achieve poses in a workspace with dimensionality 3, so in general we require *DOF=3* for the chassis. But the "holonomic" abilities to maneuver around obstacles without affecting orientation, and to track at a target while following an arbitrary path are important additional considerations. For these reasons, the particular form of holonomy most relevant to mobile robotics is that of *DDOF=DOF=3*. We define this class of robot configurations as omnidirectional: an *omnidirectional robot* is a holonomic robot with *DDOF=3*.

3.4.3 Path and trajectory considerations

In mobile robotics, we care not only about the robot's ability to reach the required final configurations, but also about *how* it gets there. Consider the issue of a robot's ability to follow paths: in the best case, a robot should be able to trace any path through its workspace of poses. Clearly, any omnidirectional robot can do this because it is holonomic in a 3D workspace. Unfortunately, omnidirectional robots must use unconstrained wheels, limiting the choice of wheels to swedish wheels, castor wheels and spheric wheels. These wheels have not yet been incorporated into designs allowing far large amounts of ground clearance and suspensions. Although powerful from a path space point of view, they are thus much less common than fixed and steerable standard wheels, mainly because their design and fabrication is somewhat complex and expensive.

Additionally non-holonomic constraints might drastically improve stability of movements. Consider an omnidirectional vehicle driving at high speed on a curve with constant diameter. During such a movement the vehicle will be exposed to a non-negligible centripetal force. This lateral force pushing the vehicle out of the curve has to be counteracted by the motor torque of the omnidirectional wheels. In case of motor or control failure, the vehicle will be thrown out of the curve. However, for a car-like robot with kinematic constraints, the lateral forces are passively counteracted through the sliding constraints, mitigating the demands on motor torque.

But recall an earlier example of high maneuverability using standard wheels: the bicycle on which both wheels are steerable, often called the *two-steer*. This vehicle achieves a degree of steerability of 2, resulting in a high degree of maneuverability: $\delta_M = \delta_m + \delta_s = 1 + 2 = 3$. Interestingly, this configuration is not holonomic, yet has a high degree of maneuverability in a workspace with *DOF*=3.

The maneuverability result, $\delta_M = 3$, means that the two-steer can select any *ICR* by appropriately steering its two wheels. So, how does this compare to an omnidirectional robot? The ability to manipulate its *ICR* in the plane means that the two-steer can follow *any* path in its workspace. More generally, any robot with $\delta_M = 3$ can follow any path in its workspace from its initial pose to its final pose. An omnidirectional robot can also follow any path in its workspace and, not surprisingly, since $\delta_m = 3$ in an omnidirectional robot, then it must follow that $\delta_M = 3$.

But there is still a difference between a degree of freedom granted by steering versus by direct control of wheel velocity. This difference is clear in the context of *trajectories* rather than paths. A trajectory is like a path, except that it occupies an additional dimension: time. Therefore, for an omnidirectional robot on the ground plane a path generally denotes a trace through a 3D space of pose; for the same robot a trajectory denotes a trace through the 4D space of pose plus time.

For example, consider a goal trajectory in which the robot moves along axis X_I at a constant speed of 1 m/s for 1 second, then changes orientation counter-clockwise 90 degrees also in 1 second, then moves parallel to axis Y_I for 1 final second. The desired 3-second trajectory is shown in Fig. 3.16 using plots of *x*, *y*, and θ in relation to time.

Can the omnidirectional robot accomplish this trajectory? We assume that the robot achieve some arbitrary, finite velocity at each wheel. For simplicity, we further assume that acceler-

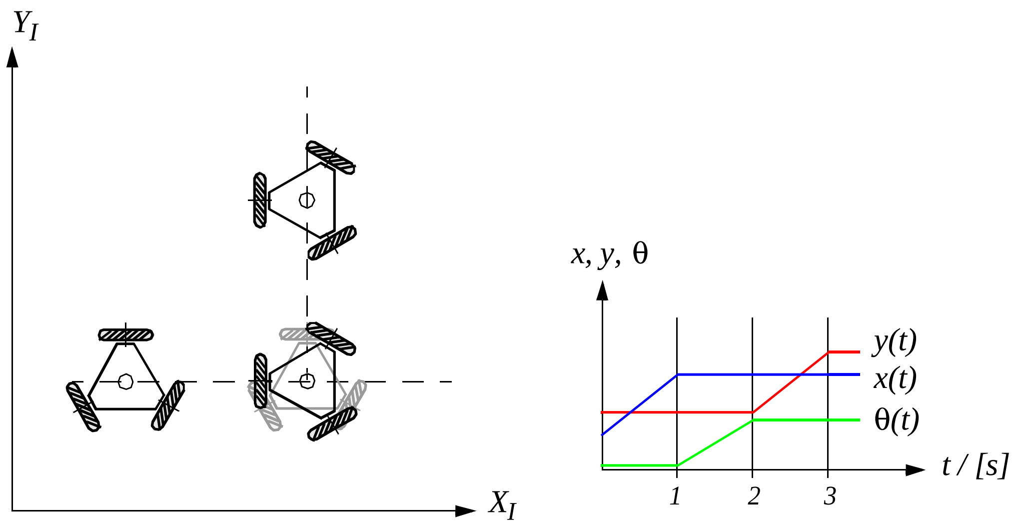

Fig 3.16 *Example of robot trajectory with omnidirectional robot:*
- move for 1 s with constant speed of 1m/s along axis X_I
- change orientation counter-clockwise 90° in 1 s
- move for 1 s with constant speed of 1m/s along axis Y_I

ation is infinite; that is, it takes zero time to reach any desired velocity. Under these assumptions, the omnidirectional robot can indeed follow the trajectory of Fig. 3.16. The transition between the motion of second 1 and second 2, for example, involves only changes to the wheel velocities.

Because the two-steer has $\delta_M = 3$ it must be able to follow the path that would result from projecting this trajectory into time-less workspace. However, it cannot follow this 4D trajectory. Even if steering velocity is finite and arbitrary, although the two-steer would be able to change steering speed instantly, it would have to wait for the angle of the steerable wheels to change to the desired position before initiating a change in the robot chassis orientation. In short, the two-steer requires changes to internal degrees of freedom and because these changes take time, arbitrary trajectories are not attainable. Figure 3.17 depicts the most similar trajectory that a two-steer can achieve. In contrast to the desired three phases of motion, this trajectory has five phases.

3.5 Beyond Basic Kinematics

The above discussion of mobile robot kinematics is only an introduction to a far richer topic. When speed and force are also considered, as is particularly necessary in the case of high speed mobile robots, dynamic constraints must be expressed in addition to kinematic constraints. Furthermore, many mobile robots such as tank-type chassis and four wheel slip-skid systems violate the kinematic models above. When analyzing such systems, it is often necessary to explicitly model the dynamics of viscous friction between the robot and the ground plane.

More significantly, the kinematic analysis of a mobile robot system provides results concerning the theoretical workspace of that mobile robot. However to effectively move in this

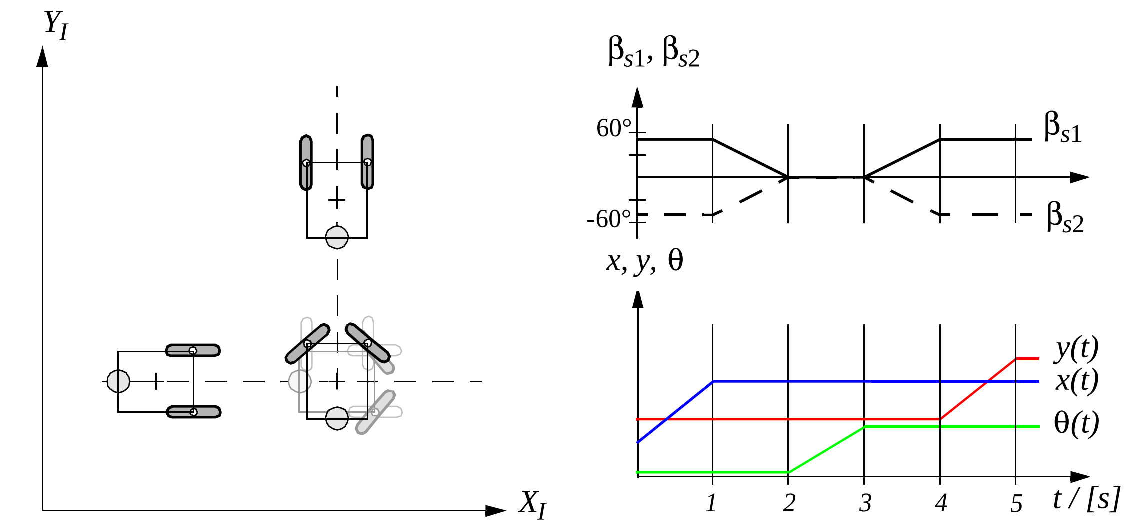

Fig 3.17 *Example of robot trajectory similar to fig. 3.16 with two steered wheels:*
- *move for 1 s with constant speed of 1m/s along axis* X_I.
- *rotate steered wheels -50° / 50° respectively*
- *change orientation counter-clockwise 90° in 1 s*
- *rotate steered wheels 50° / -50° respectively*
- *move for 1 s with constant speed of 1m/s along axis* Y_I

workspace a mobile robot must have appropriate actuation of its degrees of freedom. This problem, called motorization, requires further analysis of the forces that must be actively supplied in order to realize the kinematic range of motion available to the robot.

In addition to motorization, there is the question of controllability: under what conditions can a mobile robot travel from the initial pose to the goal pose in bounded time? Answering this question requires knowledge both knowledge of the robot kinematics and knowledge of the control systems that can be used to actuate the mobile robot. Mobile robot control is therefore a return to the practical question of designing a real-world control algorithm that can drive the robot from pose to pose using the trajectories demanded for the application.

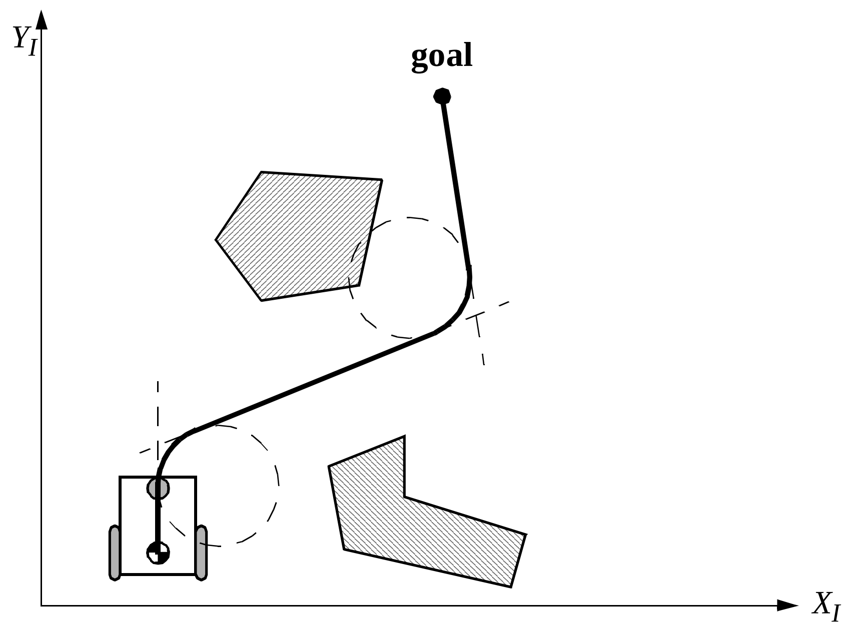

Fig 3.18 Open loop control of a mobile robot based on straight lines and circular trajectory segments

3.6 Motion Control (kinematic control)

As seen above, motion control might not be an easy task for non-holonomic systems. However, it has been studied by various research groups, e.g. [1, 35, 36, 37, 38] and some adequate solutions for motion control of a mobile robot system are available.

3.6.1 Open Loop Control (trajectory following)

The objective of a kinematic controller is to follow a trajectory described by its position and/or velocity profiles as function of time. This is often done by dividing the trajectory (path) in motion segments of clearly defined shape, e.g. straight *lines* and segments of a *circle*. The control problem is thus to pre-compute a smooth trajectory based on line and circle segments which drives the robot from the initial position to the final position (fig. 3.18). This approach can be regarded as open-loop motion control, because the measured robot position is not fed back for velocity or position control. It has several disadvantages:

- It is not at all an easy task to pre-compute a feasible trajectory if all limitations and constraints of the robots velocities and accelerations have to be considered.
- The robot will not automatically adapt or correct the trajectory if dynamic changes of the environment occur.
- The resulting trajectories are usually not smooth, because the transitions from one trajectory segment to another are for most of the commonly used segments (e.g. lines and part of circles) not smooth. This means there is a discontinuity in the robots acceleration.

3.6.2 Feedback control

A more appropriate approach in motion control of a mobile robot is to use a real state feedback controller. With such a controller the robot's path-planning task is reduced to setting

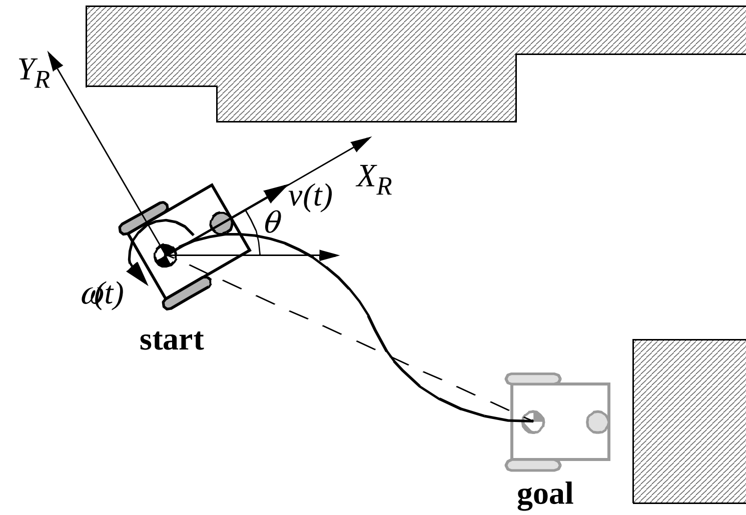

Fig 3.19 Typical situation for feedback-control of a mobile robot

intermediate positions (sub-goals) lying on the requested path. One useful solution for a stabilizing feedback control of differential drive mobile robots is presented in the following section. It is very similar to the controllers presented in [36] and [86]. Others can be found in [1, 35, 37, 38].

Problem statement

Consider the situation shown in figure 3.19, with an arbitrary position and orientation of the robot and a predefined goal position and orientation. The actual pose error vector given in the robot reference frame $\{X_R, Y_R, \theta\}$ is $e = {}^R[x, \ y, \ \theta]^T$ with x, y, and θ being the goal coordinates of the robot.

The task of the controller layout is to find a control matrix K, if it exists

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix} \qquad \text{with } k_{ij} = k(t, \ e) \tag{3.45}$$

such that the control of $v(t)$ and $\omega(t)$

$$\begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix} = K \cdot \ e = K \ {}^R\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \tag{3.46}$$

drives the error e towards zero[1].

1. *Remember that v(t) is always heading in the X_R-direction of the robot's reference frame due to the non-holonomic constraint*

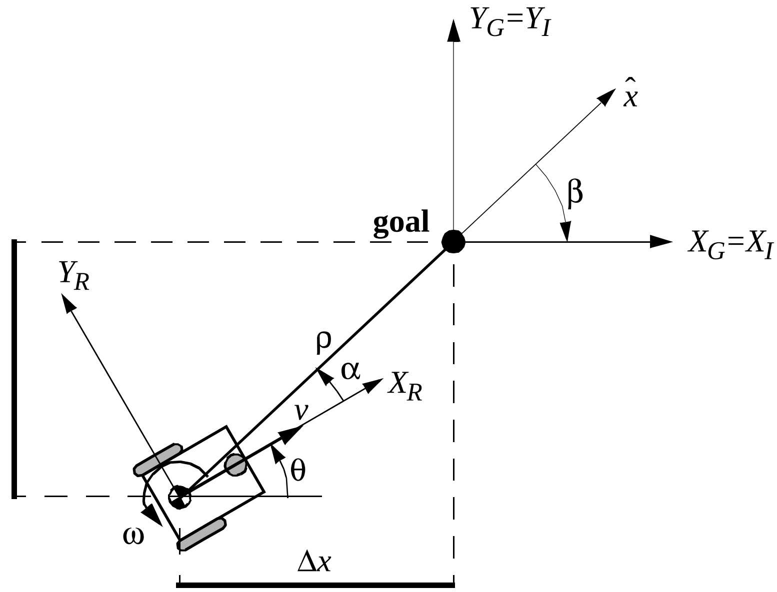

Fig 3.20 Robot kinematics and its frames of interests

$$\lim_{t \to \infty} e(t) = 0 \tag{3.47}$$

Kinematic model

We assume, without loss of generality, that the goal is at the origin of the inertial frame (fig. 3.20). In the following the position vector $[x, \; y, \; \theta]^T$ is always represented in the inertial frame.

The kinematics of a differential drive mobile robot described in the inertial frame $\{X_I, Y_I, \theta\}$ are given by,

$$^{I}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{3.48}$$

where $\dot{x}$ and $\dot{y}$ are the linear velocities in the direction of the X_I and Y_I of the inertial frame.

Let α denote the angle between the x_R axis of the robots reference frame and the vector $\hat{x}$ connecting the center of the axle of the wheels with the final position. If $\alpha \in I_1$, where

$$I_1 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \tag{3.49}$$

then consider the coordinate transformation into polar coordinates with its origin at the goal position.

$$\rho = \sqrt{\Delta x^2 + \Delta y^2} \tag{3.50}$$

$$\alpha = -\theta + \operatorname{atan}\left(\frac{\Delta y}{\Delta x}\right) \qquad mod\left(\frac{\pi}{2}\right) \tag{3.51}$$

$$\beta = -\theta - \alpha \qquad mod(\pi) \tag{3.52}$$

This yields a system description, in the new polar coordinates, using a matrix equation

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\cos\alpha & 0 \\ \frac{\sin\alpha}{\rho} & -1 \\ -\frac{\sin\alpha}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{3.53}$$

where ρ is the distance between the center of the robot's wheel axle and the goal position, θ denotes the angle between the X_R axis of the robot reference frame and the X_I axis associated with the final position and v and $\omega$are the tangent and the angular velocity respectively.

On the other hand, if $\alpha \in I_2$, where

$$I_2 = \left(-\pi, -\frac{\pi}{2}\right] \cup \left(\frac{\pi}{2}, \pi\right] \tag{3.54}$$

redefining the forward direction of the robot by setting $v = -v$, we obtain a system described by a matrix equation of the form

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 \\ -\frac{\sin\alpha}{\rho} & 1 \\ \frac{\sin\alpha}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{3.55}$$

Remarks on the kinematic model in polar coordinates (equation (3.53) and (3.55))

- The coordinate transformation is not defined at $x = y = 0$; as in such a point the determinant of the Jacobian matrix of the transformation is not defined, i.e. it is unbounded.
- For $\alpha \in I_1$ the forward direction of the robot points toward the goal, for $\alpha \in I_2$ it is the reverse direction.
- By properly defining the forward direction of the robot at its initial configuration, it is always possible to have $\alpha \in I_1$ at $t = 0$. However, this does not mean that αremains in I_1 for all time t. Hence, to avoid that the robot changes direction during approach-

ing the goal, it is necessary to determine, if possible, the controller in such a way that $\alpha \in I_1$ for all t, whenever $\alpha(0) \in I_1$. The same applies for the reverse direction (see stability issues below).

The control law

The control signals v and ω must now be designed to drive the robot from its actual configuration, say $(\rho_0, \alpha_0, \beta_0)$, to the goal position. It is obvious that equation (3.53) presents a discontinuity at $\rho = 0$, thus the theorem of Brockett does not obstruct smooth stabilizability. If we consider now the linear control law

$$v = k_\rho \rho \tag{3.56}$$

$$\omega = k_\alpha \alpha + k_\beta \beta \tag{3.57}$$

we get with equation (3.53) a closed loop system described by

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -k_\rho \rho \cos\alpha \\ k_\rho \sin\alpha - k_\alpha \alpha - k_\beta \beta \\ -k_\rho \sin\alpha \end{bmatrix} \tag{3.58}$$

The system does not have any singularity at $\rho = 0$ and has a unique equilibrium point at $(\rho, \alpha, \beta) = (0, 0, 0)$. Thus it will drive the robot to this point which is the goal position.

- In the Cartesian coordinate system the control law (3.57) leads to equations which are not defined at $x = y = 0$.
- Observe that the control signal v has always a constant sign, i.e. it is positive whenever $\alpha(0) \in I_1$ and it is always negative otherwise. This implies that the robot performs its parking maneuver always in a single direction and without reversing its motion.

In figure 3.21 you find the resulting paths when the robot is initially on a circle in the *xy*-plane. All movements have smooth trajectories toward the goal in the center. The control parameters for this simulation where set to

$$k = (k_\rho, k_\alpha, k_\beta) = (3, 8, -1.5). \tag{3.59}$$

Local stability issue

It can further be shown, that the closed loop control system (3.58) is locally exponentially stable if

$$k_\rho > 0 \ ; \quad k_\beta < 0 \ ; \quad k_\alpha - k_\rho > 0 \tag{3.60}$$

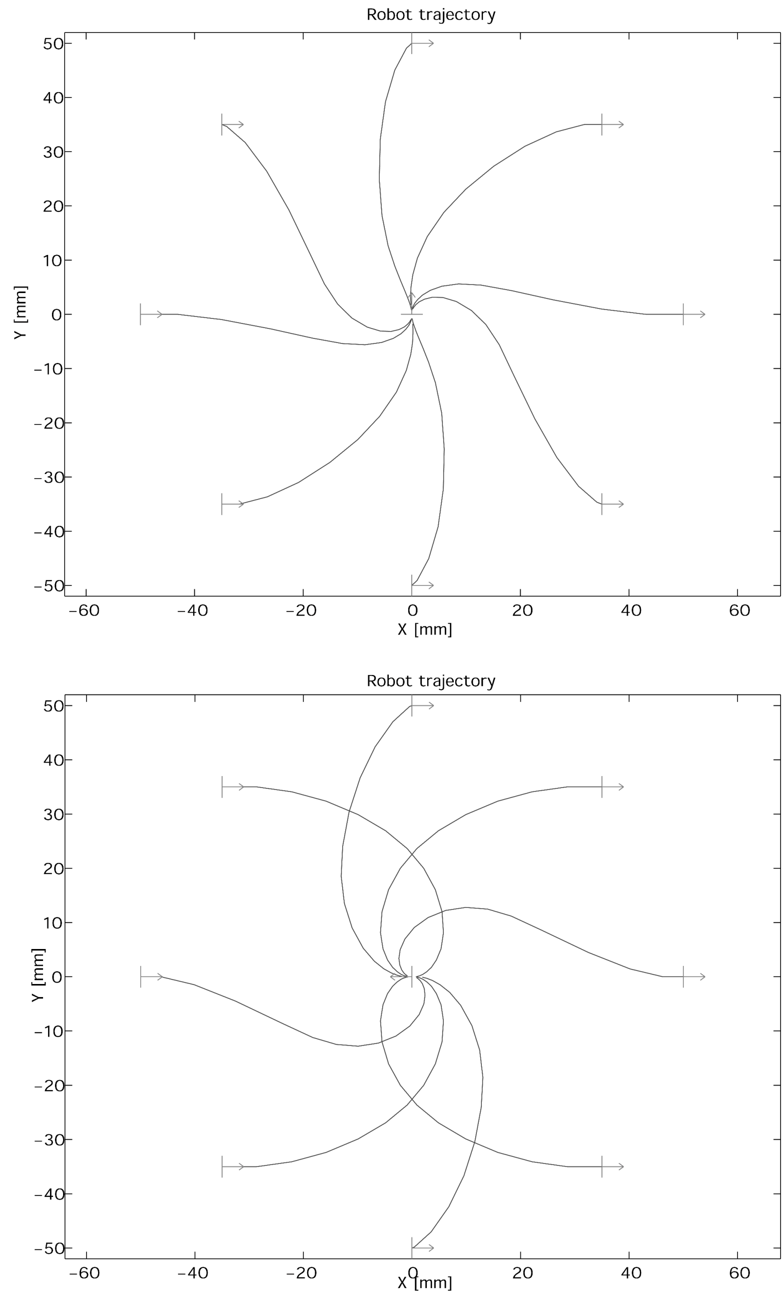

Fig 3.21 Resulting paths when the robot is initially on the unit circle in the xy-plane

Proof:

Linearized around the equilibrium ($\cos x = 1$, $\sin x = x$) position, equation (3.58) can be written as

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -k_\rho & 0 & 0 \\ 0 & -(k_\alpha - k_\rho) & -k_\beta \\ 0 & -k_\rho & 0 \end{bmatrix} \begin{bmatrix} \rho \\ \alpha \\ \beta \end{bmatrix}, \tag{3.61}$$

hence it is locally exponentially stable if the eigenvalues of the matrix

$$A = \begin{bmatrix} -k_\rho & 0 & 0 \\ 0 & -(k_\alpha - k_\rho) & -k_\beta \\ 0 & -k_\rho & 0 \end{bmatrix} \quad (3.62)$$

all have a negative real part. The characteristic polynomial of the matrix A is

$$(\lambda + k_\rho)(\lambda^2 + \lambda(k_\alpha - k_\rho) - k_\rho k_\beta) \quad (3.63)$$

and all roots have negative real part if

$$k_\rho > 0 \ ; \quad -k_\beta > 0 \ ; \quad k_\alpha - k_\rho > 0 \quad (3.64)$$

which proves the claim.

For robust position control, it might be advisable to apply the strong stability condition, that ensures that the robot does not change direction during its approach to the goal:

$$k_\rho > 0 \ ; \quad k_\beta < 0 \ ; \quad k_\alpha + \frac{5}{3}k_\beta - \frac{2}{\pi}k_\rho > 0 \quad (3.65)$$

This implies that $\alpha \in I_1$ for all t, whenever $\alpha(0) \in I_1$ and $\alpha \in I_2$ for all t, whenever $\alpha(0) \in I_2$ respectively. This strong stability condition has also been verified in applications.