Unifying Discriminative Visual Codebook Generation with Classifier Training for Object Category Recognition
(Supplemental Material for CVPR 2008 paper)

Liu Yang¹  Rong Jin¹  Rahul Sukthankar²,³  Frederic Jurie⁴
yangliu1@cse.msu.edu  rongjin@cse.msu.edu  rahuls@cs.cmu.edu  frederic.jurie@inrialpes.fr
¹Dept. CSE, Michigan State Univ. ²Intel Research Pittsburgh ³Robotics Institute, Carnegie Mellon ⁴LEAR Group - CNRS - INRIA


Appendix A : Proof of Lemma 1

Proof. \( l'(X_i, y_i) \) can be upper bounded by follows:

\[
\begin{align*}
  l'(X_i, y_i) &= \sum_{X_i} \sum_{i=1}^{n_i} e'(x_{i,j}, y_i) \\
  &= \sum_{j=1}^{n_i} \frac{e(x_{i,j}, y_i)}{\sum_{j=1}^{m} \exp(\alpha g(x_{i,j}, y_i))} \\
  &= \frac{1}{\sum_{j=1}^{n_i} \exp(\alpha g(x_{i,j}, y_i))} \\
  &\leq \frac{1}{\sum_{j=1}^{n_i} \exp(\alpha g(x_{i,j}, y_i))} \\
  &= \frac{1}{\sum_{j=1}^{n_i} q_{i,j}(y_i) \exp(\alpha g(x_{i,j}, y_i))} \\
  &= \frac{1}{\sum_{j=1}^{n_i} \exp(\alpha g(x_{i,j}, y_i))}.
\end{align*}
\]

where

\[
q_{i,j}(y_i) = \frac{e(x_{i,j}, y_i)}{\sum_{j=1}^{n_i} e(x_{i,j}, y_i)}.
\]

The above inequality gives us the result in Lemma 1.

Appendix B : Proof of Lemma 2

Proof. \( \exp(\alpha(g(x_{i,j}, y) - g(x_{i,j}, y'))(y)) \) can be upper bounded as follows:

\[
\begin{align*}
  \exp(\alpha(g(x_{i,j}, y) - g(x_{i,j}, y'))(y)) &= \exp(-3\alpha \times \frac{g(x_{i,j}, y') - g(x_{i,j}, y) + 1}{3} + 3\alpha \times \frac{1}{3} \\
  &\leq \frac{g(x_{i,j}, y') - g(x_{i,j}, y) + 1}{3} \exp(-3\alpha) \\
  &\leq \frac{1}{3} \exp(3\alpha) + \frac{g(x_{i,j}, y') - g(x_{i,j}, y) + 1}{3} \exp(-3\alpha) + \exp(3\alpha) + 1 \\
  &= \frac{1}{3} - \exp(-3\alpha) + \exp(3\alpha) + \frac{g(x_{i,j}, y') - g(x_{i,j}, y)}{3}.
\end{align*}
\]

In the above, we exploit the convexity of exponential function, namely

\[
\exp\left(\sum_{i=1}^{n} p_i q_i\right) \leq \sum_{i=1}^{n} p_i \exp(q_i),
\]

when \( p_i, i = 1, \ldots, n \) is a probability distribution (i.e., \( p_i \geq 0, i = 1, \ldots, n \) and \( \sum_{i=1}^{n} p_i = 1 \)).
Appendix C : Proof of Theorem 1

Proof. Using the results in Lemma 1 and 2, we have the loss function \( l'(X_i, y) \) upper bound as follows:

\[
\frac{l'(X_i, y)}{l(X_i, y)} \\
\leq \frac{\exp(-3\alpha) + \exp(3\alpha) + 1}{3} \sum_{j=1}^{n_i} \sum_{y'=1}^{m} q_{i,j}(y) e(X_i, y') \\
- \frac{1 - \exp(-3\alpha)}{3} \sum_{j=1}^{n_i} q_{i,j}(y) \sum_{y'=1}^{m} e(X_i, y') g(x_{i,j}, y) \\
- \frac{1 - \exp(-3\alpha)}{3} \sum_{j=1}^{n_i} q_{i,j}(y) \sum_{y'=1}^{m} e(X_i, y') g(x_{i,j}, y') \\
\]

In the above, we use the relationships

\[ \sum_{j=1}^{n_i} q_{i,j}(y) = 1, \quad \text{and} \quad \sum_{y=1}^{m} e(X_i, y) = 1. \]

Using the above inequality and the fact that \( L' = \sum_{i=1}^{N} l'(X_i, y), l(X_i, y) \), we obtain the result in Theorem 1. \( \square \)

Appendix D : Computing the Optimal \( \alpha \)

Proof. First, using Lemma 1, we have the following upper bound for the objective function \( L' \), i.e.

\[
L' = \sum_{i=1}^{N} \sum_{y \in y_i} l(X_i, y) \sum_{j=1}^{n_i} q_{i,j}(y) \frac{e(x_{i,j}, y')}{\exp(\alpha|g(x_{i,j}, y) - g(x_{i,j}, y')|)} \\
+ \sum_{i=1}^{N} \sum_{y \in y_i} \frac{n_i}{m} t_{i,j}(y, y') \delta(g(x_{i,j}, y), 1) \delta(g(x_{i,j}, y'), 0) \exp(\alpha) \\
+ \sum_{i=1}^{N} \sum_{y \in y_i} \frac{n_i}{m} t_{i,j}(y, y') \delta(g(x_{i,j}, y), 0) \delta(g(x_{i,j}, y'), 1) \exp(\alpha)
\]

where

\[ t_{i,j}(y, y') = l(X_i, y) q_{i,j}(y) e(x_{i,j}, y'). \]

By setting the derivative of the above expression with respect to \( \alpha \) to be zero, we have the expression for computing the optimal \( \alpha \). \( \square \)

Appendix E : Proof of Theorem 3

Proof. First note that \( L \) can also be written as

\[
L = \sum_{i=1}^{N} \sum_{y \in y_i} l(X_i, y) q_{i,j}(y) e(x_{i,j}, y') \\
= \sum_{i=1}^{N} \sum_{y \in y_i} \sum_{z' = 0}^{m} A_{i,j}(z) B_{i,j}(z') \\
= K_{0,0} + K_{0,1} + K_{1,0} + K_{1,1},
\]

where

\[ K_{z,z'} = \sum_{i=1}^{N} A_{i,j}(z) B_{i,j}(z'). \]

On the other hand, using the result for \( \alpha \), we have \( L' \) expressed as follows:

\[
L' = \sqrt{K_{1,0}} K_{0,1}
\]

Hence, the ratio \( L'/L \) is calculated as

\[
\frac{L'}{L} = \frac{2\sqrt{K_{1,0}} K_{0,1}}{K_{0,0} + K_{0,1} + K_{1,0} + K_{1,1}} \\
\leq 1 - \frac{K_{1,0} + K_{0,1} + K_{1,0} + K_{1,1}}{K_{0,0} + K_{0,1} + K_{1,0} + K_{1,1}} \\
= 1 - \frac{(\sqrt{K_{0,0}} - \sqrt{K_{1,0}})^2}{2K_{0,0} + K_{0,1} + K_{1,0} + K_{1,1}} \\
= 1 - \frac{(\exp(\alpha) - 1)^2}{1 + \exp(2\alpha) + \eta},
\]

where

\[ \eta = \frac{K_{1,1} K_{0,0}}{K_{1,0}} = \frac{\sum_{i=1}^{N} \sum_{j=1}^{n_i} A_{i,j}(z) B_{i,j}(z)}{\sum_{i=1}^{N} \sum_{j=1}^{n_i} A_{i,j}(1) B_{i,j}(0)}. \]

\[
L_T = L_0 \prod_{i=1}^{T} \frac{L_i}{L_{i-1}} \\
\leq \prod_{i=1}^{T} \left( 1 - \frac{(\exp(\alpha) - 1)^2}{1 + \exp(2\alpha) + \eta} \right).
\]

\( \square \)