Unifying Discriminative Visual Codebook Generation with Classifier Training for Object Category Recognition (Supplemental Material for CVPR 2008 paper)

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Appendix A : Proof of Lemma 1

Proof. $l'(X_i, y_i)$ can be upper bounded by follows:

$$\begin{aligned}
l'(X_i, y_i) &= \frac{n_i}{\sum_{j=1}^{n_i} e'(\mathbf{x}_{i,j}, y_i)} \\
&= \frac{n_i}{\sum_{j=1}^{n_i} e(\mathbf{x}_{i,j}, y_i) \frac{\exp(\alpha g(\mathbf{x}_{i,j}, y_i))}{\sum_{y=1}^{m} \exp(\alpha g(\mathbf{x}_{i,j}, y))}} \\
&= l(X_i, y_i) \frac{1}{\sum_{j=1}^{n_i} q_{i,j}(y) \frac{\exp(\alpha g(\mathbf{x}_{i,j}, y_i))}{\sum_{y=1}^{m} \exp(\alpha g(\mathbf{x}_{i,j}, y))}} \\
&\leq l(X_i, y_i) \sum_{j=1}^{n_i} q_{i,j}(y) \left(\frac{\exp(\alpha g(\mathbf{x}_{i,j}, y_i))}{\sum_{y=1}^{m} \exp(\alpha g(\mathbf{x}_{i,j}, y))} \right)^{-1} \\
&= l(X_i, y_i) \sum_{j=1}^{n_i} q_{i,j}(y) \sum_{y=1}^{m} \exp\Big(\alpha (g(\mathbf{x}_{i,j}, y) \\
&\quad - g(\mathbf{x}_{i,j} y_i)) \Big),
\end{aligned}$$

where

$$q_{i,j}(y) = \frac{e(\mathbf{x}_{i,j}, y_i)}{\sum_{j'=1}^{n_i} e(\mathbf{x}_{i,j'}, y_i)}.$$

The above inequality gives us the result in Lemma 1. □

Appendix B : Proof of Lemma 2

Proof. $\exp(\alpha(g(\mathbf{x}_{i,j}, y) - g(\mathbf{x}_{i,j}, y')))$ can be upper bounded as follows:

$$\begin{aligned}
&\exp(\alpha(g(\mathbf{x}_{i,j}, y) - g(\mathbf{x}_{i,j}, y'))) \\
&= \exp\Big(-3\alpha \times \frac{g(\mathbf{x}_{i,j}, y') - g(\mathbf{x}_{i,j}, y) + 1}{3} + 3\alpha \times \frac{1}{3} \\
&\quad + 0 \times \frac{g(\mathbf{x}_{i,j}, y) - g(\mathbf{x}_{i,j}, y') + 1}{3} \Big) \\
&\leq \frac{g(\mathbf{x}_{i,j}, y') - g(\mathbf{x}_{i,j}, y) + 1}{3} \exp(-3\alpha) \\
&\quad + \frac{1}{3} \exp(3\alpha) + \frac{g(\mathbf{x}_{i,j}, y) - g(\mathbf{x}_{i,j}, y') + 1}{3} \\
&= \frac{\exp(-3\alpha) + \exp(3\alpha) + 1}{3} \\
&\quad - \frac{1 - \exp(-3\alpha)}{3} (g(\mathbf{x}_{i,j}, y') - g(\mathbf{x}_{i,j}, y)).
\end{aligned}$$

In the above, we exploit the convexity of exponential function, namely

$$\exp\left(\sum_{i=}^{n} p_i q_i \right) \leq \sum_{i=1}^{n} p_i \exp(q_i),$$

when $p_i, i = 1, \ldots, n$ is a probability distribution (i.e., $p_i \geq 0, i = 1, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$). □

Appendix C : Proof of Theorem 1

Proof. Using the results in Lemma 1 and 2, we have the loss function $l'(X_i, y)$ upper bound as follows:

$$
\begin{aligned}
&\frac{l'(X_i,y)}{l(X_i,y)}\\
\leq\ & \frac{\exp(-3\alpha)+\exp(3\alpha)+1}{3}\sum_{j=1}^{n_i}\sum_{y'=1}^{m} q_{i,j}(y)e(X_i,y')\\
&-\frac{1-\exp(-3\alpha)}{3}\sum_{j=1}^{n_i} q_{i,j}(y)\sum_{y'=1}^{m} e(X_i,y')g(\mathbf{x}_{i,j},y)\\
&-\frac{1-\exp(-3\alpha)}{3}\sum_{j=1}^{n_i} q_{i,j}(y)\sum_{y'=1}^{m} e(X_i,y')g(\mathbf{x}_{i,j},y')\\
=\ & \frac{\exp(-3\alpha)+\exp(3\alpha)+1}{3}\\
&-\frac{1-\exp(-3\alpha)}{3}\sum_{j=1}^{n_i} q_{i,j}(y)\sum_{y'=1}^{m} \delta(y,y')g(\mathbf{x}_{i,j},y')\\
&-\frac{1-\exp(-3\alpha)}{3}\sum_{j=1}^{n_i} q_{i,j}(y)\sum_{y'=1}^{m} e(X_i,y')g(\mathbf{x}_{i,j},y').
\end{aligned}
$$

In the above, we use the relationships

$$\sum_{j=1}^{n_i} q_{i,j}(y)=1, \quad \text{and} \sum_{y=1}^{m} e(X_i,y)=1.$$

Using the above inequality and the fact that $\mathcal{L}' = \sum_{i=1}^{N} l \sum_{y\in\mathbf{y}_i} l(X_i,y)$, we obtain the result in Theorem 1. □

Appendix D : Computing the Optimal α

Proof. First, using Lemma 1, we have the following upper bound for the objective function $\mathcal{L}'$, i.e.

$$
\begin{aligned}
\mathcal{L}' \leq\ &\\
&\sum_{i=1}^{N}\sum_{y\in\mathbf{y}_i} l(X_i,y)\sum_{j=1}^{n_i} q_{i,j}(y)\sum_{y'=1}^{m}\frac{e(\mathbf{x}_{i,j},y')}{\exp(\alpha[g(\mathbf{x}_{i,j},y)-g(\mathbf{x}_{i,j},y')])}\\
=\ & \sum_{i=1}^{N}\sum_{y\in\mathbf{y}_i}\sum_{j=1}^{n_i}\sum_{y'=1}^{m} t_{i,j}(y,y')\delta(g(\mathbf{x}_{i,j},y),1)\delta(g(\mathbf{x}_{i,j},y'),0)\exp(\alpha)\\
+\ & \sum_{i=1}^{N}\sum_{y\in\mathbf{y}_i}\sum_{j=1}^{n_i}\sum_{y'=1}^{m} t_{i,j}(y,y')\delta(g(\mathbf{x}_{i,j},y),0)\delta(g(\mathbf{x}_{i,j},y'),1)\exp(-\alpha)
\end{aligned}
$$

where

$$t_{i,j}(y,y') = l(X_i,y)q_{i,j}(y)e(\mathbf{x}_{i,j},y').$$

By setting the derivative of the above expression with respect to α to be zero, we have the expression for computing the optimal α.

□

Appendix E : Proof of Theorem 3

Proof. First note that $\mathcal{L}$ can also be written as

$$
\begin{aligned}
\mathcal{L} &= \sum_{i=1}^{N}\sum_{y\in\mathbf{y}_i}\sum_{j=1}^{n_i}\sum_{y'=1}^{m} l(X_i,y)q_{i,j}(y)e(\mathbf{x}_{i,j},y')\\
&= \sum_{i=1}^{N}\sum_{j=1}^{n_i}\sum_{z=0}^{1}\sum_{z'=0}^{1} A_{i,j}(z)B_{i,j}(z')\\
&= K_{0,0}+K_{0,1}+K_{1,0}+K_{1,1},
\end{aligned}
$$

where

$$K_{z,z'} = \sum_{i=1}^{N}\sum_{j=1}^{n_i} A_{i,j}(z)B_{i,j}(z').$$

On the other hand, using the result for α, we have $\mathcal{L}'$ expressed as follows:

$$
\begin{aligned}
\mathcal{L}' &= 2\sqrt{\left(\sum_{i=1}^{N}\sum_{j=1}^{n_i} A_{i,j}(1)B_{i,j}(0)\right)\left(\sum_{i=1}^{N}\sum_{j=1}^{n_i} A_{i,j}(0)B_{i,j}(1)\right)}\\
&= 2\sqrt{K_{1,0}K_{0,1}}.
\end{aligned}
$$

Hence, the ratio $\mathcal{L}'/\mathcal{L}$ is calculated as

$$
\begin{aligned}
\frac{\mathcal{L}'}{\mathcal{L}} &= \frac{2\sqrt{K_{1,0}K_{0,1}}}{K_{0,0}+K_{0,1}+K_{1,0}+K_{1,1}}\\
&= 1-\frac{K_{0,0}+K_{0,1}+K_{1,0}+K_{1,1}-2\sqrt{K_{0,1}K_{1,0}}}{K_{0,0}+K_{0,1}+K_{1,0}+K_{1,1}}\\
&\leq 1-\frac{K_{0,1}+K_{1,0}-2\sqrt{K_{0,1}K_{1,0}}}{K_{0,0}+K_{0,1}+K_{1,0}+K_{1,1}}\\
&= 1-\frac{(\sqrt{K_{0,1}}-\sqrt{K_{1,0}})^2}{K_{0,0}+K_{0,1}+K_{1,0}+K_{1,1}}\\
&= 1-\frac{(\exp(\alpha)-1)^2}{1+\exp(2\alpha)+\eta},
\end{aligned}
$$

where

$$
\begin{aligned}
\eta &= \frac{K_{1,1}K_{0,0}}{K_{1,0}}\\
&= \frac{\sum_{i=1}^{N}\sum_{j=1}^{n_i}\sum_{z=0}^{1} A_{i,j}(z)B_{i,j}(z)}{\sum_{i=1}^{N}\sum_{j=1}^{n_i} A_{i,j}(1)B_{i,j}(0)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathcal{L}_T &= \mathcal{L}_0\prod_{t=1}^{T}\frac{\mathcal{L}_t}{\mathcal{L}_{t-1}}\\
&\leq \prod_{t=1}^{T}\left(1-\frac{(\exp(\alpha_t)-1)^2}{1+\exp(2\alpha_t)+\eta_t}\right).
\end{aligned}
$$

□