# SDS 321: Practice questions

## 1 Discrete

1. My partner and I are one of 10 married couples at a dinner party. The 20 people are given random seats around a round table.

   (a) What is the probability that I am seated next to my spouse? there are two seats next to me, and 19 people who can sit in those seats. $P(\text{spouse on left}) = 1/19$, $P(\text{spouse on right}|\text{spouse not on left})P(\text{spouse not on left}) = 1/18 \times 18/19 = 1/19$, so $P(\text{next to spouse}) = 2/19$.

   (b) What is the expected number of couples that are seated next to each other? The expected number of people sat next to their spouse is 40/19 (sum of expectations). So, the expected number of couples sat next to each other is 20/19.

2. How many unique combinations can you get by rearranging the letters MISSISSIPPI?

   11 letters so 11! permutations. 4 Is, 4Ss, 2 Ps, so 11!/(4!4!2!) unique combinations.

3. Combinatorics question:

   (a) How many different solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where $x_1$, $x_2$ and $x_3$ are positive integers? (count "$x_1 = 1, x_2 = 2, x_3 = 7$" and "$x_1 = 2, x_2 = 1, x_3 = 7$" as two separate solutions). Stars and bars... We have 9 places to put the first bar, and 8 places to put the second. But, there are 2 possible rearrangements of the bars. So, $9 \times 8/2 = 36$. Or, equivalently, there are $\binom{9}{2=36}$ ways of placing 2 bars.

   (b) How many different solutions are there to the equation $x_1+x_2+x_3 = 10$, where $x_1 < x_2$? We know there are 36 solutions in total. Of these, let's remove the solutions where $x_1 = x_2$. We have 4 such solutions ($x_1 = x_2 = 1, x_1 = x_2 = 2, x_1 = x_2 = 3, x_1 = x_2 = 4$), leaving 32 solutions with different $x_1$ and $x_2$. Of these, half have $x_1 < x_2$, so 16 solutions.

   (c) How many different solutions are there to the equation $x_1+x_2+x_3 = 10$, where $x_1 < x_2 < x_3$? We know there are 36 solutions in total. Let's first remove the solutions with repeats. We have no solutions with all three numbers the same, and 12 solutions with 2 numbers the same (3 ways each of having 2 ones, 2 twos, 2 threes, 2 fours). So, 24 solutions with no repeats. Of these, there are 3! permutations of each sequence, so there are $24/3! = 4$ solutions. Double checking, we have 1+2+7, 1+3+6, 1+4+5, 2+3+5... that's it!

4. I have three envelopes, each containing two objects. In one, there is a silver square and a gold disk. In another, there are a gold square and a gold disk. In the third, there are a silver square and a silver disk.

   (a) I pick an envelope (at random), and take out an object (at random). It is gold. What is the probability that the second object is silver?

   We have 4 possibilities: (S,S) (prob 1/3), (S,G) (prob 1/6) , (G,S) prob (1/6) , (G,G) (prob 1/3). We want $A = \{\text{2nd item is S}\}$ and $B = \{\text{1st item is G}\}$. $P(A|B) = P(AB)/P(B) = \frac{P(GS)}{P(G,S)+P(G,G)} = (1/6)/(1/2) = 1/3$.

   (b) I put the objects back in their envelope and shuffle the envelopes. I again pick an envelope (at random), and take out an object (at random). It is a gold disk. What is the probability that the second object is silver?

   Let $D$ be disk, $Q$ be square. Our sample space, after we know the first object is a gold disk, is now $(GD, SQ), (GD, GQ)$. So, the probability is 1/2.

5. On the first day of a non-leap-year, I put \$1 in a box. On the second day, I put \$2 in the box. On the third day, I put \$3 in. And so on. At the end of the year (365 days), how much money is in the box?

   The first day you put in \$1, the last day you put in \$365. The average of these two is 183. The average of the second and penultimate days is also 183. Etc. So, the total is $365 \times 183 = \$66795$.

6. Alice and Bob are playing rock-paper-scissors. If both Alice and Bob play the same hand, they play again. What is the expected number of turns before someone wins?

   This is a Geometric distribution with $p = 2/3$ (probability someone wins). The expected value of a random variable with total number of tries upto the first success is $1/p = 3/2$. Since we are interested in the expected number of turns *before* somebody wins, our answer is $3/2 - 1 = 1/2$.

## 2 Continuous

1. Let $X$ be a normal random variable with mean 3 and variance 1, and let $Y$ be a normal random variable with mean 4 and variance 2.

   (a) What is the distribution of $Z = X + Y$? $Normal(7, 3)$

   (b) What is the probability that $Z$ is between 6 and 8? $\mathbf{P}(6 \leq Z \leq 8) = \mathbf{P}(\frac{6-7}{\sqrt{3}} \leq \frac{Z-7}{\sqrt{3}} \leq \frac{8-7}{\sqrt{3}}) = \mathbf{P}(-0.577 \leq \frac{Z-7}{\sqrt{3}} \leq 0.577) = 0.44$ (from standard normal tables)

2. I am waiting for a bus, that I know will arrive at some time between 1pm and 2pm, with all times being equally likely. It gets to 1:30, and the bus has still not arrived. What is the probability that it arrives before 1:40? 1/3

3. Let $X$ be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 0.125x + 0.125 & -1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

What is the PDF of $Z = |X|$? The CDF is $P(Z \le z)$. If $z < 0$ this is zero since $Z$ is always positive. If $z > 3$ this is one. So the action is in between.

If $0 \le z \le 1$, then

$$F_Z(z) = P(|X| \le z) = P(-z \le X \le z) = \int_{-z}^{z} (0.125x + 0.125)dx = .25z.$$

If $1 < z \le 3$ then

$$F_Z(z) = P(|X| \le z) = P(-1 < X < 1) + P(1 \le X \le z) = .25 + .125(z^2/2 - 1) + .125(z - 1)$$

Differentiating we get:

$$f_Z(z) = \begin{cases} 0.25 & 0 \le z \le 1 \\ 0.125 + 0.125z & 1 < z \le 3 \\ 0 & \text{otherwise} \end{cases}$$

1. Let $X$ be a continuous random variable, with PDF:

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 0.5 & 0 \le x < 1 \\ ce^{-x} & x \ge 1 \end{cases}$$

   (a) What is $c$? We need $\int_0^\infty f_X(x)dx = 1$, so $\int_1^\infty ce^{-x}dx = 0.5$. The integral gives $c[-e^{-x}]_1^\infty = ce^{-1}$ so $c = 0.5e^1$

   (b) What is the conditional expectation of $X$, given $X < 1$?

$$E[X|X < 1] = 0.5$$

   (c) What is the conditional expectation of $X$, given $X \ge 1$? $E[X1(X \ge 1)] = \int_1^\infty cxe^{-x}dx = 1$, since $ce^{-x}$ is $f_X(x)1(x \ge 1)$.

   Note that

$$E[X|X \ge 1] = E[X1(X \ge 1)]/P(X \ge 1) = 1/(1/2) = 2$$

   Another way to do this is to note that

$$f_{X|X\ge1}(x) = f_X(x)1(X \ge 1)/P(X \ge 1) = ce^{-x}/0.5$$

   and so

$$E[X|X \ge 1] = \int_1^\infty cxe^{-x}dx/0.5 = 2c\int_1^\infty xe^{-x}dx$$

   Substituting, x-1=v

$$2c\int_0^\infty (1+v)e^{-(1+v)}dv = \frac{2c}{e}(1 + \int_0^\infty ve^{-v}dv) = 4c/e = 2$$

(d) What is the expectation of $X$?

$$E[X] = P(X < 1) \times 0.5 + P(X \geq 1) \times 2 = 0.5 \times 0.5 + 0.5 \times 2 = 1.25$$

2. Let $X$ and $Y$ be random variables with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{ay}{x^2} & x \geq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is $a$?

$$f_X(x) = \int_0^1 \frac{ay}{x^2} dy = \left[\frac{ay^2}{2x^2}\right]_0^1 = \frac{a}{2x^2}$$

$$1 = \int_1^\infty \frac{a}{2x^2} dx = \left[-\frac{a}{2x}\right]_1^\infty = 1/2$$

so, $a = 2$

(b) What is the conditional PDF $f_{Y|X}(y|x)$ of $Y$ given $X = x$?

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{ay/x^2}{a/2x^2} = 2y$$

(c) What is the conditional expectation of $Y$ given $X$?

$$E[Y|X = x] = \int_0^1 2y^2 dy = 2/3$$

So, $E[Y|X] = 2/3$.

(d) What is the expected value of $Y$? $E[Y] = E[E[Y|X]] = 2/3$

3. Let $X$ and $Y$ be uniform random variables between 0 and 1. What is the probability that:

(a) $X < Y$ 0.5

(b) $X < 2Y$ If $Y = y$, $\mathbf{P}(X \leq 2y) = 2y$ if $y \leq 0.5$, 1 if $y > 0.5$, . So, overall, $\mathbf{P}(X \leq 2Y) = \int_0^{0.5} 2ydy + \int_{0.5}^1 dy = 0.25 + 0.5 = 0.75$.

(c) $X + Y < 0.5$ If $Y = y$, $\mathbf{P}(X + Y < 0.5|Y = y) = 0.5 - y$ for $y < 0.5$. Integrating over $y$ we have:

$$\mathbf{P}(X + Y < 0.5) = \int_0^{0.5} (0.5 - y)dy = 0.125$$

(d) $\max\{X, Y\} \leq 0.7$ $\mathbf{P}(\max\{X, Y\} \leq 0.7 = \mathbf{P}(X \leq 0.7)\mathbf{P}(Y \leq 0.7) = 0.7^2 = 0.49$

4. Let $X$ be the number of ice-cream cones a vendor sells on a day. If the average temperature of a summer day is a random variable $Y$ (in Fahrenheit), where $Y \sim Uniform([95, 105])$. We also have $X \sim Poisson(Y^2/10 + Y/5 + 5)$.

(a) What is $E[X|Y]$? $E[X|Y=y] = y^2/10 + y/5 + 5$. So $E[X|Y] = Y^2/10 + Y/5 + 5$.

(b) What is $E[X]$? Remember $E[Y^2] = \mathrm{var}(Y) + E[Y]^2 = (105-95)^2/12 + 100^2 = 1008$. $E[X] = E[Y^2]/10 + E[Y]/5 + 5 = 1008/10 + 100/5 + 5 = 100.8 + 20 + 5 = 125.8$.

5. If there are no distractions, it takes me 30 minutes to walk to the store. However, if I pass someone with a dog, I stop and pet the dog and chat to the owner. The number $Y$ of dogs I pass is a Poisson random variable with mean 2. Each time I stop, the number of minutes I spend petting the dog and chatting is an exponential random variable with PDF:

$$f_X(x) = 0.5e^{-0.5x}$$

(a) If I see a single dog, what is the expectation and variance of the time spent petting the dog and chatting to its owner? $X$ is exponential, and so $E[X] = \int_0^\infty 0.5xe^{-0.5x}dx = 2$, $E[X^2] = \int_0^\infty 0.5x^2e^{-0.5x} = 8$, $\mathrm{var}(X) = E[X^2] - E[X]^2 = 4$.

(b) What is the conditional expectation of the total time spent petting dogs and chatting to their owners, as a function of $Y$? $E[X|Y=y] = 2y$, (expectation of sum), so $E[X|Y] = 2Y$

(c) Using the law of iterated expectation calculate $E[X]$. $E[X] = E[E[X|Y]] = 2E[Y] = 2 \times 2 = 4$.

## 3 Statistics

1. Consider $n$ iid Poisson random variables $X_i \sim Poisson(\lambda)$.

(a) What is the MLE $\hat{\lambda}$ of $\lambda$? Calculate log likelihood $\ell_\lambda = \sum_i \log(e^{-\lambda}\lambda^{X_i}/X_i!) = -n\lambda + \sum_i X_i \log\lambda + const$. The constant term does not depend on $\lambda$, so we dont need to worry about it. Differentiate and set to zero and solve. $-n + \sum_i X_i/\hat{\lambda} = 0$. So $\hat{\lambda} = \sum_i X_i/n$.

(b) Is it unbiased or biased? Unbiased.

2. Consider $n$ iid Exponential random variables $X_i \sim Exp(\lambda)$.

(a) What is the MLE $\hat{\lambda}$ of $\lambda$? We did this in class. $\hat{\lambda} = 1/\bar{X}_n$.

(b) Is $1/\hat{\lambda}$ an unbiased or biased estimator of $1/\lambda$? $E[1/\hat{\lambda}] = \sum_i E[X_i]/n = 1/\lambda$. So unbiased.

(c) What is the variance of $1/\hat{\lambda}$? $\mathrm{var}(1/\hat{\lambda}) = \mathrm{var}(\bar{X}_n) = \mathrm{var}(X_1)/n = \frac{1}{n\lambda^2}$

(d) Using CLT approximate the distribution of $1/\hat{\lambda}$. $\frac{1}{\hat{\lambda}} \sim N\left(\frac{1}{\lambda}, \frac{1}{n\lambda^2}\right)$

(e) Using CLT, compute how big $n$ had to be in order to have $P(|1/\hat{\lambda} - 1/\lambda| \leq 0.1/\lambda) = .95$. $P(\frac{|1/\hat{\lambda}-1/\lambda|}{\frac{1/\lambda}{\sqrt{n}}} \leq 0.1\sqrt{n}) \approx P(|Z| \leq 0.1\sqrt{n}) = .95$. So $.1\sqrt{n} = 1.96$ and so $n \approx 400$.

(f) Using Chebyshev's inequality, use the $n$ you found in the last question to upper bound $P(|1/\hat{\lambda} - 1/\lambda| \geq 0.1/\lambda)$. $P(|1/\hat{\lambda} - 1/\lambda| \geq 0.1/\lambda) \leq \text{var}(1/\hat{\lambda})/(.01/\lambda^2) = 1/(n \times .01) = 1/4$.

3. On a given day, a $Poisson(100)$ number of insects fly through my yard. Using an appropriate approximation, what is the the probability that, over the month of May (31 days), the average number of insects is between 98 and 102? You may use the fact that a $Poisson(\lambda)$ random variable has mean and variance $\lambda$.

   $E[X_i] = \lambda$ so $E[\bar{X}] = \lambda$. $\text{var}(X_i) = \lambda$, so $\text{var}(\bar{X}) = \lambda/n$. We can approximate the distribution as $Normal(\lambda, \lambda/n) = Normal(100, 100/31)$.

$$\mathbf{P}(98 \leq \bar{X} \leq 102) \approx \mathbf{P}(\frac{-2}{\sqrt{100/31}} \leq Z \leq \frac{2}{\sqrt{100/31}} = 0.73$$

4. I am interested in seeing whether a sequence of 16 observations has zero mean. My null hypothesis is $H_0 : \mu = 0$, and my alternative is $H_1 : \mu \neq 0$. I know the variance is 1. What is an appropriate rejection region for the null hypothesis at 0.05 significance?

   I want to pick $\gamma$ s.t. $P(\bar{X} > \gamma) = 0.025$ under the null. Under the null, $\bar{X} \sim Normal(0, 1/16)$, so $\mathbf{P}(\bar{X} > \gamma) = \mathbf{P}(4\bar{X} > 4\gamma) = 0.025$. We have $P(Z > 1.96) = 0.025$, so $\gamma = 1.96/4 = 0.49$. So, reject if $|\bar{X}| > 0.49$.

5. I set up a motion sensor to look for intruders. The signal when there is no intruder is a normal random variable with mean 0 and variance 1. The signal where there is an intruder is a normal random variable with mean 1 and variance 2.

   (a) If I raise the alarm if the signal is greater than 0.7, what is my Type I error? Type I error is probability of false rejection of the null, i.e. $P(X > 0.7)$ under the $Normal(0, 1)$ null. From tables, we get $\alpha = 1 - 0.7580 = 0.242$

   (b) What is the corresponding Type II error? Type II error is probability of false acceptance of the null, i.e. $P(X < 0.7)$ under the $Normal(1, 2)$ null. This is the same as $P((X - 1)/\sqrt{2} < -0.3/\sqrt{2}) = P(Z < -0.21) = 1 - 0.5832 = 0.4168$ (from the standard normal tables)

6. I have a coin with unknown probability of heads $\theta$. I throw the coin $n$ times, and count the number $k$ of heads. What is the MLE estimator of $\theta$? Is it biased or unbiased? *hint: we can write the likelihood of a single Bernoulli random variable as* $f_X(x_i; \theta) = \theta^{x_i}(1 - \theta)^{1 - x_i}$

$$\begin{aligned} \text{log likelihood } \ell_\theta &= \sum_i \log f_X(x_i; \theta) \\ &= \sum_i (x_i \log \theta + (1 - x_i) \log(1 - \theta)) \\ & \log \theta \sum_i x_i + \log(1 - \theta)(n - \sum_i x_i) \end{aligned}$$

   differentiating and setting to zero,

   $\hat{\theta} = \sum_i X_i/n$ and by linearity of expectation, since $E[X_i] = \theta$, this is unbiased.

7. I have 10 samples $x_1, \dots, x_{10}$ from a normal distribution with unknown mean $\mu$ and variance 4. The mean $\bar{x} = \frac{\sum_{i=1}^{10} x_i}{10}$ of the samples is 5.33.

   (a) What is the mean and variance of $\bar{x}$?
   $E[\bar{x}] = \mu$, $\mathrm{var}(\bar{x}) = 4/10 = 2/5$.

   (b) Give a 95% confidence interval for the mean. We want $\mathbf{P}(|\bar{X} - \mu| < \gamma) = 0.95$. So, $\mathbf{P}(\bar{X} - \mu < \gamma) = 0.975$. So, $\mathbf{P}(\frac{\bar{X}-\mu}{\sqrt{2/5}} < \gamma/\sqrt{2/5}) = 0.975$. We know $\mathbf{P}(Z < 1.96) = 0.975$, so $\gamma = 1.96\sqrt{2/5} = 1.24$.
   So, our C.I. is $(\bar{X} - 1.24, \bar{X} + 1.24)$

   (c) If I see 10 more observations, will the confidence interval be narrower or wider?
   Narrower - because the variance of the sample mean will be smaller

8. I want to estimate the mean water level of a lake. I assume that, on any given day, the water level will be a normal random variable with unknown mean corresponding to the mean water level, and unknown variance $\sigma^2$.

   - Show that $\hat{s}_n^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$ is an unbiased estimator for the variance. Bookwork

   - You record the water height over three days as 115, 121, 118. Calculate an unbiased estimator for the mean and variance of the water level.

$$\bar{X} = (115 + 121 + 118)/3 = 118$$
$$\hat{s}_n^2 = \frac{(115 - 118)^2 + (121 - 118)^2 + (118 - 118)^2}{2} = 9$$

   - Using the attached $t$-distribution tables, give a 0.95 confidence region for the true mean.
   We want $\mathbf{P}(|\bar{X} - \mu| > \gamma) = 0.05$, i.e. $\mathbf{P}(\bar{X} - \mu > \gamma) = 0.025$.
   $(\bar{X} - \mu)/\sqrt{3} \sim t_2$
   **Do not forget to divide the sample standard deviation by $\sqrt{n}$ to get the standard deviation of the average or $\bar{X}$. Thanks to Mert Hizli for catching this!**
   Translating to the standard normal, this is the same as $\mathbf{P}(\frac{\bar{X}-\mu}{\sqrt{3}} = \gamma/\sqrt{3}) = 0.025$. From the t-table with 2 degrees of freedom, we have $\gamma/\sqrt{3} = 4.303$ so $\gamma = 7.45$. So, an appropriate CI is $(110.55, 125.45)$

9. I have 500 samples from a Poisson distribution with pdf $p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}$. My null hypothesis is that $\lambda = 1$; my alternative hypothesis is that $\lambda > 1$.

   (a) Is this a one-sided or two-sided test? one-sided

   (b) Under the null hypothesis, use a normal approximation to estimate the probability that $\bar{X} < 0.9$. You may use the fact that the mean and variance of a Poisson random variable are both $\lambda$.

Under the null,

$$E[\bar{X}] = \lambda \qquad\qquad \text{var}(\bar{X}) = \lambda/500$$

.

$$\begin{aligned}\mathbf{P}(\bar{X} < 0.9) &= \mathbf{P}\left(\frac{\bar{X}-1}{1/\sqrt{500}} < -0.1\sqrt{500}\right) \\ &\approx \mathbf{P}(Z < -2.23) = 1 - \mathbf{P}(Z < 2.23) = 1 - 0.9871 = 0.0129\end{aligned}$$

(c) We want to choose a critical value $\xi$ such that we reject the null hypothesis if $\bar{X} > \xi$. Find a value of $\xi$ that gives a significance of 0.01.

We are looking for $\xi$ s.t. $\mathbf{P}(\bar{X} > \xi) = 0.01$. Translating to the standard normal, this is the same as $\mathbf{P}\left(\frac{X\bar{-}1}{1/\sqrt{500}} < \sqrt{500}(\xi-1)\right) = 0.01$. From the standard normal tables, $\sqrt{500}(\xi - 1) = 2.33$ so $\xi = 1.10$.

10. I have two independent sequences, $X$ and $Y$ (which are also mutually independent), with known variance $\sigma^2$ and unknown means $\mu_X$ and $\mu_Y$. I want to test the null hypothesis that $\mu_X = 2\mu_Y$ using a two-sided significance test.

(a) Suggest an appropriate statistic – specifically, a function of $\bar{X}$ and $\bar{Y}$ – that has zero mean under the null hypothesis. $\bar{X} - 2\bar{Y}$

(b) What is the variance of this statistic? $\text{var}(\bar{X}) + \text{var}(2\bar{Y}) = \text{var}(\bar{X}) + 4\text{var}(\bar{Y}) = 5\sigma^2$

(c) Construct an appropriate significance test for the null hypothesis, at significance level $\alpha = 0.05$

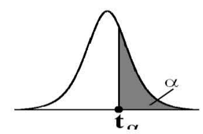
We want $\mathbf{P}(|\bar{X} - 2\bar{Y}| > \gamma) = 0.05$ under the null. This is the same as $\mathbf{P}(\bar{X} - 2\bar{Y} > \gamma) = 0.025$, which in turn is the same as $\mathbf{P}\left(\frac{\bar{X}-2\bar{Y}}{\sqrt{5}\sigma} < \frac{\gamma}{\sqrt{5}\sigma}\right) = 0.025$. This gives $\frac{\gamma}{\sqrt{5}\sigma} = 1.96$ so $\gamma = 4.38\sigma$

**Standard normal table**

| | 0 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |

**Percentage Points of the $t$ Distribution; $t_{\nu,\alpha}$**

**$P(T > t_{\nu,\alpha}) = \alpha$**




| | α | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **ν** | **0.40** | **0.30** | **0.20** | **0.15** | **0.10** | **0.05** | **0.025** | **0.02** | **0.015** | **0.01** | **0.0075** | **0.005** | **0.0025** | **0.0005** |
| **1** | 0.325 | 0.727 | 1.376 | 1.963 | 3.078 | 6.314 | 12.706 | 15.895 | 21.205 | 31.821 | 42.434 | 63.657 | 127.322 | 636.590 |
| **2** | 0.289 | 0.617 | 1.061 | 1.386 | 1.886 | 2.920 | 4.303 | 4.849 | 5.643 | 6.965 | 8.073 | 9.925 | 14.089 | 31.598 |
| **3** | 0.277 | 0.584 | 0.978 | 1.250 | 1.638 | 2.353 | 3.182 | 3.482 | 3.896 | 4.541 | 5.047 | 5.841 | 7.453 | 12.924 |
| **4** | 0.271 | 0.569 | 0.941 | 1.190 | 1.533 | 2.132 | 2.776 | 2.999 | 3.298 | 3.747 | 4.088 | 4.604 | 5.598 | 8.610 |
| **5** | 0.267 | 0.559 | 0.920 | 1.156 | 1.476 | 2.015 | 2.571 | 2.757 | 3.003 | 3.365 | 3.634 | 4.032 | 4.773 | 6.869 |
| **6** | 0.265 | 0.553 | 0.906 | 1.134 | 1.440 | 1.943 | 2.447 | 2.612 | 2.829 | 3.143 | 3.372 | 3.707 | 4.317 | 5.959 |
| **7** | 0.263 | 0.549 | 0.896 | 1.119 | 1.415 | 1.895 | 2.365 | 2.517 | 2.715 | 2.998 | 3.203 | 3.499 | 4.029 | 5.408 |
| **8** | 0.262 | 0.546 | 0.889 | 1.108 | 1.397 | 1.860 | 2.306 | 2.449 | 2.634 | 2.896 | 3.085 | 3.355 | 3.833 | 5.041 |
| **9** | 0.261 | 0.543 | 0.883 | 1.100 | 1.383 | 1.833 | 2.262 | 2.398 | 2.574 | 2.821 | 2.998 | 3.250 | 3.690 | 4.781 |
| **10** | 0.260 | 0.542 | 0.879 | 1.093 | 1.372 | 1.812 | 2.228 | 2.359 | 2.527 | 2.764 | 2.932 | 3.169 | 3.581 | 4.587 |
| **11** | 0.260 | 0.540 | 0.876 | 1.088 | 1.363 | 1.796 | 2.201 | 2.328 | 2.491 | 2.718 | 2.879 | 3.106 | 3.497 | 4.437 |
| **12** | 0.259 | 0.539 | 0.873 | 1.083 | 1.356 | 1.782 | 2.179 | 2.303 | 2.461 | 2.681 | 2.836 | 3.055 | 3.428 | 4.318 |
| **13** | 0.259 | 0.538 | 0.870 | 1.079 | 1.350 | 1.771 | 2.160 | 2.282 | 2.436 | 2.650 | 2.801 | 3.012 | 3.372 | 4.221 |
| **14** | 0.258 | 0.537 | 0.868 | 1.076 | 1.345 | 1.761 | 2.145 | 2.264 | 2.415 | 2.624 | 2.771 | 2.977 | 3.326 | 4.140 |
| **15** | 0.258 | 0.536 | 0.866 | 1.074 | 1.341 | 1.753 | 2.131 | 2.249 | 2.397 | 2.602 | 2.746 | 2.947 | 3.286 | 4.073 |
| **16** | 0.258 | 0.535 | 0.865 | 1.071 | 1.337 | 1.746 | 2.120 | 2.235 | 2.382 | 2.583 | 2.724 | 2.921 | 3.252 | 4.015 |
| **17** | 0.257 | 0.534 | 0.863 | 1.069 | 1.333 | 1.740 | 2.110 | 2.224 | 2.368 | 2.567 | 2.706 | 2.898 | 3.222 | 3.965 |
| **18** | 0.257 | 0.534 | 0.862 | 1.067 | 1.330 | 1.734 | 2.101 | 2.214 | 2.356 | 2.552 | 2.689 | 2.878 | 3.197 | 3.922 |
| **19** | 0.257 | 0.533 | 0.861 | 1.066 | 1.328 | 1.729 | 2.093 | 2.205 | 2.346 | 2.539 | 2.674 | 2.861 | 3.174 | 3.883 |
| **20** | 0.257 | 0.533 | 0.860 | 1.064 | 1.325 | 1.725 | 2.086 | 2.197 | 2.336 | 2.528 | 2.661 | 2.845 | 3.153 | 3.850 |
| **21** | 0.257 | 0.532 | 0.859 | 1.063 | 1.323 | 1.721 | 2.080 | 2.189 | 2.328 | 2.518 | 2.649 | 2.831 | 3.135 | 3.819 |
| **22** | 0.256 | 0.532 | 0.858 | 1.061 | 1.321 | 1.717 | 2.074 | 2.183 | 2.320 | 2.508 | 2.639 | 2.819 | 3.119 | 3.792 |
| **23** | 0.256 | 0.532 | 0.858 | 1.060 | 1.319 | 1.714 | 2.069 | 2.177 | 2.313 | 2.500 | 2.629 | 2.807 | 3.104 | 3.768 |
| **24** | 0.256 | 0.531 | 0.857 | 1.059 | 1.318 | 1.711 | 2.064 | 2.172 | 2.307 | 2.492 | 2.620 | 2.797 | 3.091 | 3.745 |
| **25** | 0.256 | 0.531 | 0.856 | 1.058 | 1.316 | 1.708 | 2.060 | 2.167 | 2.301 | 2.485 | 2.612 | 2.787 | 3.078 | 3.725 |
| **26** | 0.256 | 0.531 | 0.856 | 1.058 | 1.315 | 1.706 | 2.056 | 2.162 | 2.296 | 2.479 | 2.605 | 2.779 | 3.067 | 3.707 |
| **27** | 0.256 | 0.531 | 0.855 | 1.057 | 1.314 | 1.703 | 2.052 | 2.158 | 2.291 | 2.473 | 2.598 | 2.771 | 3.057 | 3.690 |
| **28** | 0.256 | 0.530 | 0.855 | 1.056 | 1.313 | 1.701 | 2.048 | 2.154 | 2.286 | 2.467 | 2.592 | 2.763 | 3.047 | 3.674 |
| **29** | 0.256 | 0.530 | 0.854 | 1.055 | 1.311 | 1.699 | 2.045 | 2.150 | 2.282 | 2.462 | 2.586 | 2.756 | 3.038 | 3.659 |
| **30** | 0.256 | 0.530 | 0.854 | 1.055 | 1.310 | 1.697 | 2.042 | 2.147 | 2.278 | 2.457 | 2.581 | 2.750 | 3.030 | 3.646 |
| **40** | 0.255 | 0.529 | 0.851 | 1.050 | 1.303 | 1.684 | 2.021 | 2.123 | 2.250 | 2.423 | 2.542 | 2.704 | 2.971 | 3.551 |
| **60** | 0.254 | 0.527 | 0.848 | 1.045 | 1.296 | 1.671 | 2.000 | 2.099 | 2.223 | 2.390 | 2.504 | 2.660 | 2.915 | 3.460 |
| **120** | 0.254 | 0.526 | 0.845 | 1.041 | 1.289 | 1.658 | 1.980 | 2.076 | 2.196 | 2.358 | 2.468 | 2.617 | 2.860 | 3.373 |
| **∞** | 0.253 | 0.524 | 0.842 | 1.036 | 1.282 | 1.645 | 1.960 | 2.054 | 2.170 | 2.326 | 2.432 | 2.576 | 2.807 | 3.291 |