Midterm 2

SDS321

Spring 2016

You may use a two (2 sided) pages of notes, and you may use a calculator.

This exam consists of five questions, containing multiple sub-questions. The assigned points are noted next to each question; the total number of points is 25. You have 75 minutes to answer the questions.

Please answer all problems in the space provided on the exam. Use extra pages if needed. Of course, please put your name on extra pages.

Read each question carefully, show your work and clearly present your answers. Note, the exam is printed two-sided - please don't forget the problems on the even pages!

Good Luck!

Name: ____

UTeid: ____

1. (5 pts) Let $X \sim N(70, 100)$ be a normal random variable with $\mu = 70$ and $\sigma = 10$.

   (a) (1 pt) Find $P(X < 60)$.

   (b) (2 pts) Find $P(60 < X < 80)$.

   Solution: $P(60 < X < 80) = P(-1 \leq Z \leq 1) = .6827$

   Grading: 1/2 pt for standardizing. 1 pt for realizing $P(-1 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq -1) = 1 - 2P(Z \leq -1)$***. 1/2 pt for correct table lookup.***

   (c) (2 pts) Find q such that $P\left(70 - 2q < X < 70 + 2q\right) = 0.6827$.

   Solution: *q=5.*

   Grading: 1 pt for realizing that this the same as the last question. 1 pt for correct answer.

2. (5 pts) A continuous random variable X has the following density function.

$$f_X(x) = \begin{cases} 0.5 & -1 < x < 0 \\ 0.5\, e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) (2 pts) Find $E[X|X \geq 0]$

Solution: *$f_{X|X\geq 0}(x) = e^{-x}$ when $x > 0$ and zero otherwise. This is just an exponential. So $E[X|X \geq 0] = 1$*

Grading: 1 pt for getting the conditional pdf. 1 pt for calculating the expectation of that. Take 1 pt off if they just calculated $E[X1(X < 0)]$

(b) (2 pts) Find $E[X|X < 0]$

Solution: *$f_{X|X<0}(x) = 1$ when $x \in [-1, 0]$ and zero otherwise. This is just an uniform. So $E[X|X \leq 0] = -.5$*

Grading: 1 pt for getting the conditional pdf. 1 pt for calculating the expectation of that. Take 1 pt off if they just calculated $E[X1(X > 0)]$

(c) (1 pt) Find $E[X]$

Solution: *$P(X < 0) = 0.5$ and so $E[X] = E[X|X \leq 0]/2 + E[X|X > 0]/2 = .25$*

Grading: 1/2 pt for understanding that this is a total expectation question. 1/2 pt for $P(X \leq 0)$

3. (5 pts) The median of a continuous r.v. with CDF F_X is that value m such that $F_X(m) = P(X \leq m) = 1/2$, i.e. a random variable is just as likely to be larger than the median as it is to be smaller. Find the median of X if X is:

(a) (1 pt) Uniform over $[a, b]$

Solution: $(a + b)/2$

Grading: 1 pt for correct answer.

(b) (1 pt) Normal with parameters (μ, σ^2)

Solution: μ

Grading: 1 pt for correct answer.

(c) (3 pts) $f_X(x) = e^{-x}$ when $x \geq 0$ and zero otherwise.

Solution: $P(X \geq m) = e^{-m} = 0.5$ *and so* $m = .693$

Grading: 1 pt for setting up the problem. 1 pt for correct cdf of exponential and 1 pt for correct answer.

4. The joint pdf of two random variables X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} e^{-(0.5x+2y)} & x \geq 0, y \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

(a) (1 pt) Are X and Y independent? Explain your answer.

Solution: *Yes. The joint factorizes.*

Grading: 1 pt for correct explanation.

(b) (3 pts) What is the pdf of $\max(X, Y)$?

Solution: $f_{X,Y}(x,y) = 0.5e^{-0.5x} \times 2e^{-2y}$ *i.e.* $X \sim Exp(.5)$ *and* $Y \sim Exp(2)$. *Let* $Z = \max(X,Y)$. $P(Z \leq z) = P(X \leq z, Y \leq z) = P(X \leq z)P(Y \leq z) = (1 - e^{-.5z})(1 - e^{-2z})$. $f_Z(z) = 0.5e^{-0.5z} + 2e^{-2z} - 2.5e^{-2.5z}$ *when* $z > 0$ *and 0 otherwise.*

Grading: 1 pt for writing the CDF of max in terms of $P(X \leq z, Y \leq z)$***. 1/2 pt for using independence. 1 pt for plugging in the CDF. 1/2 pt for taking a derivative.***

(c) (1 pt) What is $\mathrm{var}(2X - 3Y)$?

Solution: $4var(X) + 9var(Y) = 4 \times 4 + 9 \times .25.$

Grading: 1/2 pt for using independence and 1/2 for using the right variance of an exponential.

5. The joint pdf of two random variables X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{c}{x} & 0 < y < x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

(a) (1 pt) What is c?

Solution: $\int_0^1 \int_0^x \frac{c}{x} dx dy = c \int_0^1 dx = c = 1$

Grading: 1/2 pt for setting up normalization. 1/2 pt for correct answer.

(b) (1.5 pts) What is $f_X(x)$?

Solution: $f_X(x) = \int_0^x \frac{1}{x} dy = 1$ *for* $x \in (0, 1]$

Grading: 1/2 pt for correct marginal formula. 1/2 pt for correct limit. 1/2 pt for correct answer.

(c) (1.5 pts) What is $f_Y(y)$?

Solution: $f_Y(y) = \int_y^1 \frac{1}{x} dx = -\log y$ *for* $y \in (0, 1]$

Grading: same as the last one.

(d) (1 pt) Are X and Y independent? Explain your answer.

Solution: *The joint does not factorize. So not independent.*

Grading: 1 pt for correct explanation.

Useful distributions

All PDFs/PMFs are zero outside the range specified.

	PDF/PMF	$E[X]$	$\text{var}(X)$
Bernoulli	$p^x(1-p)^{1-x}, x = 0, 1$	p	$p(1-p)$
Binomial	$\binom{n}{k}p^k(1-p)^{n-k}$, k=0,1,...,n	np	$np(1-p)$
Geometric	$p(1-p)^{k-1}, k = 1, 2, 3, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	$\frac{\lambda^k e^{-\lambda}}{k!}, k = 1, 2, 3, \ldots$	λ	λ
Normal	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Exponential	$\lambda e^{-\lambda x}, x \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Uniform	$\frac{1}{b-a} \qquad a \leq x \leq b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$

Standard normal table

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990